A TEST OF AN ECONOMETRIC MODEL FOR
THE UNITED STATES, 1921-1947*

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This paper presents a revision of Lawrence Klein's sixteen-equation Model III for the United States.¹ The starting point of the revision is a test of Klein's model for 1946 and 1947 carried out by Andrew W. Marshall (17), which rejected several of Klein's equations. The equations of the revised model are estimated from a sample consisting of Klein's sample plus the two years 1946 and 1947. The estimates of the equations of the revised model are tested against the 1948 data.

In Sections 1-4 I have drawn freely and without specific acknowledgment on definitions and theorems from the published and unpublished literature, particularly on Anderson and Rubin (1), Haavelmo (5, 6, 7), and Koopmans (14, 15, 16).

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1 ECONOMETRIC MODELS: GENERAL CONCEPTS AND DEFINITIONS
When one thinks of science, one usually thinks also of experiments. In a typical experiment, there is one variable whose behavior is studied under various conditions. The experimenter fixes at will the values of all the other variables he thinks are important, and observes the one in which he is interested. He then repeats the process, fixing different values of the other variables each time. Some of these "other variables" may not be

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¹ Klein (11, 13). Numbers in parentheses in contexts like this indicate references listed in Appendix G.
under his control; the important thing is that they are fixed in advance of the experiment. The variables that are fixed in advance are called independent; the one that the experimenter merely observes is called dependent. The experimenter hopes to find a single equation that describes closely the relationship he has observed.

In more complicated situations there may be more than just one relationship among the variables studied. This is the case when there is a determinate result and, at the same time, there are two or more important variables that are not fixed in advance; there may even be no experimenter. Economics abounds with such situations. The simplest is of course a competitive market, in which neither price nor quantity is fixed in advance. The economist assumes that two relations between these variables, a supply equation and a demand equation, must be simultaneously satisfied.

The econometric work discussed here is based on the belief that we will do well to make our theory conform to this state of affairs. Accordingly we deal with systems of simultaneous equations, called structural equations. Each structural equation is assumed to describe an economic relation exactly except for random shocks; hence each contains a non-observable random disturbance (with mean assumed to be zero). Effects of errors in measuring variables are here assumed to be small relative to the disturbances (this kind of model is called a shock model, as distinct from an error model).

Time must enter into the equations if they are to describe a dynamic process. The work discussed here treats time as if it came in discrete chunks of equal size, called periods. The raw materials are time series for the variables considered. A given equation is supposed to represent a relation that holds, for any time period \( t \), among a given set of variables evaluated as of the time period \( t \), where \( t \) runs over a sequence of periods from 1 to \( T \).

Note that so far we have used the term 'variable' to denote something

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3 They may be constants fixed in advance or random variables with probability distributions fixed in advance (such as weather).

4 Structural equations are divided by Koopmans (14) into four classes: equations of economic behavior such as the consumption function, technical constraints such as the production function, institutional constraints such as tax schedules or reserve requirements, and definitional identities such as income equals consumption plus investment. Another possible type is made up of market adjustment equations, of which equilibrium conditions are a special case.

5 This means we believe that either (1) there are systematic discoverable causes for all the observed variation of the variables but we are satisfied for the time being if we can explain enough of the variation so that the residual appears random, or (2) there really are random elements in economic affairs. For present purposes we do not care which of these two is the case.

6 Identities are meant to be perfectly exact and hence contain no disturbances.
like national income $Y$ or price level $p$, which can have different values from one period to the next. We shall continue to use ‘variable’ in this way, but at times we shall use it instead to denote something like national income in period $t$, denoted by $Y_t$, or price level in period $t$, denoted by $p_t$. In the first sense, $Y_t$ and $Y_{t+1}$ are different values of the same variable $Y$ in different periods; in the second sense, $Y_t$ and $Y_{t+1}$ are different variables. It will usually be easy to tell from the context the sense in which the term is being used.

We might have $G$ structural equations containing $G + K$ variables (second sense) at time $t$. Suppose that of the variables at time $t$, $K$ are fixed in advance of time $t$, not by an experimenter to be sure, but by society, or by nature, or even by the past operation of the system of equations; they are called predetermined variables at time $t$. The $G$ variables, which are not fixed in advance but are determined by the economic process we seek to describe (analogous to the dependent variable in the simple experimental case at the beginning), are called jointly dependent variables at time $t$ (sometimes they are called current endogenous). More precisely, the definitions are as follows. (1) Variables (in the first sense) that are stochastically independent of the random disturbances in the equations are called exogenous; they may be arbitrarily fixed by some agency or process, or they may themselves be random. All other variables (in the first sense) are called endogenous. (2) Variables (in the second sense) which for time $t$ are either values of exogenous variables at times $t, t-1, t-2, \ldots$, or lagged values of endogenous variables, i.e., values at times prior to $t$, are called predetermined variables at time $t$. Variables (in the second sense) which for time $t$ are current values of endogenous variables, i.e., values at time $t$, are called jointly dependent variables at time $t$. Sometimes the phrase “at time $t$” is omitted, but whenever we speak of jointly dependent or predetermined variables, it is always understood.

Such a system of structural equations is called a structure, provided (1) that each equation is completely specified as to form and as to numerical values of parameters, and (2) that it is accompanied by a similarly completely specified joint probability distribution function of the disturbances.

One objective will be to find a structure or structures that will enable us to rationalize past observations of economic variables and to predict future ones.\(^6\) This will be difficult because we cannot observe structures or disturbances directly. However, we can observe samples from the joint

\(^6\) Or, more generally, a structural relation or a distribution may be completely specified by a graph, if it is not expressible in terms of simple functional forms. But such relations are very difficult to work with.

\(^7\) Here the “future” includes the part of the past that was not consulted in the process of finding the structure in question.
conditional probability distribution function \( \phi(y | z) \) of the jointly dependent variables \( y = (y_1, y_2, \ldots) \), given the predetermined variables \( z = (z_1, z_2, \ldots) \). It is clear that any given structure generates exactly one such distribution function, i.e., to any given structure there corresponds exactly one distribution function \( \phi(y | z) \) which is consistent with it. It is natural to ask whether this correspondence is one to one in both directions; in other words whether, if we knew only the conditional distribution \( \phi(y | z) \) of the jointly dependent variables given the predetermined variables, we could proceed backwards and find a unique structure that generates it. In general the answer is no, because in general there are several (or an infinite number of) structures consistent with a given \( \phi(y | z) \). Thus even apart from the sampling problem of estimating \( \phi(y | z) \) from a finite sample, it is in general not possible to find a unique structure by studying observations alone. This would not be serious were it not for the fact that in general the structures generating a given \( \phi(y | z) \) are not identical in their implications about the effects of economic policy decisions (see Sec. 2 for a more detailed discussion).

Sometimes it is possible, on theoretical grounds, i.e., on the basis of knowledge derived ultimately from other observations not used to estimate \( \phi(y | z) \), to find a set of restrictions that we believe must be satisfied by any structure that can make good predictions. Such a set of restrictions defines a model, i.e., the set of exactly those structures that satisfy the given restrictions.

A model is said to be structure-identifying, or simply identifying, if each possible distribution function \( \phi(y | z) \) is generated by exactly one structure belonging to the model. A structure is said to be identified (or identifiable) within a given model if the model contains no other structures generating the distribution \( \phi(y | z) \) that is generated by the given structure. It is important to note that the problem of the identification of structures is completely separate from the problem of estimating probability distributions from finite samples, and would exist (in the context of simultaneous equations) even if there were no random elements. It is of course to be hoped that enough theoretical restrictions are available to permit the construction of identifying models.

It is desirable that the models be more general rather than less. But at this early stage of the development of econometrics it is convenient to impose further restrictions in addition to (or even at the expense of some of) those derived from theory, in order to keep the models fairly simple. For instance, it is customary for simplicity's sake (though not conceptually necessary) to choose a model that is a parametric family of structures, i.e., a set of \( G \) simultaneous equations in \( G \) jointly dependent variables and a joint distribution function of disturbances, both having a specified
form but unspecified parameters. Such a model is preferable because of its relative ease of handling: first, the equations and probability distribution function of the model can be set up on the basis of previous knowledge; then observations of appropriate variables can be obtained; then the parameters can be estimated by straightforward statistical procedures (see Secs. 3 and 4). \(^6\)

Revisions of a model in the light of its performance in forecasting are of course permitted and expected; to this subject we shall return in Section 7.

To simplify computational procedures, further restrictions are placed on the model: it is assumed to be linear in the unknown parameters (though not necessarily in the variables); it deals with macro-variables (aggregates) as distinct from micro-; its disturbances are usually assumed to be normally distributed and serially uncorrelated. Obviously some of these restrictions make the model a poor approximation to the actual world. It can be expected that they will be made more realistic as statistical and economic theory, computational facilities, and data permit.

\(^6\)Consider the model represented by the following system of equations and restrictions:

1. \(D = \alpha_0 + \alpha_1 Y + \alpha_2 + u\)
2. \(S = \beta_0 + \beta_1 w + \beta_2 + v\)
3. \(u\) and \(v\) are normally distributed, with distribution function \(\phi(u,v)\) and means zero.
4. Successive drawings from \(\phi(u,v)\) are independent.
5. \(E(u \mid Y, w) = E(v \mid Y, w) = 0.\)

This model will be changed if we restrict any of its parameters to specific values or to specific ranges, or add new terms, or change the assumptions about the distribution \(\phi(u,v)\), etc. Two hypothetical structures belonging to it are:

6. \(D = -2\rho + .10Y + 1.5 + u\)
7. \(S = 3\rho - 2.6w - 0.8 + v\)
8. \(\sigma_u^2 = 1, \quad \sigma_v^2 = 4, \quad \sigma_{uw} = -1.5\)
9. restrictions (3) to (5) above

and

10. \(D = .16Y + 1.2 + u\)
11. \(S = 2.8\rho - 3w - 1.3 + v\)
12. \(\sigma_u^2 = 2, \quad \sigma_v^2 = 3, \quad \sigma_{uw} = 0\)
13. restrictions (3) to (5) above.

\(^9\)In practice, the tendency is to select a model after looking at the data to be used to estimate its parameters. This is useful and legitimate, even necessary, as a means of suggesting hypotheses. However, the effect is to make spuriously small the estimated standard errors we obtain from the usual formula, i.e., to give us excessive confidence in our estimates, because this formula assumes that all the restrictions implied in our model were derived from some a priori source of knowledge before we examined the data, whereas in fact some were derived from an examination of the data.
With these restrictions, a model consists of $G$ simultaneous equations in $G$ endogenous variables and a distribution function of disturbances, thus:

$$\begin{align*}
\left\{ \sum_{j=1}^{J} a_{gj} f^j (y_1', \ldots, y_g'; z_1', \ldots, z_h') &= u_{gt}, \quad g = 1, \ldots, G, \quad G \leq J, \\
\phi(u_t, \ldots, u_G) &= \text{joint normal distribution with mean zero.}
\end{align*}$$

where: $y_h'$ = jointly dependent variables at time $t$, $h = 1, \ldots, G$. $z_m'$ = predetermined variables at time $t$, $m = 1, \ldots, M$. $f^j$ = functions of the $y_h'$ and the $z_m'$, which are of a given form and contain no unknown parameters, $j = 1, \ldots, J$ (for example, $X$ or $(pX - c)/q$ in Klein's model; see Sec. 5). As a special case, $f^j$ might equal $y_h'$ for $j = 1, \ldots, G$ and equal $z_m'$ for $j = G + 1, \ldots, J$, if so, $J = G + M$ and we have linear equations; we do not restrict the model to this extent. $a_{gj}$ = known or unknown parameters (some of which may be zero).

successive drawings from $\phi(u_t, \ldots, u_G)$ are independent of one another and of current and previous values of $z_m'$, $m = 1, \ldots, M$.

each variable and disturbance is understood to carry the subscript $t$ to indicate that it is evaluated as of period $t$, $t = 1, \ldots, T$.

To illustrate (A) concretely, consider the following simple income-consumption model, where $C$, $V$, and $Y$ are respectively consumption, investment, and income:

$$\begin{align*}
(a') \quad a_{11} C/Y + a_{12} Y + a_{15} &= u_t \\
Y - C - V &= 0 \\
\phi(u_t) &= N(0, \sigma^2)
\end{align*}$$

Here $Y$ and $C$ are the endogenous variables $y_h'$ and $y_g'$, and $V$ is exogenous. $G = 2$ and $J = 5$; $f$ ($C$, $Y$, $V$) is equal to $C/Y$ when $j = 1$, to $Y$ when $j = 2$, to $-C$ when $j = 3$, to $-V$ when $j = 4$, and to 1 when $j = 5$. $a_{18} = a_{21} = a_{26} = 0; \ a_{22} = a_{23} = a_{24} = 1; \ u_2 = 0$. $N(0, \sigma^2)$ is the normal distribution with mean zero and variance $\sigma^2$. By checking these statements, the reader can verify that (A') is a special case of (A). Similarly, any system of equations that is linear in the unknown parameters can be expressed in the form (A).

We can rewrite (A) in a more convenient form by separating all the $f^j$ into two classes: (1) those which involve some jointly dependent variables, i.e., some subset of the $y_h'$ (whether or not they also involve any of the $z_m'$), and (2) those which are completely predetermined, i.e., involve
only the $z_m$. We call the first group $y_i, i = 1, \ldots, l$, and the second group $z_k, k = 1, \ldots, K$. Then the model becomes

\[
(B) \begin{cases}
\sum_{i=1}^{l} \beta_i y_i + \sum_{k=1}^{K} \gamma_k z_k = u_g, \ g = 1, \ldots, G.
\end{cases}
\]

$\phi(u_1, \ldots, u_G)$ = joint normal distribution with mean zero.

where: $y_i$ = jointly dependent variables at time $t$ (including functions $f^i$ which depend on any of the $y_i$, and have no unknown parameters), $i = 1, \ldots, l$.

$z_k$ = predetermined variables at time $t$ (including functions $f^j$ which depend on no $y_j$, and have no unknown parameters), $j = 1, \ldots, J$.

$\beta_i = \alpha_i$ for all $i, j$ such that $f^i = y_i, g = 1, \ldots, G, i = 1, \ldots, l, j = 1, \ldots, J$.

$\gamma_{jk} = \alpha_{jk}$ for all $j, k$ such that $f^j = z_k, g = 1, \ldots, G, j = 1, \ldots, J, k = 1, \ldots, K$.

successive drawings from $\phi(u_1, \ldots, u_G)$ are independent of one another, and of current and previous values of $z_k, k = 1, \ldots, K$.

each variable and disturbance is again understood to carry the subscript $t$.

This is the form in which we shall use the model. In general $l \geq G$, so that there appear to be more jointly dependent variables than equations. To complete the model, it is necessary and sufficient to include the identities that define the $y_i$ in terms of the $y_i$ — thus in the case of Klein’s model mentioned above, there would be an equation defining, say, $y_2$, thus: $y_2 = (pX - \epsilon)/q$. There are $l$ such identities.

To illustrate (B) concretely, rewrite (A') as follows, together with the identities defining the $y_i$:

\[
(B') \begin{cases}
\beta_{11} y_1 + \beta_{12} y_2 + \gamma_{12} z_2 = u_1 \quad & y_1 = C/Y \\
y_2 + y_3 + z_1 = 0 \quad & y_2 = Y \\
y_3 = -C \quad & y_3 = -C \\
\phi(u_1) = N(0, \sigma^2)
\end{cases}
\]

Here $G = 2$ again, of course, $l = 3$, and $K = 2$; $z_1 = -V$, and $z_2 = 1$. $\beta_{11} = \alpha_{11}, \beta_{12} = \alpha_{12}, \beta_{13} = \alpha_{13} = 0, \beta_{21} = \alpha_{21} = 0, \beta_{22} = \alpha_{22} = 1, \beta_{23} = \alpha_{23} = 1, \gamma_{11} = \alpha_{14} = 0, \gamma_{12} = \alpha_{15}, \gamma_{21} = \alpha_{24} = 1, \gamma_{22} = \alpha_{25} = 0$. The reader can verify that (B') is a special case of (B), and that any set of equations of the form (A) can be rewritten in the form (B).

Simple procedures are available for determining whether a model like
(B) is structure-identifying, i.e., whether the structures belonging to (B) are identified. A structure is said to be identified within the model (B) if and only if all its equations are. A necessary condition for the identification of an equation of (B) (if the covariance matrix of the disturbances at time \( t \) is completely unknown) is the order condition: \( K^* \), the number of predetermined variables \( z_k \) in the other equations of the model but not in that equation, must be greater than or equal to \( H - 1 \), where \( H \) is the number of jointly dependent variables \( y_i \) in that equation. (If \( K^* \) is greater than \( H - 1 \), the equation is sometimes said to be overidentified, and to have \( K^* \) \( H + 1 \) overidentifying restrictions.) There is a necessary and sufficient condition as well (the rank condition); it is more difficult to apply to an equation in a system where there are unknown parameters, but if it is not satisfied we expect to be notified of the fact when we reach the estimation stage by the presence of large estimated sampling variances of our estimates of the parameters (Koopmans, 15, 16).

There remains one more general remark, concerning the generation of cyclical patterns by an econometric model of the type defined here. Such a model contains lagged values of many of its variables, and therefore is a set of simultaneous difference equations. Whereas the solution of a set of ordinary simultaneous equations (given the values of the exogenous variables) is simply a set of numbers, each giving a single value for one endogenous variable, the solution of a set of difference equations (given the values of the exogenous variables) is a set of functions of time, each giving a path in time for one endogenous variable. Such a time-path gives the future history that an endogenous variable of the model would have, as a function of future values of exogenous variables, if future disturbances were zero. This history may behave in one of several ways as \( t \) increases indefinitely:

1) approach a finite limit monotonically.
2) approach a finite limit with oscillations of diminishing amplitude.
3) oscillate indefinitely with constant finite amplitude.
4) approach infinity (positive or negative) monotonically.
5) approach infinity (positive and/or negative) with oscillations.\(^{10}\)

The oscillations, if any, will have a constant period and a constant 'damping ratio' (which may be less than, equal to, or greater than one according as \( 2, 3, \) or \( 5 \) above is the case) if future disturbances are zero and future exogenous variables are constant. Their limits as \( t \) increases indefinitely can be computed. Derivatives of period, damping ratio, and

\(^{10}\)Except during periods of hyperinflation, etc., we expect the solutions of the equations we construct for our economy to be of the first, second, or third kind, i.e., not to 'explode'. This expectation is not included among the restrictions used in the estimation procedures of this paper, however.
amplitude with respect to exogenous variables and parameters can also be computed.\footnote{11}

Since econometric models can thus generate cyclical fluctuations that respond to changes in exogenous variables, they recommend themselves as promising analytical tools for business cycle research.\footnote{12}

2 PREDICTION: THE REDUCED FORM OF A SYSTEM OF STRUCTURAL EQUATIONS

The ultimate test of an econometric model, as of any theory, comes with checking its predictions. A model of the form of equations (B) in Section 1 is not ready to make predictions of the jointly dependent variables, even if its parameters are known. It must first be solved algebraically, so that each jointly dependent variable is expressed in terms of things that can be at least approximately known when the predictions are being made, i.e., parameters and predetermined variables. If (A) is linear in the \( y_h \), then \( y_h = y_i \) for \( h = i \), and (A) and (B) are just alike. Then the solution of (B) or (A) for the jointly dependent variables is simply the solution of a set of \( G \) linear equations for \( G \) unknowns \( y_i \). It is called the reduced form of the model, and looks like this:

\[
\begin{cases}
    y_i = \sum_{k=1}^{K} \pi_{ih} z_h + \nu_i, i = 1, \ldots, I. \\
    \phi(v_1, \ldots, v_I) = \text{joint distribution function of } v_1, \ldots, v_I.
\end{cases}
\]

where: \( \pi_{ih} \) = parameters dependent upon the structural parameters \( \beta_{gi} \) and \( \gamma_{ghi} \), \( g = 1, \ldots, G, i = 1, \ldots, I, k = 1, \ldots, K. \)

\( \nu_i \) = a random disturbance, equal to a function of the \( \beta_{gi} \) and \( u_g \), \( g = 1, \ldots, G, i = 1, \ldots, I. \)

all other symbols have the meaning given in Section 1; in particular the \( y_i \) are jointly dependent variables at time \( t \) (including functions thereof with no unknown parameters), and the \( z_h \) are predetermined variables at time \( t \) (including functions thereof with no unknown parameters), as defined in Section 1.

We still call (C) the reduced form of (B), even if (A) is not linear in the \( y_h \), though in that case (C) is not actually the solution of (B), but is only a kind of linear approximation of it. The exact solution of (B) for either the \( y_h \) or the \( y_i \) is nonlinear in the \( z_h \) if (A) is not linear in the \( y_h \).

(C) is a solution of (B) in the sense that each equation of (B) is a linear combination of the equations of (C), as the reader may verify. Since the linear form (C) is more convenient for later purposes than the exact solu-

\footnote{11}{This applies to nonoscillatory solutions too, except for the period.}

\footnote{12}{See, for example, Frisch (4), Kalecki (10), and Tinbergen (23).}
tion of (B), we shall use it, and unless otherwise specified, the term reduced form hereafter will refer to it.

We have encountered the reduced form in Section 1, but not under its present name: it specifies the conditional probability distribution \( \phi(y \mid z) \) of the jointly dependent variables \( y \) given the predetermined variables \( z \).

If the parameters of the reduced form (C) are known or estimated, predictions of any desired jointly dependent \( y \) for time \( t \) may be made simply by substituting the values of the predetermined \( z \) for time \( t \) into the reduced-form equation for the desired \( y \).\(^{13}\) (It is also possible to predict from the structural equations, by the following process: first substitute into the structural equations the known or assumed values of the predetermined variables, and the estimated values of the structural parameters; then solve the system simultaneously to get the predicted values of the jointly dependent variables. This method in effect uses the exact reduced form rather than the linear approximation for predicting; it gives predictions identical with those of the linearized reduced form if and only if the model is linear and just-identifying; in other cases it presumably gives better predictions because it ignores fewer of the available \textit{a priori} restrictions, but in non-linear models it is more difficult to apply.) As we shall find in Sections 3 and 4, it is easier to obtain suitable estimates for the parameters of the reduced form (C) than for the parameters of the structural equations (B). Thus it appears that in order to predict the jointly dependent variables \( y \), we need only know the values of the parameters of the reduced form, not bothering with the structural equations at all. This is true if, between the period for which the reduced form equations are estimated and the period for which predictions are to be made, no change occurs in the distribution \( \phi(y \mid z) \), which means no change occurs in the structure that generates the distribution \( \phi(y \mid z) \). But if the structure does change, \( \phi(y \mid z) \) and hence the reduced-form parameters \( \pi_{ik} \) will change also, and this will invalidate any predictions based upon knowledge of the old \( \pi_{ik} \). To be valid, predictions must be based upon knowledge of the new \( \pi_{ik} \). To obtain this we must have (besides knowledge of the old \( \pi_{ik} \)) knowledge of the structural parameters \( \beta_{ip} \) and \( \gamma_{ik} \) before the structural change, and knowledge of the effect upon them of the structural change. Thus, for prediction under known structural change, the reduced form is not enough; we must know the structural parameters as well.\(^{14}\)

\(^{13}\) Strictly, a prediction of this kind specifies a probability distribution, not a number. Loosely, we shall use the terms 'predicted value' or 'prediction' to mean 'expected value of predicted distribution'.

\(^{14}\) Even if the change in a structural parameter is known in \textit{direction} only, not in magnitude, it is still true except in special cases that to find the \textit{direction} of the resulting change in some variable (such as national income) it is necessary to know the \textit{magnitudes} of other parameters of the system. See Samuelson (22), pp. 12-14.
One of the most interesting uses of econometric models will, I hope, be to predict the consequences of alternative public policy measures in order that enlightened decisions can be made. Institution of a new policy can often be interpreted as a change of structure. Information about the effect of such a structural change is likely to be available. Thus prediction under structural change can be expected to assume a very important role.

3 Statistical estimation procedures: the structural parameters

As we have said, we hope to be able to construct a structure-identifying model, i.e., a model containing exactly one structure able to generate each distribution function \( \phi(y \mid z) \) of the jointly dependent variables given the predetermined variables. It is the job of a statistical estimation procedure, given the observations, to find estimates of that structure. We mention three such procedures: the full information maximum likelihood method (which we shall call full information for short), the limited information single-equation maximum likelihood method (which we shall call limited information for short),\(^{15}\) and the least squares method.

Before describing these three estimation procedures, we shall briefly discuss maximum likelihood estimates in general. Given a sample of observations, the maximum likelihood estimate \( \hat{\theta} \) of a parameter \( \theta \) of the distribution of the observations is the value of \( \theta \) among all possible values of \( \theta \) that yields the highest probability density for the given sample of observations. It is obtained by two steps: (a) forming the likelihood function of the unknown parameters given the sample of observations (this is the probability density function of the observable variables, with the actual observed values of the variables substituted into it, so that it is a function only of parameters); (b) maximizing this likelihood function with respect to the unknown parameters, treating them as variables for the moment.

For a wide class of distribution functions (including asymptotically normal distributions), maximum likelihood estimates are asymptotically normally distributed. Furthermore, under certain assumptions, they are consistent,\(^{16}\) and efficient\(^{16}\) compared with any other estimates that are both consistent and asymptotically normally distributed.\(^{17}\)

\(^{15}\) The limited information method is treated in Anderson and Rubin (1).

\(^{16}\) An estimate \( t \) of a parameter \( \theta \) is said to be consistent if \( \lim_{N \to \infty} P_b(\mid t - \theta \mid > \varepsilon) = 0 \) for any \( \varepsilon > 0 \), where \( N \) is the sample size and \( P_b(x) \) means the probability of \( x \) occurring. A consistent estimate \( t \) is said to be efficient compared with another consistent estimate \( \tilde{t} \) if \( \lim_{N \to \infty} N^{-1}E(\mid t - \theta \mid^2) \leq \lim_{N \to \infty} N^{-1}E(\mid \tilde{t} - \theta \mid^2) \).

\(^{17}\) The proof that maximum likelihood estimates have these optimal properties is based on the assumption that there is a true structure, which belongs to the model used. Hence the optimal properties of maximum likelihood estimates may not exist if an unrealistic model is used. This is mentioned here as a caution, even though it is
1) The full information maximum likelihood method (full information for short) treats all the jointly dependent variables alike, considering them all dependent as a group on the predetermined variables. It consists of two steps: (a) forming the joint likelihood function of all the parameters, given the observations of all the jointly dependent and predetermined variables in the model; (b) maximizing this likelihood function with respect to all the parameters simultaneously, subject to all the restrictions implied in the model. Its application therefore requires complete specification of the model, plus observed values for all the variables included in the model. Estimates so derived are consistent, and efficient compared to any other estimates which are both consistent and asymptotically normally distributed.

2) The limited information single-equation maximum likelihood method (limited information for short) treats each equation of the model separately, not the whole model at once, but nevertheless recognizes the simultaneous-equations character of the model. For any given equation, it consists of two steps: (a) forming the joint likelihood function of the parameters in the equation, given the observations of the jointly dependent variables in the equation and all the predetermined variables in the model; (b) maximizing this likelihood function with respect to the parameters in the given equation, subject to the restrictions implied in the probability distribution function of the disturbances and in the given equation (including in particular the restrictions stating which predetermined variables do not enter the given equation). Its application to a particular equation therefore uses complete specification of that equation, plus observed values for the jointly dependent variables appearing in that equation, plus observed values for all the predetermined variables appearing in the model. Estimates so derived are consistent, and efficient compared with any other

a commonplace that the results of any statistical analysis may fall down if the original assumptions are not fulfilled.

The proof is based also on the assumption of normally distributed disturbances. If this assumption is not true, estimates computed as if it were true still retain the property of consistency; such estimates are called quasi-maximum likelihood estimates.

Proofs of consistency in estimation employ also the assumption that the matrix of moments of predetermined variables is bounded in the limit as the sample size increases.

In the process of maximizing this likelihood function, what is done essentially is first to compute the least squares estimates of the parameters of the reduced form (see Sec. 4), and second to transform those estimates into estimates of the structural parameters by means of the inverse of the transformation used to obtain the reduced form (C) from the structural equations (B). This is a complex process only in the case of overidentified structural equations.
estimates that (a) are consistent, (b) are asymptotically normally distributed, and (c) use the same information, i.e., the same a priori restrictions and the same observations. The limited information estimates are not efficient compared with the full information estimates (in cases where there is any difference), because the latter use more restrictions and more observations.

In another variant of the limited information method the likelihood function that is maximized is an abbreviated version of the one assumed to be the true one: it is conditional not upon all the predetermined variables appearing in the model, but only on a subset of them. The subset must include all the predetermined variables appearing in the equation to be estimated, and at least \( H - 1 \) others (where \( H \) is the number of jointly dependent variables in the equation to be estimated), but otherwise it is arbitrary. In other words, this subset must be large enough to ensure that the given equation satisfies the necessary condition for identification stated in Section 1. This abbreviated variant of the limited information method uses observations on only the jointly dependent and predetermined variables in the given equation plus the \( H - 1 \) (or more) other predetermined variables included in the likelihood function. Thus for overidentified equations it yields estimates that (for finite samples) are not unique because the choice of the \( H - 1 \) or more other predetermined variables is arbitrary. Estimates of structural parameters obtained by the abbreviated variant of the limited information method are consistent. They are less efficient than ordinary limited information estimates (because the latter use more observations and correct instead of incorrect restrictions), but they are efficient compared with any other consistent and asymptotically normal estimates that use the same observations and restrictions.\(^{10}\)

3) The least squares method treats each equation of the model completely separately, as if there were no other equations. It is not a maximum likelihood method except in special cases. For any given equation, it consists of three steps: (a) choosing arbitrarily one variable to be regarded as dependent upon the others; (b) forming the likelihood function of the parameters in the equation, given the observed values of the dependent variable and the other (independent) variables in the equation; (c) maximizing this likelihood function with respect to the parameters in the equation. Its application to a particular equation requires complete specification of the equation, plus observed values for only those variables appearing in the equation. In the general case of a model with more than one equation,

\(^{10}\) There are still other variants, not discussed here, called limited information subsystem maximum likelihood methods, in which a proper subset of two or more equations of the structure is estimated simultaneously; see Rubin (20).
least squares estimates of the parameters are biased and inconsistent, as Haavelmo (7) has proved. They are also arbitrary within certain limits because of the arbitrary choice of a dependent variable.

In certain cases, depending upon the restrictions implied in the model, the full information method is equivalent to the least squares method, and all three methods lead to identical computation procedures and estimates.\textsuperscript{20}

The full information method is most expensive, and therefore has never (to my knowledge) been used for a system of more than three equations, except where for simplicity all disturbances at time \( t \) were assumed to be independent of one another. The limited information method, though less efficient, is considerably less expensive and has been more extensively used. The least squares method, though known to give biased estimates except in special cases, is computationally much the simplest, and has been traditionally used.

The justifications advanced for the continued use of the least squares method in cases where it yields biased estimates are of two kinds. First, assuming that estimates having the asymptotic properties of consistency and efficiency are in fact superior (in terms of the expected value of the square of the error) for small samples as well as for large (although this superiority is proved only for large), the cost of superior estimation may be too high. Second, in the interesting cases the least squares method's bias may be small and the convergence of its estimates as the sample size increases may be rapid, so that its expected errors may be smaller for small samples than those of the full or limited information maximum likelihood methods.\textsuperscript{21}

\textsuperscript{20} An example is the following model, due to Hurwitz:

\[
\begin{align*}
y_t + \beta y_{t-1} &= u_t \\
y_{t-1} + \gamma z &= u_{t-1}
\end{align*}
\]

\( u_t \) and \( u_{t-1} \) are normally distributed, and serially independent

\[
E(u_t | z) = E(u_{t-1} | z) = 0.
\]

The full information estimate of \( \beta \) from a sample of \( T \) is

\[
\hat{\beta} = -\frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_t^2}
\]

which is also the limited information estimate and the least squares estimate. Note that \( y_t \) is exogenous to the first equation of the model because its distribution depends only on \( z \) and \( u_{t-1} \), so that it is independent of \( u_t \).

\textsuperscript{21} Hurwitz has pointed out informally that when estimates of some parameters are obtained using incorrect assumptions about the values of others, there is in general both a gain and a loss in accuracy of estimation (measured by the expected value of the squared difference between the estimate and the parameter), as compared with
The least-squares method and the abbreviated variant of the limited information method will be used in this paper. Of these two, the latter appears preferable because it preserves the simultaneous-equations character of economic theory, rather than distorting it by forcing it into a framework designed for only one dependent variable. But until we can recognize in advance the cases in which least-squares error, i.e., bias plus sampling error, in small samples is in fact so large as to make the least-squares method poorer for small samples than the limited information method, it may be just as well to use both methods and compare their results.

4 Statistical Estimation Procedures: The Parameters of the Reduced Form

Since the equations of the reduced form contain only one dependent variable each, they are automatically identified. We mention two procedures for estimating the parameters of the reduced form. One is the ordinary least-squares method, and the other is a modification of it which for the purposes of this paper we shall call the restricted least squares method.

1) The ordinary least-squares method is equivalent to forming for each reduced-form equation the likelihood function of its parameters, given the observed values of its dependent variable and its predetermined variables, and then maximizing this likelihood function with respect to the parameters of the equation. Ordinary least-squares estimates of the parameters of any reduced-form equation are unbiased and consistent, provided either that none of the predetermined variables in the model is excluded from the reduced-form equation or that no excluded predetermined variable is correlated with any of the other predetermined variables. But the resulting estimate of the expected value of the jointly dependent variable will remain unbiased and consistent even if some of the predetermined variables are excluded and are correlated with other included ones.22

2) The restricted least squares method is the same as the ordinary least-squares method except that for a given reduced-form equation the maxi-
mization is performed subject to a restriction or restrictions implied in the form of a (proper or improper) subset of the set of all those structural equations that contain the jointly dependent variable appearing in the given reduced-form equation. The procedure yields estimates that (for finite samples) are not unique because of the arbitrary choice of the subset of structural equations. If limited information single-equation estimates of the structure have already been computed, it is much the simplest to choose a one-element subset of structural equations because most of the computations have already been made (this is the procedure followed in this paper). The restricted least-squares estimates of the parameters of the reduced form are unbiased and consistent, with the same qualifications as apply to the ordinary least-squares estimates (see the preceding paragraph). They can be expected to be more efficient than the ordinary least-squares estimates because they use more restrictions.

5 KLEIN'S MODEL III

Klein's model III has 15 equations, of which 3 are definitional identities containing no disturbances and no unknown parameters. Thus there are 12 stochastic equations to be estimated.

There are 15 endogenous variables (in the sense of the \( y' \) in equations (A) in Sec. 1):

- \( C = \) consumer expenditures, in billions of 1934 dollars.
- \( D_1 = \) gross construction expenditure for owner-occupied one-family nonfarm housing, in billions of 1934 dollars.
- \( D_2 = \) gross construction expenditure for rented nonfarm housing, in billions of 1934 dollars.
- \( H = \) inventories at year end, in billions of 1934 dollars.
- \( I = \) net private producers' investment in plant and equipment, in billions of 1934 dollars.
- \( i = \) average corporate bond yield, in per cent.
- \( K = \) stock of private producers' fixed capital at year end, in billions of 1934 dollars.
- \( M_1^D = \) active cash balances = demand deposits + currency outside banks, in billions of current dollars.
- \( M_2^D = \) idle cash balances = time deposits, in billions of current dollars.
- \( p = \) general price level, 1934:1.0.
- \( r = \) nonfarm rent index, 1934:1.0.
- \( v = \) fraction of nonfarm housing units occupied at year end, in per cent.
- \( W = \) private wages and salaries, in billions of current dollars.
- \( X = \) private output (except housing services), in billions of 1934 dollars.
- \( Y = \) disposable income, in billions of 1934 dollars.

\( ^{33} \) Appendix E gives a fuller description of the nature of these restrictions and how they are applied.
Fourteen variables are assumed to be exogenous (in the sense of the exogenous $z_{k}'$ in equation (A) in Sec. 1):

$D_3$ = gross construction expenditures for farm housing, in billions of 1934 dollars.

$D''$ = depreciation on all housing, in billions of 1934 dollars.

$\varepsilon$ = excise taxes, in billions of current dollars.

$\varepsilon_{ep}$ = excess bank reserves, in millions of current dollars.

$\Delta F$ = increase in number of nonfarm families, in thousands.

$G$ = government expenditures (except transfers and net government interest) + net exports + net investment of nonprofit institutions, in billions of 1934 dollars.

$N^s$ = nonfarm housing units at year end, in millions.

$q$ = price index of capital goods, 1934:1.0.

$q_1$ = construction cost index, 1934:1.0.

$R_1$ = nonfarm housing rents, paid or imputed, in billions of current dollars.

$R_2$ = farm housing rents, paid or imputed, in billions of current dollars.

$T$ = government revenues — net government interest — transfers + corporate saving,

= net national product — disposal income, in billions of 1934 dollars.

$t$ = time, in years; $t = 0$ in 1931.

$W_2$ = government wages and salaries, in billions of current dollars.

Data for these variables for 1921-41 are presented in Klein (11, 13) and in Appendix A of this paper.

The 12 equations and 3 identities are as follows (they are here grouped as they were by Klein for his limited information estimation, and renumbered by me):

1. demand for investment

$$I = \beta_0 + \beta_1 \frac{pX - \varepsilon}{q} + \beta_2 \frac{pX - \varepsilon}{q}_{-1} + \beta_3 X_{-1} + u_2$$

2. demand for inventory

$$H = \gamma_0 + \gamma_1 (X - \Delta H) + \gamma_2\Delta p + \gamma_3 H_{-t} + u_9$$

3. output adjustment

$$\Delta X = \mu_0 + \mu_1 (u_3)_{-1} + \mu_2 \Delta \rho + u_{12}$$

4. demand for labor

$$W_1 = \alpha_0 + \alpha_1 (pX - \varepsilon) + \alpha_2 (pX - \varepsilon)_{-1} + \alpha_3 t + u_1$$
(6) demand for consumer goods  
\[ C = \delta_0 + \delta_1 Y + \delta_2 t + u_4 \]

(7) demand for owned housing  
\[ D_1 = \varepsilon_0 + \varepsilon_1 r \frac{1}{q_1} + \varepsilon_2 (Y + Y_{-1}) + \varepsilon_3 \Delta F + \varepsilon_4 \rho s + u_5 \]

(8) demand for dwelling space  
\[ \nu = \eta_0 + \eta_1 r + \eta_2 Y + \eta_3 t + \eta_4 N + u_7 \]

(9) rent adjustment  
\[ \Delta r = \theta_0 + \theta_1 \nu_{-1} + \theta_2 Y + \theta_3 \frac{1}{r_{-1}} + u_8 \]

(10) demand for rental housing  
\[ D_2 = \zeta_0 + \zeta_1 r_{-1} + \zeta_2 (q_1)_{-1} + \zeta_3 (q_1)_{-2} + \zeta_4 Y + \zeta_5 \Delta F_{-1} + u_9 \]

(15) demand for active dollars  
\[ M_1^D = \lambda_0 + \lambda_1 Y + \lambda_2 T + \lambda_3 t + \lambda_4 p (Y + T) t + \lambda_5 + u_9 \]

(16) demand for idle dollars  
\[ M_2^D = \kappa_0 + \kappa_1 Y + \kappa_2 Y_{-1} + \kappa_3 (M_2^D)_{-1} + \kappa_4 t + \kappa_5 \Delta F + u_{10} \]

(11) interest adjustment  
\[ \Delta i = \lambda_0 + \lambda_1 i + \lambda_2 i_{-1} + \lambda_3 t + \lambda_4 \Delta i + \lambda_5 + u_{11} \]

(12) definition of net national product  
\[ Y + T = C + I + \Delta H + D_1 + D_2 + D_3 - D' + G \]

(13) definition of \( X \)  
\[ X = Y + T - \frac{1}{p} (W_2 + R_1 + R_2) \]

(14) definition of \( K \)  
\[ \Delta K = I \]

Observe in Klein's model the following nonlinear functions involving no unknown parameters: \((pX - \bar{e})/q, pX - \bar{e}, r/q_1, 1/r_{-1}, p(Y + T), p(Y + T) t, and (1/p) (W_2 + R_1 + R_2)\).

Equations 1, 2, 3, and 6 are related to the market for goods and services, excluding labor and the construction of housing (these two markets will be treated separately immediately below). Demand for consumer goods (6) is a linear function of income and trend.

Demand for net investment in plant and equipment (1) is a linear function of (a) present and lagged values of deflated (by capital goods prices)
privately produced national income at factor cost excluding housing, which is similar to profits, and of (b) the stock of plant and equipment at the beginning of the year. This function is meant to show the dependence of demand for investment upon (a) anticipated profits and (b) existing capital.

Demand for inventory stocks to hold \((2)\) is a linear function of sales, of expected price change (assumed to be given by a linear combination of current and lagged prices), and of the stock of inventories at the end of the year (an inertia factor). Lagged prices do not appear because Klein found them to be unimportant statistically.

Equation 3 expresses the change in private nonhousing output as a linear function of unintended inventory accumulation (assumed to be measured by \((u_0)_{-1}\), the lagged disturbance in the demand-for-inventory-stocks equation 4), and of the rate of change in general prices. It is essentially a supply equation.

Equation 4 gives the demand for labor, measured by the total wage bill, as a linear function of trend, and of current and lagged values of privately produced national income at factor cost excluding housing (which is supposed to reflect anticipated receipts from sales, net of excises). Observe that this equation could be omitted without impairing the completeness of the model, because the variable \(W_1\) (wage-bill) does not appear in any other equation; in other words, if this equation were omitted, a system of 14 equations in 14 variables would remain.

Equations 7-11 pertain to the housing market. Demand for owner-occupied one-family nonfarm housing construction (7), which is purchased by consumers, is a linear function of the real value of rents (where the deflator is construction costs), of accumulated cash balances (assumed to be proportional to the sum of incomes during the 3 most recent years), and of the increase in the number of nonfarm families.

Demand for rented nonfarm housing construction (10), which is purchased by entrepreneurs, is a linear function of lagged rents, of anticipated prices of housing (assumed to be given by a linear combination of construction costs lagged one and two years), of corporate bond yield, and of lagged increase in the number of nonfarm families.

Equation 9 describes the determination of the nonfarm rent level, which occurs in the housing-construction equations 7 and 10, by a linear function of lagged rents, lagged occupancy rate, and income.

Equation 11 describes the change in corporate bond yield, which occurs in the rental housing construction equation 10 and in the idle balances equation 16, as a linear function of excess reserves, of lagged interest rate, and of trend. Note that it has only one dependent variable.
Nonfarm occupancy rate, $v$ (8), which occurs lagged in the rent adjustment equation 9, is a linear function of rents, of income, of trend, and of the supply of nonfarm dwelling units. It could, like $W_1$, be dropped together with its equation 8, since $v$ occurs nowhere else in the model except in lagged form.

Equations 12-14 are definitions containing no disturbances. Equation 12 is an identity defining net national product as a sum of demand for consumer goods, net investment, increase in inventories, housing construction (net), and goods for government use. This sum might be regarded as an aggregate demand; the fact that it is called the definition of net national product indicates that implicit in the model is an assumption that quantity supplied always equates itself to quantity demanded, except for unintended inventory; see equation 3.

Equation 13 defines privately produced real output excluding housing services, which appears in equations 1-4.

Equation 14 defines stock of capital, which appears lagged in the demand for investment equation.

Klein included in his model an equation defining $R_1$, nonfarm rent, which he classified as endogenous. $R_1$ is actually exogenous, however, according to the way he treats it, and we shall so regard it.

Equations 15 and 16 could, like 4 and 8, be omitted without impairing the completeness of the model, since the variables $M_1^D$ and $M_2^D$ (active and idle balances, respectively) occur in no other equations. Demand for active balances (15) is a nonlinear function of disposable money income and trend. Demand for idle balances (16) is a linear function of current and lagged corporate bond yield, of lagged idle balances, and of trend.

Equations 1, 2, 4, 6, 7, 8, 10, 15, and 16 are demand equations, describing the behavior of various economic groups in the population. Equations 3, 9, and 11 are market adjustment equations describing responses of certain market variables to disequilibria. Equations 12-14 are identities describing definitional relationships.

Klein's estimates of the parameters of his model, for both least squares and limited information methods, appear in Section 10 below.

The results I am interested in presenting are those flowing from my revision of Klein's model. This revision is based upon a test of Klein's model carried out by Andrew W. Marshall. The next section discusses Marshall's test and its findings.

\textsuperscript{24} Its time series is obtained as shown in Appendix B and in Klein (13); its defining equation is not used at all except to obtain estimates of its value for 1919-20, for which data are lacking.
6 MARSHALL’S TEST OF KLEIN’S MODEL III

Marshall (17) tested Klein’s model III, together with Klein’s limited information estimates of its structural parameters, in two ways. Both ways use the calculated disturbances to Klein’s structural equations for 1946 and 1947. For any time period \( t \), these calculated disturbances (called \( u^t \)) are obtained from the structural equations by substituting into them the limited information estimates of the structural parameters, together with the values of all the jointly dependent and predetermined variables at time \( t \).

1) Marshall’s first test examines each \( u^t \) for 1946 and for 1947 to see whether it is larger than would be expected under the hypothesis that Klein’s model and estimates describe 1946 and 1947 as well as they describe the sample period. This is done for each structural equation separately by means of a tolerance interval\(^{25}\) for the calculated disturbances \( u^t \): the hypothesis is accepted for a given equation and a given post-sample year if the value of \( u^t \) for that equation and that year falls inside its tolerance interval. Marshall chooses \( \gamma = 0.99 \) and \( P = 0.99 \), which means that under the hypothesis the probability is 0.99 that the tolerance interval for a given equation will include at least 0.99 of the population of calculated disturbances \( u^t \).

A tolerance interval is of the form \( \bar{x} \pm ks \), where \( \bar{x} \) and \( s \) are the mean and standard deviation computed from a sample of \( N \), and \( k \) is a number depending upon \( \gamma \), \( P \), and \( N \).\(^{26}\) In this case, \( \bar{x} \) is \( \bar{u}^t \), which is zero by the construction of the estimates of the structural parameters. For each structural equation, Marshall uses in place of \( s \) an estimated approximation to the standard deviation of the calculated disturbance \( u^t \), analogous to the Hotelling (9) formula for the standard error of forecast from a regression. This approximation, which we call \( \sigma^t \), is given by Rubin (21). For the \( g^{th} \) structural equation and the year \( t \), it looks like this:

\[
\sigma^2(g, t) = E(u^t^2) = \sigma^2 + \frac{\sigma^2}{T} \text{tr} \Delta \Omega + \frac{\sigma^2}{T} z_t^t M^{-1} z_t^t + z_t^t \Pi''\Pi' z_t^t \sigma^2
\]

\(^{25}\) A tolerance interval is a random variable; it is an interval that encloses, with a certain probability \( \gamma \), at least a certain proportion \( P \) of the individuals in a given probability distribution. This, and not a confidence interval, is what we want here: we are interested in predicting a future drawing from our population of years, not in the true mean. Tolerance limits for the normal distribution have been developed by Wald and Wolfowitz (24), and tables have been prepared for constructing them; see Eisenhart, Hastay, and Wallis (3). The size of the tolerance interval depends upon an estimate of the variance of the calculated disturbances in the sample period, i.e., it depends partly upon the estimates of the parameters of the equation.

where: 
\[ \sigma^2 = E(u^2) \]
\[ T = \text{number of years in sample.} \]
\[ \Lambda = \text{covariance matrix of the estimates of parameters of those endogenous variables } y_i \text{ appearing in the } g^{th} \text{ structural equation.} \]
\[ \Omega = \text{covariance matrix of disturbances } \nu_i \text{ of reduced-form equations containing those endogenous variables } y_i \text{ appearing in the } g^{th} \text{ structural equation.} \]
\[ z_t^* = \text{vector of values in year } t \text{ of all those predetermined variables } z_{\Delta} \text{ appearing in the } g^{th} \text{ structural equation, measured from their sample means.} \]
\[ M_{z \cdot z^*} = \text{moment matrix of } z^* \text{ with } z^*; \quad m_{ij} = \frac{1}{T} \sum (z_i - \bar{z}_t)(z_j - \bar{z}_t) \]
\[ z_t^{*0} = \text{vector of residuals at time } t \text{ of the regressions of the } z^{**} \text{ (i.e., predetermined variables } z_{\Delta} \text{ appearing in the system as a whole but not in the } g^{th} \text{ structural equation, measured from the sample mean) on the } z^*; \text{ i.e., } z_t^{*0} = z_t^{**} - M_{z \cdot z^{**}} M^{-1} z_t^{**}. \]
\[ \Pi^{**} = \text{matrix of reduced-form parameters of the } z^{**} \text{ in those reduced-form equations containing those endogenous variables } y_i \text{ appearing in the } g^{th} \text{ structural equation.} \]

For each structural equation the values of \( T, z_t^*, \text{ and } z_t^{**} \) are known, and estimates are available for \( \sigma^2, \Lambda, \Omega, M_{z \cdot z^*}, M_{z \cdot z^{**}}, \text{ and } \Pi^{**}. \) Thus an estimate of \( \sigma^* \) is available for each structural equation. We call this estimate \( s^* \).

The test for year \( t \) then takes the form of constructing a tolerance interval, \( \pm ks^* \), for each structural equation, and rejecting the equation if its calculated disturbance \( u^* \) falls outside the interval. I shall call it the structural equation tolerance interval test, provisionally, or the SETI test for short.²⁸

In applying the SETI test, Marshall computed \( ks^* \) in five steps, \( ks_1^*, ks_2^*, ks_3^*, ks_4^*, \text{ and } ks_5^* = ks^* \), corresponding to the first term of the estimate of 17, the first two terms, \( \ldots, \text{ and all five terms. For each equation he compared each of these successively with } u^*, \text{ and stopped as soon as he got a region } \pm ks_i^* \text{ which enclosed } u^*. \text{ In this way he saved some computational effort, because he did not have to compute all the terms of } s^* \text{ for every equation.} \)

2) Marshall's second test examines each calculated disturbance \( u^* \) for 1946 and 1947 to see whether it is larger than the error one would expect to make by using what he calls "naive models". Naive model I says that next year's value of any variable will equal this year's value plus a random normal disturbance; naive model II says it will equal this year's value plus

²⁷ See Anderson and Rubin (1).
²⁸ The name was suggested by John Gurland.
the change from last year to this year plus a random normal disturbance.

For each naive model and each Klein structural equation, Marshall compares the calculated disturbances $u'$ of 1946 and 1947 with a tolerance interval for the calculated disturbance of the one naive-model equation that contains the variable appearing on the left side of the given Klein equation. If both $u'$'s for a given Klein equation are outside the interval, Marshall rejects the Klein equation; if one is outside, he puts the Klein equation on probation; if neither is outside, he accepts the Klein equation.

From the viewpoint of this paper, the naive model tests should be applied to the calculated disturbances of the reduced form, not to those of the structural equations: the naive model tests are best suited to compare different methods of predicting (because their disturbances are their errors of prediction), and the predictions made by an econometric model are obtained from its reduced form (see Sec. 2), not directly from its structural equations. But if a naive model test is applied to the calculated disturbances of the reduced form, and if it is to be a fair comparison between methods of prediction, then the treatment of the errors of the naive model should be symmetrical with the treatment of the errors, i.e., the calculated disturbances of the reduced form of the econometric model. This means that a direct comparison of errors should be used, instead of a tolerance interval procedure such as Marshall's which will not reject an equation of the reduced form of the econometric model unless the latter's errors are about three times as large as the naive model's errors (because Marshall's value of $k$ in his naive model tests is about 3).

The results of Marshall's SETI test are shown in Table 1. Marshall did not apply the SETI test to equations 3, 6, or 16, because he had already rejected them on the basis of his naive model tests. The SETI test obviously would have rejected them, however. In 1946 and 1947 they have by far the largest calculated disturbances in the model. Also, for each of these equations in 1946 and 1947 the disturbance is between 5 and 6 times as large as its maximum value in the sample period, and between 6 and 18 times as large as its estimated standard error.

Therefore, we conclude that by the SETI test equations 3, 6, and 16 are

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90 Milton Friedman too has suggested these naive models, though not under this name.

91 Furthermore, the size of a structural equation's disturbance is not an invariant for this purpose because a structural equation can be normalized arbitrarily on any endogenous variable, but the size of a reduced-form equation's disturbance is a definite quantity because there is only one dependent variable on which to normalize a given reduced-form equation. Marshall comes close to realizing this when he comments that the verdict of a naive model test of a structural equation depends on which variable is selected from the equation as a basis for the test. He always chooses the one Klein has placed on the left side, and he does realize that this is an arbitrary choice. See Marshall (17).
Table 1

**Results of Marshall's SETI Test of Klein's Model III**

<table>
<thead>
<tr>
<th>Eq.</th>
<th>Var. at left</th>
<th>Yr.</th>
<th>Calc. dist.</th>
<th>$k_{5\alpha}$</th>
<th>$k_{5\alpha}^*$</th>
<th>$k_{10\alpha}$</th>
<th>Verdict*</th>
</tr>
</thead>
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<td>$I$</td>
<td>46</td>
<td>$-5.6$</td>
<td>2.0</td>
<td>3.4</td>
<td>8.0</td>
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<tr>
<td></td>
<td></td>
<td>47</td>
<td>$-2.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$H$</td>
<td>46</td>
<td>$-7$</td>
<td></td>
<td>2.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>47</td>
<td>$-1.9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\Delta X$</td>
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<td>$-37.9$</td>
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</table>

Source: Marshall (17).

* R means reject; a blank space means accept.

* Marshall did not apply the SETI test to this equation because he rejected it on the basis of his naive model test.

* Less than .05 in absolute value, and negative.

rejected; equations 4 and 15 are on probation for having been rejected for either 1946 or 1947; and equations 1, 2, and 7-11 have a clear record so far.

Since neither Klein or Marshall made any explicit computations of the
reduced form, results of naive model tests of the reduced form of Klein's model III are not presented here.

7 REVISIONS OF KLEIN'S EQUATIONS

This section presents several equations designed to replace those of Klein's which fared badly in Marshall's SETI test.

The SETI test, as indicated in Section 6, would have rejected three equations: (3) output adjustment, (6) demand for consumption, and (16) demand for idle cash balances. It cast doubt upon two others: (4) demand for labor and (15) demand for active cash balances. These five equations are the ones to be revised or changed here. If theoretically justified, it is permissible to change the number of variables and equations in the model, but the number of equations must not exceed the number of endogenous variables, and the two must be the same if the system is to be complete.

Consider first the demand for money equations (15 and 16). Their function is to determine two variables, $M^D_1$ and $M^D_2$ (active and idle money balances, respectively), which are purely symptomatic in Klein's model. Since they do not enter into any other equations, $M^D_1$ and $M^D_2$ cannot mathematically affect the other variables of the model but can only be affected by them. We are not interested in the quantity of money per se unless it has some effect. Therefore, we drop equations 15 and 16, together with the variables $M^D_1$ and $M^D_2$.

This cuts the number of equations by two but still leaves a complete model.

The demand for dwelling space, equation 8, is in the same position as the demand for money equations. It determines the nonfarm housing occupancy rate, $v$, whose current value does not appear elsewhere in the model. Therefore, it can be dropped, along with variable $v$. We have now removed three equations and three variables jointly dependent at time $t$, without affecting the completeness of the model.

Consider the consumption function next. Klein's equation 6 underestimated consumption in 1946 and 1947 by some 13 and 14 billions of 1934 dollars, or about 15 and 16 per cent. The real value of the stock of money at the beginning of each of these years was $110 and $105 billion, respectively, approximately twice the largest value attained during 1921-39. (The real value of the stock of money is here defined as the sum of currency outside banks plus demand deposits adjusted plus time deposits, but not

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81 I believe that a good economic theory will not say that the quantity of money is merely a symptom having no effect upon economic affairs. Accordingly, Klein's theory is amended below, at least with respect to the consumption function. Only lack of time prevented further changes involving the quantity of money in other parts of the model and dictated the dropping of 15 and 16 rather than their revision.

82 See Marshall (17).
including government deposits, deflated by the 1934-base price index of output as a whole.) For the interwar years Klein (12) was unable to reject the hypothesis that consumption is not dependent upon real cash balances, but this was to be expected because real balances were almost constant during that period except for a smooth trend, so their effect, if any, could not be discovered. The postwar data suggest that real balances may have been important in the consumption function all along. The skewness of the distribution of ownership of real balances among consumers may also be important; we might expect to find that an increase in the holdings of richer people would stimulate consumption less than an equal aggregate increase in the holdings of poorer people (of course the same might be true of income). Time series are not available for this ownership distribution, however.

The proper definition of cash balances for this purpose is total consumer holdings of currency, demand deposits, time deposits, and probably also U. S. Savings Bonds (Series E) as long as they are guaranteed to be immediately redeemable in cash at no loss and yield negligible interest. Holdings by individuals and unincorporated businesses might be a good approximation, but suitable figures do not exist as far as I know, especially if Series E bonds are included. Therefore the definition used in the preceding paragraph seems best.

Lagged disposable income has often been mentioned as a candidate for membership in the consumption function. It is recommended by the fact that people do not adjust themselves immediately to changes in income. Lagged consumption has also been suggested, for a similar reason.\textsuperscript{33} As lagged income and lagged consumption are highly correlated (through the consumption function) it is best not to use both.

Accordingly we experiment with fitting the following consumption functions:

\begin{align*}
(6.0) \quad C &= \delta_0 + \delta_1 Y + \delta_2 Y_{-1} + \delta_3 \left( \frac{M}{p} \right)_{-1} + \delta_4 t + \mu_6 \\
(6.1) \quad C &= \delta_0' + \delta_1' Y + \delta_2' Y_{-1} + \delta_3' t + \mu_6' \\
(6.2) \quad C &= \delta_0'' + \delta_1'' Y + \delta_2'' \left( \frac{M}{p} \right)_{-1} + \delta_3'' t + \mu_6'' \\
(6.3) \quad C &= \delta_0''' + \delta_1''' Y + \delta_2''' t + \mu_6'''
\end{align*}

\begin{align*}
(6.4) \quad C &= \delta_0^{IV} + \delta_1^{IV} Y + \delta_2^{IV} C_{-1} + \delta_3^{IV} \left( \frac{M}{p} \right)_{-1} + \delta_4^{IV} t + \mu_6^{IV} \\
(6.5) \quad C &= \delta_0^V + \delta_1^V Y + \delta_2^V C_{-1} + \delta_3^V t + \mu_6^V
\end{align*}

\textsuperscript{33} The suggestion was made informally by Klein and by Franco Modigliani.
where \( M = \) currency outside banks + demand deposits adjusted + time deposits, at the end of the year; and other symbols are defined in Section 5. Observe that \( (M/p)_{-1} \) is a predetermined variable since it is lagged (the same is of course true of \( Y_{-1} \) and \( C_{-1} \)). Thus we have not added any new current endogenous variables to the system by these modifications of the consumption function.

There remain two equations, (3) output adjustment (which is really a supply function, as mentioned before) and (4) demand for labor. They are closely related theoretically, because under the assumption of profit maximizing, the firm's demand-for-factor equations are deductible from the profit function and the production function; the supply function is then deductible from these demand-for-factor equations and the production function.\footnote{Equivalently, if the demand-for-factor equations and the supply equation are given, the production function is determined. Thus if we are concerned only with the logical completeness of the model, it does not matter whether it is the production function or the supply function that we include, provided the demand-for-factor equations are present.\footnote{Suppose we are given competitive conditions, a production function
\begin{equation}
(1) \quad x = \phi(y_1, \ldots, y_n),
\end{equation}
and a profit function
\begin{equation}
(2) \quad \pi = px - \sum_{i=1}^{n} q_i y_i,
\end{equation}
where \( x \) is output and \( p \) is its price, \( y_i \) is the input of a factor of production and \( q_i \) its price, \( i = 1, \ldots, n \), and \( \pi \) is profit. Then the firm maximizes (2) with respect to the \( y_i \), subject to the restraint (1), to get
\begin{equation}
(3) \quad \frac{\partial \phi}{\partial y_i} - q_i = 0, \quad i = 1, \ldots, n.
\end{equation}
If the set of simultaneous equations (1) is solved for the \( y_i, i = 1, \ldots, n \), the result is the demand-for-factor equations
\begin{equation}
(4) \quad y_i = f_i \left( \frac{q_1}{p}, \ldots, \frac{q_n}{p} \right), \quad i = 1, \ldots, n.
\end{equation}
The supply equation is obtained by substituting \( y_i \) from (4) into (1), \( i = 1, \ldots, n \). Results are similar in the noncompetitive case, but elasticities of product demand and factor supply enter in then.}}

\footnote{Under the assumption of profit-maximizing, with a profit function such as (2) in the preceding note, and with a set of demand-for-factor equations for the firm that can be uniquely solved for the factor prices, the production function for the firm can be derived from given demand-for-factor equations, uniquely except for a boundary condition such as \( \phi(0, \ldots, 0) = 0 \), even with no knowledge of the supply function, as follows: By hypothesis it is possible to pass uniquely from (4) to (3) of the preceding note, which can then be divided through by \( p \) and integrated to obtain \( \phi \) uniquely except for a constant term (subject to certain integrability conditions which in our case are satisfied). Q.E.D. This proof is due to Koopmans.}

This system is not likely to be made overdetermined by including a production function (or alternatively a supply function), however, since an additional variable \( x \) is brought along at the same time.
choose here to use a production function, because Klein's output adjustment equation is so far off (overestimating output by 61 and 38 billions of 1934 dollars in 1946 and 1947, respectively) and because the production function is less likely to be affected by possible structural changes.

Variables must be chosen to represent capital input and labor input in the production function. Capital input can be measured by depreciation charges, which would be ideal if depreciation really reflected the services of capital accurately. But since depreciation is a very arbitrary thing, subject to various legal and accounting pressures, it is not a satisfactory measure of capital use. Another possible measure is the stock of producers' capital at the beginning of the year, defined as the sum over time of net investment. This is not free from the effects of the arbitrariness of depreciation charges but it is less sensitive to them because stock of capital is so large in relation to depreciation charges for any one year. It measures capital existing, not capital in use, which is unfortunate, but we shall try it anyway, perhaps together with some device for indicating the extent to which available capacity is being used.

Labor input, which might appear also in the demand for labor equation, should ideally be measured in man-hours. But data difficulties deter us here; the BLS series for average weekly working hours before 1932 is for manufacturing and railroads only, and does not cover all industries even now. The concept of full time equivalent persons engaged in production, used by Simon Kuznets and the Department of Commerce, is the next best thing. However, it does not regard overtime work as an increase of labor input: it measures roughly the number of persons engaged full time or more (where full time for any person means simply the current customary work week in his job, whether it is 35 hours or 55), plus an appropriate fraction of the number of persons engaged part time (to convert them to full time equivalents). A time trend term will then approximately take care of the secular decrease in weekly working hours that has occurred.87

We might choose any one of several forms for the production function. The Cobb-Douglas function, linear in logarithms, is one possibility; a simple linear or quadratic function is another. Investment during the current year might be included on the theory that new capital, because of

88 It is private labor input that concerns us here, by the way, not total, because only in the private sector is production assumed to be guided by the desire for profit.

87 Cyclical fluctuations in weekly working hours will be an important source of error here unless their effect is largely explainable by cyclical changes in full time equivalent persons engaged plus a time trend, i.e., if data on weekly working hours (which we do not have), full time equivalent persons engaged, and time trend are not approximately linearly related.
improvements in the design of equipment, is more productive than old capital even after depreciation has been deducted.

In attempting to make the production function reflect the fact that output can be increased if existing capital is used more intensively, we might break our sample into two samples—one containing boom years in which capital was being used at approximately full capacity, and the other containing slack years in which it was not—and then fit two production functions, one to each. The sign of net investment could be used as a crude indicator for classifying the years: in boom years one would expect demand for capital services to exceed existing supply, thereby stimulating an increase in the stock of capital, so that net investment would be positive, and in slack years the opposite. This scheme is undesirable because it sets up a dichotomy where there should be a continuum, and because it reduces the already too small sample. An alternative, suggested during discussions with Jacob Marschak, is to make each parameter of the production function a linear function of net investment, thus:

\[(3.0) \quad X = (\mu_0 + \mu_1 I) + (\mu_2 + \mu_3 I)N + (\mu_4 + \mu_5 I)K_{-1} + \mu_6 t + u_b\]

where \(N\) = private labor input, in millions of full time equivalent man-years (endogenous), and other variables are defined in Section 5. We would expect \(\mu_5\) to be positive: a large positive net investment \(I\) can be presumed to indicate that capital is being used at a high percentage of capacity, and existing capital \(K_{-1}\) can be expected to contribute more to output than otherwise, so that its coefficient \((\mu_4 + \mu_5 I)\) should be high. We might expect \(\mu_6\) to be negative because the marginal product of labor is probably less in boom times than otherwise. Of course we expect \(\mu_2, \mu_4,\) and \(\mu_6\) to be positive (though \(\mu_6\) would be negative if the above mentioned secular drop in working hours were enough to overbalance the increase in per man-hour productivity). We have no presumptions about \(\mu_0\) and \(\mu_1\), except that \(\mu_1\) should probably be positive and not very important.

Another way of trying to solve the problem of unused capacity is to set up a production function in which output depends upon both labor input and existing capital in boom years, but only upon labor input in slack years. This again unfortunately requires a dichotomous classification of all years as either boom or slack. Mainly because of lack of time, estimates of this kind of production function are not presented in this paper; it would be interesting to return to this idea in the future.

\[\text{ Such a production function is dependent upon the assumption that net investment occurs in response to near-capacity use of existing capital. If something happens so that this is no longer true, the production function changes. But nothing of this sort is likely to happen unless the profit maximizing assumption becomes invalid, in which case several other equations will go by the board too.}\]
Besides 3.0, we try the following production functions:

\[(3.1) \quad X = \mu_0 + \mu_1 I + \mu_2 N + \mu_3 K_{-1} + \mu_4 t + u_3\]
\[(3.2) \quad X = \mu_0'' + \mu_1'' N + \mu_2'' K_{-1} + \mu_3'' t + u_3''\]
\[(3.3) \quad \log X = \mu_0''' + \mu_1''' \log N + \mu_2''' \log K_{-1} + \mu_3''' t + u_3'''\]
\[(3.4) \quad X = \mu_0 IV + \mu_1 IV N + \mu_2 IV t + u_3 IV\]
\[(3.5) \quad X = \mu_0 V + \mu_1 V N + \mu_2 V N K_{-1} + \mu_3 V K_{-1} + \mu_4 V t + u_3 V\]
\[(3.6) \quad X = \mu_0 VI + \mu_1 VI N + \mu_2 VI N^2 + \mu_3 VI N K_{-1} + \mu_4 VI K_{-1}^2 + \mu_5 VI K_{-1} + \mu_6 VI t + u_3 VI\]

In 3.1 to 3.4 we would expect all parameters (except possibly the \(\mu_0\)’s) to be positive. In 3.5 and 3.6 we would expect \(\mu_2, \mu_4, \mu_8, \mu_9\) to be positive, and \(\mu_3, \mu_4, \mu_5, \mu_6\) to be negative (this can be seen more easily by examining the expressions for marginal productivity of labor and capital implicit in the two equations).

Equation 3.3 is a Cobb-Douglas function with a time trend to take approximate account of technological improvements. 3.2 is a linear approximation. 3.1 is like 3.2 except that it treats new and old capital differently. 3.4 is a linear approximation which attempts to account for the existence of unused productive capacity by (1) disregarding the quantity of existing capital and (2) assuming (more or less plausibly) that capital input (not measurable) is proportional to labor input, so that output can be expressed as a function of labor input alone. 3.5 and 3.6 are attempts at more accurate approximation than a linear function provides: they have marginal productivity functions that vary with inputs instead of being constant.

Observe that by replacing Klein’s output adjustment equation with any of the production functions 3.0 to 3.6, we have added a new endogenous variable, \(N\). Before we finish our revisions, we must therefore find a corresponding additional equation, if we are to end with a complete system.

Now that the wage-salary bill and labor input are both in the model, it is natural to include the wage rate too:

\[(18) \quad w = \frac{W_1}{N}\]

where \(w\) = private wage-salary rate, in thousands of current dollars per full time equivalent man-year (endogenous), and \(W_1\) and \(N\) are defined in Section 5 and in this section, respectively.

Adding 18 will not affect the completeness of the system. We have here one new equation and another new endogenous variable, \(w\), so we still need to find an additional equation.
If the wage rate is to mean merely total labor earnings per unit of labor input, the definition is satisfactory in the simple form (18). However, if it is to mean the hourly wage, the thing over which workers and employers bargain, overtime payments, premiums for night-shift work, etc., must be allowed for; furthermore, labor input must be measured in man-hours. The advantage of using the hourly wage is that it enables us to introduce an equation describing the bargaining process and its dependence upon price movements, level of employment, and any other relevant variables. This wage adjustment equation could serve also as the additional one required by the introduction of the two new endogenous variables, \( w \) and \( N \), with only the one equation 18. But existing data do not permit us to incorporate overtime payments and premiums for shift-work into the wage rate or, as we have seen, to define labor input in man-hours.

Accordingly we retain 18 as it stands. We assume that our \( w \) is closely representative of hourly wage, and use a wage adjustment equation such as:

\[
(5.0) \quad w = \kappa_0 + \kappa_1 \Delta p + \kappa_2 (N_L - N) + \kappa_3 w_{-1} + \kappa_4 (N_L - N)_{-1} + \kappa_5 t + \mu_6
\]

or

\[
(5.1) \quad w = \kappa_0' + \kappa_1' \Delta p + \kappa_2' (N_L - N) + \kappa_3' w_{-1} + \kappa_4' t + \mu_6'
\]

where \( N_L \) = labor force, including work relief employees but excluding other government employees, in full time equivalent man-years (exogenous), and other variables are as defined above. These wage adjustment

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\(^{40}\) A study in the *Monthly Labor Review* for November 1942, pp. 1053-56, shows the estimated average number of overtime hours per worker per week in manufacturing in 1942 as a function of average total hours per worker per week. Using this study and the BLS series for average weekly hours in manufacturing, and assuming that the 1942 study is valid for all years and that all time over forty hours is paid at time and a half, one concludes that if overtime pay had been the sole cause of the difference between our \( w \) and the straight-time hourly wage, this difference would have been less than 2 per cent in all interwar years and less than 3 per cent in 1946 and 1947. Thus we are not risking more than about 3 per cent from this cause. Shift premiums probably do not contribute a larger error than this. And we are more comfortable if we remember that the manufacturing industries probably had more extensive shift premiums and more complete observance of the time and a half for overtime rule than did the economy as a whole.

\(^{41}\) \( N_L - N \) is meant to measure unemployment including relief workers, and \( N \) excludes government workers. Therefore if \( N_L - N \) is to be a correct measure, \( N_L \) must exclude government nonrelief workers.

\(^{42}\) Labor force is the only measure we have for labor supply, and it is expressible in man-years, but not in man-hours except by some trick assumption. Hence unemployment and employment, which add up to labor force, must also be in man-years instead of man-hours. Hence, there is another advantage in defining labor input \( N \) in man-years as we have done.
equations tell us that the wage level depends upon the past wage level (reflecting the downward rigidity of wages), upon price changes (reflecting wage increases following increases in the cost of living and in the prices received by employers), upon unemployment (reflecting the state of the labor market), and upon trend (reflecting the growth in productivity and/or in the strength of unions).

By adding a wage adjustment equation, we have completed our system again.

The demand for labor equation (4) is still to be considered. It was put on probation, not completely rejected. Therefore we try it again, but we also try two alternatives which express the demand for labor in real terms as a function of the real wage rate, of real output, and possibly of trend. The real wage rate enters as a result of the profit maximizing assumption. Output is relevant on the theory that if producers receive more orders they will demand more labor even if the real wage does not fall. Trend may be necessary to reflect the long-term rise in per man-hour productivity. Our alternative equations are:

\begin{align}
W_1 &= a_0 + a_2(px - \varepsilon) + a_2(px - \varepsilon)_{-1} + a_3t + u_4 \\
N &= a_0 + a'_2\frac{w}{p} + a_2X + a_3t + u_4 \\
N &= a_0'' + a'_2\frac{w}{p} + a_2''X + u_4''
\end{align}

Klein's equation 4.0 is not as different from the others as it looks at first: if we divide 4.0 through by \( w \) we get

\begin{equation}
\frac{W_1}{w} = N = a_0 + \frac{a_2X}{w} - a_2 - + \ldots
\end{equation}

which also depends on real wage \( w/p \) and on real output \( X \), though there is only one parameter, \( a_1 \), to take care of both, and there are other terms involving \( w \) and \( \varepsilon \) and lagged quantities meant to account for expectations.

Whether we finally choose 4.0 or 4.1 or 4.2, we still have a complete system of fourteen equations (including four definitional identities) in fourteen jointly dependent variables: \( I, H, X, W_1, w, C, D_1, r, D_2, i, Y, p, K, \) and \( N \). It should be understood that there are additional identities which define as additional variables the following nonlinear functions in

\( ^* \) The dependence of the demand for labor upon output cannot be found from the profit maximizing assumption and the usual production function, which may indicate a weakness in one of these two.

\( ^* \) Klein (13) at least hints at most of the changes made in this section, and even includes exploratory computations on some.
the system: \((pX - \mathcal{E})/q, pX - \mathcal{E}, w/p, M/p, r/q_1, 1/r_{-1}, \) and \((1/p)(W_2 + R_1 + R_2)\).

8 \textbf{DESCRIPTION OF TESTS USED}

Several tests are available for application to a model or structure obtained by the methods described in this paper. They may be divided into two groups according to the information required for their use. The first group comprises tests dependent only on observations and restrictions available for use in the estimation process; these are essentially tests of internal consistency. The second group comprises tests that use observations concerning events outside (usually subsequent to) the sample period; these are tests of success in extrapolation and prediction, and therefore are of higher authority. We describe here the tests applied in this paper.

\textit{a) Tests of internal consistency}

First, there are certain qualitative procedures that perhaps should not be called tests at all: the estimates of the structural parameters can be examined to see whether they have the approximate magnitudes and particularly the algebraic signs to be expected on the basis of theoretical and other information about elasticities, marginal propensities, etc. The estimated sampling variance of each estimate can be examined to see how much confidence can be placed in its sign or in its approximate size. The calculated disturbances can be examined to see whether they are very large according to some intuitive standard of how large they are expected to be. This last procedure is of doubtful usefulness because it is not always possible to tell whether disturbances are due to the existence of several systematic factors that have been neglected, or to a real randomness in the phenomenon studied, especially if the disturbances appear to be random.

Second, for any equation of the model there is a test of all the restrictions used in the limited information estimation of that equation. The test is applied to the largest characteristic root \(\lambda_1\) of the equation

\[ (20) \quad \text{det}[W(W^* - W)^{-1} - \lambda I] = 0 \]

which is used in the estimation process. Here \(W\) is the covariance matrix of disturbances to the regressions of the \(H\) jointly dependent variables in the equation to be estimated on the predetermined variables assumed to be known to appear in the entire model; \(W^*\) is the covariance matrix of disturbances to the regressions of the same \(H\) jointly dependent variables in the equation to be estimated on the predetermined variables in the equation to be estimated; the roots \(\lambda_1 \cong \lambda_2 \cong \cdots \cong \lambda_H\) are scalars; and \(I\) is the identity matrix. Anderson and Rubin (1) have shown that under the assumptions of the limited information method, the quantity \(T \log (1 + 1/\lambda_1)\) has the \(\chi^2\) distribution asymptotically as the sample size \(T\) in-
creases, with the number of degrees of freedom equal to the number of overidentifying restrictions, i.e., to the excess of $K^\ast$, the number of predetermined variables assumed to be known to enter the model but not the given equation, over $H-1$, where $H$ is the number of jointly dependent variables in the given equation. $1 + 1/\lambda_1$ can never be less than 1, and if it is close to 1 in an overidentified model it means that the effect of excluding the excluded predetermined variables is only slightly detrimental to the variances, i.e., increases them only slightly, which is what we want. This $\chi^2$ test of the largest root $\lambda_1$ is a sort of over-all test of the totality of restrictions and assumptions applied in estimating an equation; if in a particular equation, $\lambda_1$ takes a value that is very improbable under the hypothesis that all these assumptions are true, then for that equation we have only a very generalized alarm signal which cannot point to a specific remedy. (Of course, if we have a high degree of a priori confidence in some specific set of identifying restrictions, this test can be regarded as a test of the remaining, overidentifying, restrictions.) The test is of questionable usefulness for this paper because it is derived on the assumptions of the ordinary limited information method, and this paper uses the abbreviated variant of the limited information method.

Third, for any equation of the model there is a test of the assumption that the disturbances are serially uncorrelated, based on the distribution of the statistic $\delta^2/S^2$. For a given equation and sample period, $\delta^2$ is the mean square successive difference of the disturbances $u$, given by

$$\delta^2 = \frac{1}{T - F - 1} \sum_{t=2}^{T} (u_t - u_{t-1})^2$$

and $S^2$ is the variance of the disturbances $u$ over the sample, given by

$$S^2 = \frac{1}{T - F} \sum_{t=1}^{T} u_t^2$$

where $T$ is the sample size and $F$ is the number of parameters to be estimated in the equation. The question of the proper number of degrees of freedom is not solved, so we follow Marshall in using $T - F$ arbitrarily, as if we were dealing with least squares. The distribution of $\delta^2/S^2$ has been tabulated by Hart and von Neumann (8).44

$b)$ Tests of success in extrapolation and prediction

First, there is the SETI test, described in Section 6. As stated there, the SETI test tells whether each structural equation describes events in future

44 Since we never observe the disturbances, being forced to calculate their values on the basis of estimates of the structural parameters, there may be some bias in using the tables given by Hart and von Neumann. In fact, Orcutt and Cochrane (18) have found in sampling experiments that there is a high probability of bias against finding serial correlation, especially when the number of parameters to be estimated is large.
periods as well as it does those in the past sample period. A similar test could be applied to the estimates of the equations of the reduced form; it might be called the reduced form tolerance interval test, or the RFTI test for short. The RFTI test is a test of predictions, but in case of poor prediction it cannot tell us which structural equations should be changed.\textsuperscript{45} The SETI test examines each structural equation separately, and is therefore more useful in this respect.\textsuperscript{46}

Second, there are the naive model tests of the predicting ability of the reduced form of the econometric model, mentioned in Section 6. We want the errors of prediction made by the reduced form, i.e., the calculated disturbances to the reduced form equation in the years for which predictions are made, to be at least as small in absolute value as the errors made by the noneconomic naive models.\textsuperscript{47} If this condition is not met, we cannot have much confidence in the predicting ability of the econometric model. But observe that even if such a naive model does predict about as well as our econometric model, our model may still be preferable because it may be able to predict consequences of alternative policy measures and of other exogenous changes, while the naive model cannot.

Third, a comparison can be made to see whether the limited information method or the least squares method yields smaller calculated disturbances to the structural equations in the years for which predictions are made.

\textsuperscript{45} 'Autonomy' of an equation is the name given to a concept that is useful here. It is not numerically defined, but corresponds to the degree to which the equation is invariant under possible or probable changes in structure. Structural equations are the most autonomous, since each depends on the structural parameters of only one equation, namely itself. Reduced form equations are the least autonomous. The advantage of autonomous equations is obvious for prediction under changes of structure. See Haavelmo (6).

\textsuperscript{46} If a structural equation with limited information estimates of its parameters fails to pass the SETI test, we can be reasonably confident that the trouble (apart from sampling variation) lies with the form of that equation and not with the other equations of the model, because in estimating that equation no information from the rest of the model was used, except for observations on a list of predetermined variables. This statement could fail to be true only if the rest of the model contained a seriously wrong set of predetermined variables. But observe this caution: even if all the calculated disturbances fell inside their tolerance intervals, we still might not have a good structure; we might instead have a poor structure which, however, is not worse in the prediction period than it was in the sample period (this remark arose in discussions with Harry Markowitz).

\textsuperscript{47} To test this, we can make point predictions (as opposed to tolerance interval predictions) with both methods for a number of years, and apply a simple t-test to the hypothesis that the means of the absolute values of their errors are the same, using as an alternative the hypothesis that the mean of the absolute values of the econometric model's errors is larger. As we are likely to have very small samples for this test as well as for the SETI test (Marshall would have had a sample of two, for instance) its results will not be conclusive.
9 PLAN OF COMPUTATIONS

Klein has estimated the equations of his model (Sec. 5) by the least squares and limited information methods; the estimates are given in Klein (13). Certain of these equations have been rejected by Marshall's SETI test on the basis of Klein's limited information estimates and the data for 1946 and 1947; the results are given in Section 6 above. The rejected equations have been revised, replaced, or eliminated (see Sec. 7).

The estimates presented here are for the unrejected Klein equations, and for the new or revised equations of Section 7. They are based in each case on a sample consisting of the years used by Klein for his limited information estimates plus 1946 and 1947, which were added in order to bring the estimates up to date and give the model a fairer chance to do a good job of describing 1948. The war years 1942-45 were omitted because some of the ordinary economic relationships were set aside in favor of direct government controls during that period. Some controls, e.g., rent controls, continued after 1945, however, and some period of readjustment may be required before the postwar economy finds its stride. After a few years, when the sample of postwar years has grown, it may be wise to drop 1946 as well as 1942-45.

All the unrejected and new equations are estimated by least squares, and the estimated standard errors of the disturbances and of the estimates are computed. Then one form is chosen from the theoretically acceptable alternative forms of each equation, e.g., one production function from equations 3.0 to 3.6, etc., and estimated by the limited information method.

The estimates appearing in Klein (11) have been revised because of the discovery of an error in the time series for X. The revised series is used in Klein (13) and in this paper.

This means that my sample is 1922-41 plus 1946-47 for all equations except 10.0 and 11.0, for which it is 1921-41 plus 1946-47.

See Appendix C for a discussion of certain peculiarities in the time series obtained for 1946 and 1947.

The choice is based partly on theoretical grounds (but not wholly, or else it could be made before any empirical work is done), and partly on the least squares estimates. There is a presumption that if an equation fits well by least squares, i.e., if its residuals and the estimated standard errors of the estimates of parameters are small, there is likely to be a relation among its variables that can be consistently estimated by the limited information method. This is particularly true if the variance of the disturbance to the equation is small; see Jean Bronfenbrenner (2). I realize that this procedure is not satisfactory to the uncompromising advocate of consistency in estimation. Ideally all the alternative forms of each equation should be estimated by the limited information method, but as this is an expensive process the least squares estimates are used as a kind of screening device. How misleading they can be is shown in the cases of equations 1.0 and 4.0, discussed below.
For each equation estimated by the limited information method, estimates are prepared for: the standard error of the disturbance; the covariances of the estimates of the parameters; the successive values of $k_s$; required for the SETI test, where $P = .95$ and $\gamma = .99$; the value of the ratio $\bar{e}/S^2$; and the quantities needed for the characteristic root test. The calculated disturbances for 1948 are computed for Klein's limited information estimates, for my least-squares estimates, and for my limited information estimates. The SETI test is applied to the last.

The parameters of the reduced form are estimated by the ordinary least-squares method and by the restricted least-squares method. The naive model tests are applied to both sets of estimates, with the single year 1948 as a sample.

The results of all these computations appear in the next section.

10 RESULTS OF COMPUTATIONS

Table 2 shows the computational results that are applicable directly to structural equations (as opposed to equations of the reduced form): the estimates of parameters and variances, the calculated 1948 disturbances, and the quantities needed for the SETI test, the serial correlation test, and the characteristic root test.

Table 3 presents results pertaining to the equations of the reduced form of the revised model. For each of the endogenous variables, it shows: (1) the observed 1948 value; (2) the change in the observed value from 1947 to 1948; (3) the average absolute value of the annual change in the observed value, over the 24 periods 1920-21 to 1940-41 and 1945-46 to 1947-48; (4) and (5) the two 1948 predictions made by the reduced form of the revised model, as estimated by the ordinary least squares method and by the restricted least squares method, respectively; (6) and (7) the 1948 predictions made by naive models I and II; (8) and (9) the errors of the two reduced form predictions, i.e., the observed values minus the predicted values; (10) and (11) the errors of the naive models; (12) to (15) a comparison of each reduced form error with each naive model error, to see in each case which is smaller in absolute value; (16) the percentage error of the least squares prediction, using the 1948 observed

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41 In this paper each reduced form equation includes only the predetermined variables that appear in the corresponding group of structural equations. See Appendix D for the grouping and Section 4 for remarks about the properties of the estimates.

42 Endogenous in the sense of the $\gamma_j$ in section 2. The equations of the reduced form are linear regressions on certain predetermined variables. Therefore, the predicted value of a nonlinear function such as $w/p$ cannot be expected to be the same when obtained from the quotient of the predictions of $w$ and $p$ as when obtained directly from a regression. Predictions of such nonlinear functions are not presented here.
Table 2: Results of Computations on Structural Equations of Revised Model

<table>
<thead>
<tr>
<th>Eq.</th>
<th>Estimates of Parameters and Standard Errors&lt;sup&gt;b&lt;/sup&gt;</th>
<th>$\delta$</th>
<th>$\delta^*/S^e$</th>
<th>Calc. value 1948 (11)&lt;sup&gt;e&lt;/sup&gt;</th>
<th>Calc. dist.</th>
<th>$k_S^*$</th>
<th>SETI getting a small result $\lambda_C$</th>
<th>10 q&lt;sub&gt;d&lt;/sub&gt;</th>
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<td>.07</td>
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<td>(.02)</td>
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<td>(.023)</td>
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<td>.87</td>
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<td></td>
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<tr>
<td>3.0&lt;sup&gt;e&lt;/sup&gt;</td>
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<td>X</td>
<td>I</td>
<td>IN</td>
<td>N</td>
<td>$IK_N$</td>
<td>$K_N$</td>
<td>t</td>
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<td>I</td>
<td>N</td>
<td>$K_N$</td>
<td>t</td>
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<tr>
<td>CLS</td>
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<td>2.657</td>
<td>−.303</td>
<td>.62</td>
<td>−19.75</td>
<td>3.33</td>
<td>90.26</td>
</tr>
</tbody>
</table>

<sup>a</sup> The table presents results of computations on structural equations of a revised model. Columns include estimates of parameters and standard errors, $\delta$, $\delta^*/S^e$, calculation values for 1948, calculation distribution, and SETI results for obtaining a small result $\lambda_C$. The final column indicates 10 quantiles ($q_d$).
<table>
<thead>
<tr>
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<th>( N )</th>
<th>( K_{-1} )</th>
<th>( t )</th>
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<td>.55</td>
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<td>.63</td>
<td>-12.18</td>
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<tr>
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<td>.70</td>
<td>-11.77</td>
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<th>( \log K_{-1} )</th>
<th>( t )</th>
<th>( 1 )</th>
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<table>
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<th>( t )</th>
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<table>
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<th>( NK_{-1} )</th>
<th>( K_{-1} )</th>
<th>( t )</th>
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<th>( W_1 )</th>
<th>( (pX - \varepsilon)_{-1} )</th>
<th>( t )</th>
<th>( 1 )</th>
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<td>.13</td>
<td>.17</td>
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<td>Estimates of Parameters and Standard Errors^b</td>
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<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
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<td>$t$</td>
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<td>1</td>
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<td>$.35.20$</td>
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<td>5.0 Wage</td>
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<td>$C \ Y \ t$</td>
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<td>1</td>
<td>1</td>
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<td>(.10)</td>
<td>(.09)</td>
<td>(.10)</td>
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</table>

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^b Estimates of parameters and standard errors are shown in parentheses.

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get-ting a smaller $\lambda$
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<th>6.2 Consumption</th>
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<th>Y</th>
<th>(M/p)_t</th>
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<td>Y</td>
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<td>(M/p)_t</td>
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<td>r/q_t</td>
<td>Σ_s* Y</td>
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<td>(5)</td>
</tr>
</tbody>
</table>
|     | Estimates of Parameters and Standard Errors | \( S \) | \( \delta^2 / \sigma^2 \) | \( \text{O
Notes to Table 2

The equations are numbered here just as they are in the text except that equations 1, 2, 4, 7, 9, 10, and 11 in the text appear here as 1.0, 2.0, · · · , 11.0. The variables in each equation are listed in the same row with the equation number, jointly dependent variables first and predetermined variables next. Each equation has a short title.

The units in which each variable is measured are given in Section 5 except that $\xi_k$ is here converted to billions of current dollars, so that all quantities whose dimensions are current or 1934 dollars are measured in billions.

$KLS$ and $KLI$ refer to Klein's estimates by the least squares and limited information methods, respectively, based on a sample period ending in 1941, and found in Klein (13). (The sample was 1921-41 for all $KLS$ equations except 1.0 and 9.0, and for $KLI$ equations 10.0 and 11.0; the sample was 1922-41 for all other $KLI$ and $KLS$ equations.)

$CLS$ and $CLI$ refer to my estimates by the least squares and limited information methods, respectively, based on the $KLI$ sample plus 1946 and 1947. (Thus the sample was 1921-41 and 1946-47 for $CLS$ and $CLI$ equations 10.0 and 11.0, and 1922-41 and 1946-47 for all other $CLS$ and $CLI$ equations.)

In the interest of not wasting effort in accurate computation of small quantities which will be added to larger and less accurate ones, relatively few significant figures are given for estimates of parameters attached to variables having small numerical values.

The numbers in parentheses in columns 1-7 are estimates of the standard errors of the estimates of the parameters. The numbers not in parentheses are the estimates of the parameters. They are arranged in such a way that any equation may be read off directly in the form in which it is given in the text. For example, the $CLI$ estimate of the consumption equation 6.2 is seen to be

$$ C = .543Y + .315 (M/p) - .27t + 8.56. $$

Column 10 gives the observed 1948 value of the variable appearing on the left side of each equation, i.e., the variable in column 1 of the table. Column 11 gives the value of the linear combination on the right side of each equation. Column 12 is column 10 minus column 11, the calculated disturbance. If this is positive, the equation has underestimated the variable on its left side. Column 10 minus 11 may not equal column 12 exactly because of rounding.

The values of $k$ for $\gamma = 0.99$ and $P = 0.95$, from Eisenhart, Hastay, and Wallis (3), p. 102, are as follows:

<table>
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<tr>
<th>d.f.</th>
<th>k</th>
<th>d.f.</th>
<th>k</th>
<th>d.f.</th>
<th>k</th>
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<td>15</td>
<td>3.507</td>
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<td>3.279</td>
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<td>3.168</td>
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$A = $ accept; $R = $ reject ($CLI$ equations only).

Column 15 gives the approximate probability of obtaining a value of $\lambda_k$ smaller than was in fact observed. A low probability indicates that our confidence in the a priori restrictions imposed must be low. See Table 4 for more details.

The constant term in the $CLS$ estimate of equation 3.0 is $-84.76$. $CLS^*$ and $CLS^{**}$ under equation 3.2 refer to some special exploratory computations based on different series for $K_i$. They are discussed in the text below.
The constant term in the CLS estimate of equation 3.6 is 233.45.

The value of $k^*$ for equation 5.1 is .181, which is larger than .17, the calculated disturbance. Hence the verdict is acceptance.

The value of $k^*$ for equation 6.2 is 6.39, which is smaller than 6.88, the calculated disturbance. Hence the verdict is rejection.

The LS and Li estimates of equation 11.0 are identical since it has only one dependent variable. No value of $\lambda$, is available for this equation.

value as a base; (17) the least-squares predicted change from 1947 to 1948; (18) a notation as to whether this predicted change was in the right direction.

11 Discussion of Results of Computations

We shall look first at the results of (a) the SETI test and (b) the naïve model tests. Also we shall (c) compare the 1948 calculated disturbances obtained from different estimates of each structural equation. Then we shall go back and look at the results of the tests of internal consistency described in the first part of Section 8: (d) the serial correlation test; (e) the characteristic root test; and (f) the qualitative examination of the estimates, in particular those for equations 1.0 and 4.0, where anomalous results appear.

a) There are ten stochastic equations in our revised econometric model, namely those estimated by the limited information method with a sample including 1946–47: the CLI equations 1.0, 2.0, 3.4, 4.0 or 4.2, 5.1, 6.2, 7.0, 9.0, 10.0, and 11.0. All are accepted for 1948 by the SETI test with $P = 0.95$ and $\gamma = 0.99$, except for the consumption function 6.2. If $P$ and $\gamma$ are both relaxed to 0.95, only one additional equation, the wage adjustment equation 5.1, is rejected by the SETI test. Even if $P$ and $\gamma$ are both relaxed to 0.75, all equations except 3.4, 5.1, and 6.2 are accepted by the SETI test with room to spare. This means that nearly every equation fits 1948 as well as could be expected on the basis of its performance during the sample period.

b) For 1948, each of the two naïve models predicts 7 out of 13 endogenous variables better, i.e., has smaller errors, than do the equations of the reduced form as estimated by the ordinary least squares method. Naïve model I predicts better in 15 cases out of 21 than the reduced form as estimated by the restricted least-squares method, and naïve model II pre-

53 The results of all the computations in this paper of course depend upon the time series used for the variables for 1946 and 1947. See Appendix C for a discussion of certain peculiarities in those time series.

54 To verify this, compare Table 2, columns 8 and 12, and Eisenhart, Hastay, and Wallis (3), p. 102.
### Table 3

**Results of Naive Model Tests**

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<td>-38</td>
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<td>Private output ( X )</td>
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<td>90.3</td>
<td>2.96</td>
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<td>107.6</td>
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<td>Fed. rate ( w )</td>
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<td>2.08</td>
<td>2.05</td>
<td>2.22</td>
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<td>Disposable income ( Y )</td>
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<td>1.13</td>
<td>2.86</td>
<td>80.2</td>
<td>67.8</td>
<td>81.7</td>
<td>76.9</td>
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<td>Owned housing ( D_1 )</td>
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<td>1.74</td>
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<td>70.9</td>
<td>78.0</td>
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<td>Rental housing ( D_2 )</td>
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<tr>
<td>Interest rate ( I )</td>
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<td>22</td>
<td>.38</td>
<td>2.73</td>
<td>2.59</td>
<td>2.86</td>
<td>2.98</td>
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</tbody>
</table>

Fraction of cases in which naive model error is less than reduced form error

7/13 7/13 5/13 7/15/21

* Only the endogenous variables in the sense of the \( y' \) in Section 1 appear in this table. Each reduced form prediction is obtained from a reduced form equation containing all the predetermined variables in the group to which the predicted variable belongs. See Appendix D for the grouping.


* LS means least-squares estimates; RLS means restricted least-squares estimates.

* Error equals observed value minus predicted value. A positive error means underestimation.

* RF means that the reduced form's error is smaller than the naive model's error; N means the reverse.
dicts better in 13 cases out of 21 than the reduced form as estimated by the restricted least-squares method.\textsuperscript{65}

These results do not permit us to say that there is any significant difference between the predicting abilities of the ordinary least squares estimates of the equations of the reduced form on the one hand and the naive models on the other. They suggest that, at least in the absence of structural change, predictions by the restricted least-squares estimates of the reduced form are inferior, both to predictions by the ordinary least-squares estimates of the reduced form and to those made by the naive models. The econometric model used here has failed, at least in our sample consisting of the one year 1948, to be a better predicting device than the incomparably cheaper naive models, even though almost every structural equation performs as well, i.e., has just as small an error, in extrapolation to 1948 as it does in the sample period.

It might be noted that the variables that are predicted better for 1948 by naive model I than by the reduced form (as estimated by either of the two ways) are almost exactly the same as those for which the change from 1947 to 1948 was less than the average (absolute value) of the annual changes over the sample period (see Table 3, col. 2, 3, 12, and 14). In other words, roughly speaking, naive model I predicted better the variables that changed less than usual, and the econometric model through its reduced form predicted better the variables that changed more than usual. This is not surprising, because naive model I assumes no change, and so of course will do well when there are only small changes, and poorly when there are large changes. On the other hand, the variables that are predicted better for 1948 by naive model II than by the reduced form include some for which the 1947-48 change was greater than average and some for which it was less (see Table 3, col. 2, 3, 13, and 15). But the variables whose predicted 1947-48 changes (based on the reduced form) were greater than average are not uniformly better predicted by the reduced form than by either naive model (see Table 3, col. 2, 12, 13, and 17). We conclude that it is not possible to tell in advance which variables are likely to be predicted better by the reduced form and which by a naive model.

However, an econometric model may be preferable, even though a naive model predicts equally well, because an econometric model may be able to predict the effects of alternative policy measures or other exogenous changes (including changes in structure if they are known about beforehand), while the naive model can only say that there will be no effect. Unfortunately we do not know how to tell rigorously in advance whether

\textsuperscript{65} Incidentally, neither naive model is shown to be superior to the other; naive model II predicts better than naive model I in 7 out of 13 cases.
this will be true in a particular case, but it appears likely to be true when
large or irregular changes occur in the exogenous variables, because it is
then that the naive models are at their greatest disadvantage.

c) Table 2 shows that for every structural equation whose $KLI$ and $CLS$ 
estimates were both computed, the $CLS$ estimates yield the smaller cal-
ulated disturbance. This suggests that for small samples a short extrapola-
tion of least-squares estimates (i.e., from 1947 to 1948) may be more
reliable than a prolonged extrapolation of limited information estimates
(i.e., from 1941 to 1948). Table 2 shows also that for the eleven equations
whose $CLI$ and $CLS$ estimates were both computed, the $CLS$ estimates
yield appreciably smaller calculated disturbances in four cases, the $CLI$
yield smaller ones in two cases, and there is approximately a tie in five
cases. This suggests that short extrapolations based on least-squares es-

imates may be more reliable for samples as small as 22 than those based
on limited information estimates.

We have here two comparisons of ordinary least-squares estimates with
others known to be asymptotically superior (three if we recall that the
ordinary least-squares estimates of the reduced form equations yield better
predictions for 1948 than do the restricted least-squares estimates). In
these comparisons the results suggest that in our problem the least-squares
estimates lead to smaller errors in extrapolation. Now this is not surprising
if there is no change in the underlying mechanism generating the observa-
tions, i.e., no change in structure. The argument is as follows: The least-
squares method yields an estimate of the expected value of the conditional
probability distribution of one variable, the one chosen to be "dependent",
given the others. This distribution remains fixed as long as there is no
change in structure. Therefore the least-squares estimates, which by con-
struction produce the smallest possible calculated root-mean-square residu-

al over the sample period, will continue to produce small residuals in
extrapolation to subsequent periods as long as there is no change in
structure. But if the structure changes after the sample period and before
the prediction period, the conditional probability distribution of the chosen
dependent variable, given the others, will change in a complicated way,
depending on the old and new structures. Then the least-squares estimates
will no longer yield small errors in extrapolation, because they are es-

cimates of the expected value of a distribution that is no longer relevant.

---

$^{66}$ The cases are, respectively: 1.0, 4.0, 9.0, 10.0; 3.4, 4.2; 2.0, 5.1, 6.2, 7.0, 11.0.
Equation 11.0 must produce a tie because the $CLS$ and $CLI$ estimates are identical.

$^{67}$ The size of the error in extrapolation by any method will increase with the length
of the extrapolation. For the case of least squares this is described by the Hotelling
(9) formula for the standard error of forecast.
This is why it is desirable to estimate structural relations as well as simple regressions.

d) The limited information estimates presented here, as indicated in Section 1, are computed on the assumption that disturbances to the structural equations are not serially correlated. For a sample of 22, if there is no serial correlation, the probability is 0.95 that the value of $\delta^2/S^2$ will lie between approximately 1.26 and 2.93. For the $CLI$ estimates of equations 4.2, 6.2, 7.0, and 9.0, $\delta^2/S^2$ is less than 1.26, indicating positive serial correlation of their disturbances. Klein's limited information estimates give evidence of positive serial correlation of the disturbances to equations 6.3 and 9.0. There is no obvious relation between the performance of an equation in the SETI test and the serial correlation of its disturbances; no attempt has been made here to assess the error incurred by assuming zero serial correlation of disturbances.

**Table 4**

**Characteristic Root Test Results**

<table>
<thead>
<tr>
<th>Eq.</th>
<th>$K^{**}$</th>
<th>$H - 1^*$</th>
<th>$\nu$</th>
<th>$T$</th>
<th>$T \log (1 + 1/\nu) = d.f^*$</th>
<th>$Pb$ of getting a smaller $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>6</td>
<td>1</td>
<td>1.662</td>
<td>22</td>
<td>4.50</td>
<td>5</td>
</tr>
<tr>
<td>2.0</td>
<td>7</td>
<td>2</td>
<td>1.160</td>
<td>22</td>
<td>5.94</td>
<td>5</td>
</tr>
<tr>
<td>3.4</td>
<td>7</td>
<td>1</td>
<td>0.450</td>
<td>22</td>
<td>11.18</td>
<td>6</td>
</tr>
<tr>
<td>4.0</td>
<td>6</td>
<td>1</td>
<td>0.605</td>
<td>22</td>
<td>9.32</td>
<td>5</td>
</tr>
<tr>
<td>4.2</td>
<td>8</td>
<td>2</td>
<td>0.345</td>
<td>22</td>
<td>13.00</td>
<td>6</td>
</tr>
<tr>
<td>5.1</td>
<td>6</td>
<td>2</td>
<td>0.587</td>
<td>22</td>
<td>9.50</td>
<td>4</td>
</tr>
<tr>
<td>6.2</td>
<td>4</td>
<td>1</td>
<td>1.012</td>
<td>22</td>
<td>6.57</td>
<td>3</td>
</tr>
<tr>
<td>7.0</td>
<td>5</td>
<td>2</td>
<td>1.189</td>
<td>22</td>
<td>22.14</td>
<td>3</td>
</tr>
<tr>
<td>9.0</td>
<td>4</td>
<td>1</td>
<td>0.686</td>
<td>22</td>
<td>8.60</td>
<td>3</td>
</tr>
<tr>
<td>10.0</td>
<td>2</td>
<td>1</td>
<td>12.96</td>
<td>23</td>
<td>0.71</td>
<td>1</td>
</tr>
</tbody>
</table>

* $K^{**}$ is the number of predetermined variables that are assumed to be known to be in the model but not in the equation to be estimated; $H$ is the number of jointly dependent variables in the equation to be estimated; and $K^{**} - H + 1$ is the number of overidentifying restrictions, i.e., the number of degrees of freedom of $T \log (1 + 1/\nu)$; see Section 8.

e) Table 4, an expanded version of column 15 in Table 2, gives the results of the characteristic root test as applied to each equation of the revised model. At the 95 per cent significance level four equations are

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*Chernoff and Rubin have developed a consistent method of estimation, as yet unpublished, that does not require this assumption, but no computations have as yet been made with it.

* $\delta^2/S^2$ is defined in Section 8, and its distribution is tabulated in Hart and von Neumann (8), p. 213. See also note 44 above.

* At the 90 per cent significance level the interval containing $\delta^2/S^2$ is smaller, but no additional equations show serial correlation.
rejected by the characteristic root test: 4.2, 5.1, 7.0, and 9.0. Furthermore, at the 90 per cent significance level three other equations are rejected as well: 3.4, 4.0, and 6.2. Again, there is no obvious relation between the performance of an equation on this test and its performance on the SETI test, but this test rejects all equations rejected by the test of $\hat{\sigma}^2/S^2$.

f) In the following discussion of the estimates presented in Table 2 we use abbreviated designations, such as "KLS 1.0" for "the Klein least squares estimates of equation 1.0."

In the demand for investment equation, $CLS\ 1.0$, the estimate of the coefficient of \[
\left(\frac{pX - \bar{X}}{q}\right)_{-1},
\]
which is closely related to lagged profits, has become negative, though not significantly. This may be due to sampling variation, or it may mean that entrepreneurs invest partly in response to increases in profits rather than only in response to high present and past profits. Thus, by means of the identity $\Delta x = x - x_{-1}$, $CLS\ (1.0)$ can be equivalently written as

\[
(1.0') \quad I = .089 \left(\frac{pX - \bar{X}}{q}\right) + .041\Delta \left(\frac{pX - \bar{X}}{q}\right) - .040K_{-1} + .18.
\]

In $CLS\ 3.0, 3.1, 3.2,$ and 3.3, i.e., in all the $CLS$ production functions containing $K_{-1}$ but no cross-product term in $NK_{-1}$, the marginal product of capital emerges as negative, though not significantly. At first this seemed to be due to the fall in $K_{-1}$ from 110.1 in 1941 to 98.8 in 1946, coupled with the tremendous rise in $X$; it did not seem reasonable that the stock of productive capital in private hands had decreased 10 per cent during the war. But $CLS'$ 3.2 and $CLS^{**}$ 3.2, each based on an upward-revised postwar series for $K_{-1}$,\(^{61}\) yield even more strongly negative estimates of the marginal product of capital than $CLS\ 3.2$.\(^{62}\) An examination of the

---

\(^{61}\) The values of $K_{-1}$ used in $CLS, CLS'$, and $CLS^{**}$ 3.2 are, respectively,

<table>
<thead>
<tr>
<th>Year</th>
<th>$K_{-1}$</th>
<th>$K_{-1}'$</th>
<th>$K_{-1}^{**}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1946</td>
<td>98.787</td>
<td>102.600</td>
<td>101.098</td>
</tr>
<tr>
<td>1947</td>
<td>101.098</td>
<td>108.168</td>
<td>116.968</td>
</tr>
</tbody>
</table>

$K_{-1}'$ is like $K_{-1}$ except that additions are made to correct understatements during the war years due to the amortization of war plants in five years or less, allowed under the wartime revenue acts; the transfer of surpluses producer goods from government to private hands; and (less defensible) the fact that the joint SEC-Department of Commerce series for plant and equipment expenditures, used in defining $K$, is smaller beginning in 1941 than the Department of Commerce series that appears in the national income accounts (the two series are almost identical before 1941). $K_{-1}^{**}$ is like $K_{-1}$ except that it assumes that the 1946 value is 111.4, as assumed in Klein (11), p. 135.

\(^{62}\) $CLS'$ 3.2 and $CLS^{**}$ 3.2 have smaller calculated disturbances in 1948 than $CLS\ 3.2$ or $CLS\ 3.4$ or $CLT\ 3.4$, despite their negative marginal products of capital, because all the production functions have negative disturbances which are made smaller in magnitude by the presence of a larger negative term in stock of capital.
time series for \(X\) and \(K_{-1}\) shows that for about half of the sample period the two variables move in opposite directions; consequently, given the time series the result is not unreasonable. The conclusion to be drawn is either that the \(K_{-1}\) series does not measure stock of capital, as it is meant to, or that the stock of capital sometimes does not limit output. A less aggregative theory might be helpful in solving the problem. Equation 3.4 is the immediate solution chosen here.\(^65\)

In CLS 3.0 the estimated coefficient of \(IK_{-1}\) is nearly zero when it is expected to be positive. But this is not a new cause for alarm, given the fact that the coefficient of \(K_{-1}\) is negative.

CLS 3.0 has one additional independent variable besides those in CLS 3.2, yet its disturbance has a larger estimated standard error, \(S\). The same is true of CLS 6.0 and CLS 6.2, respectively, and of CLS 6.4 and CLS 6.5, respectively. This seems odd because when a new independent variable is added to a regression, it cannot increase the sum of squares of residuals. The answer lies in the fact that the reduction in the number of degrees of freedom caused by the introduction of the new variable more than uses up the reduction in the sum of squares brought about by the same cause. In such a case the additional variable is not worth its extra cost in degrees of freedom, except in larger samples where the cost is negligible.

The time trend term in the demand for labor equation CLS 4.1 has a very small coefficient not significantly different from zero, and therefore might reasonably be omitted. But the coefficient of \(w/p\) in CLS 4.1 and CLS 4.2 is very sensitive to the presence or absence of the trend term. As it too has a coefficient not significantly different from zero, however, its sensitivity might be attributed to sampling variation. It is apparent that in both 4.1 and 4.2 the chief relationship being estimated is that between \(X\) and \(N\), namely the backbone of the production function. Indeed 3.4 is almost identical with either 4.1 or 4.2, numerical estimates and all — the term in \(w/p\) contributes relatively little to 4.1 or 4.2. It may be noted that 4.1 is not identified if 3.4 is in the model at the same time. Since 3.4 is the most satisfactory of our production functions, except for 3.5 and 3.6, which were tried later, we want to keep it, and so we replace 4.1 by 4.0 or 4.2.\(^64\)

\(^64\) The nonlinear equations 3.5 and 3.6, theoretically preferable to 3.2 and 3.4 because of having non-constant marginal productivities, were not estimated by the limited information method because of lack of time. Their least-squares estimates, particularly for 3.6, yield smaller calculated residuals for 1948 than any of the other production equations, however.

\(^65\) This decision to drop 4.1 from the model is open to criticism because it is made in order to satisfy the necessary conditions for the identification of all equations, and not on theoretical or empirical grounds.
dion) for identifiability with room to spare, the probability is high that
they meet the necessary and sufficient (rank) condition as well (see Sec. 1).

The consumption functions $CLS\ 6.0$, $CLS\ 6.2$, $CLI\ 6.2$, and $CLS\ 6.4$
show significantly positive coefficients for real cash balances $(M/p)_{-1}$. 
The addition of this term alone is enough to reduce the calculated disturbance in 1948 almost half — see $CLI\ 6.2$ — as compared with that of
equation 6.3 which does not contain $(M/p)_{-1}$. But apparently the intro-
duction of $(M/p)_{-1}$ is not sufficient to correct the consumption function,
for $CLI\ 6.2$ is rejected by the SETI test. Another indication that $(M/p)_{-1}$
is not sufficient is that consumption has not fallen relative to disposable
income since 1944, but $(M/p)_{-1}$ has been falling since 1946. Thus a term
in $(M/p)_{-1}$ can explain the high postwar average level of consumption
relative to income as compared with prewar, but it cannot explain the
fact that consumption has remained high in 1947 and 1948, even exceeding
disposable income, while $(M/p)_{-1}$ has been declining. It is evident
that some other variable in addition to or in place of $(M/p)_{-1}$ is needed.

As can be seen from an examination of $CLS\ 6.4$ and $CLS\ 6.5$, lagged
consumption expenditure $C_{-1}$ appears to help matters, and more so when
used instead of $(M/p)_{-1}$ than when used in addition to it.\textsuperscript{65} $CLS\ 6.5$ has
a smaller estimated standard error $S$ and a smaller calculated 1948 dis-
turbance than any of the other consumption equations estimated from the
sample that includes 1946 and 1947.\textsuperscript{66}

The estimated coefficients of disposable income $Y$ in the rent adjust-
ment equation $CLS\ 9.0$ and $CLI\ 9.0$ are negative. The explanation may lie
in sampling variation, since the standard errors are of the same order of
magnitude as the estimates. However, the controlled rise in postwar rents
and the fall of $Y$ from 1946 to 1947 may be responsible (see App. C).

The estimated coefficients of twice lagged construction costs $(q_2)_{-2}$
are negative in all four estimates of the demand for the construction of
rental housing, equation 10.0. As their standard errors are about as large
as the estimates, this need not be taken seriously. However, some response
to expected costs, based on the past behavior of costs, may be indicated.

The $CLI$ estimates of equations 1.0 and 4.0, as we have seen, are far out
of line with our expectations. Unlike the $CLI$ estimates of the other equa-
tions, they do not remotely resemble the $CLS$, $KLS$, and $KLI$ estimates.
Their calculated disturbances are often of the same order of magnitude as

\textsuperscript{65} This is because the addition of $(M/p)_{-1}$ to 6.5 costs more in degrees of freedom than it is worth in reducing the sum of squares of residuals, as discussed in the third paragraph above.

\textsuperscript{66} Limited information estimates were not computed for equations 6.4 and 6.5 because of lack of time; these equations were not considered until all the other computations were finished and the inadequacy of 6.2 became obvious.
the variables they contain, but they do not show clear signs of serial correlation.

One obvious possibility must be rejected immediately, namely that for each of these two equations the 1946 and 1947 observations may be nowhere near the line fitted to the 1922-41 sample, so that the line is radically changed by the addition of the 1946 and 1947 points. If this were true, the least squares estimates would be radically changed and the estimated standard error of disturbances greatly increased; but neither happens. Equations 1.0 and 4.0 are clearly cases where there is an approximately linear empirical relation among the variables (as evidenced by the least squares fits) but where the limited information method yields a straight line very different in slopes and intercepts from this empirical relation.

Sampling variation cannot be excluded as a possible explanation, especially since the estimated standard errors are so large that the CLI estimates do not differ significantly, i.e., by more than two or three times their respective standard errors, from the other estimates. Furthermore, nothing in the derivation of the limited information method requires it to yield small residuals and estimates close to the least-squares estimates, even though it has usually done so in the past.

There are two differences between the two procedures used in obtaining the KLI and the CLI estimates of equations 1.0 and 4.0. One is, obviously, that 1946 and 1947 are in the CLI sample but not in the KLI sample. The other is that the list of predetermined variables \( z^* \) (explained in App. D) for the CLI estimates differs from the list for the KLI estimates in that the variables \( X_{-1} \) and \( H_{-2} \) are omitted and the variables \( w_{-1} \) and \( (N_{L} - N)_{-1} \) added. In other words, certain of the reduced form equations in the CLI case are regressions on a set of predetermined variables which differs from the corresponding set in the KLI case by containing \( w_{-1} \) and \( (N_{L} - N)_{-1} \) instead of \( X_{-1} \) and \( H_{-2} \), and accordingly the estimates of the parameters of equations 1.0 and 4.0 depend upon observations of a slightly different set of predetermined variables.

To separate the effects of these two changes, equation 4.0 was estimated four times: (1) (KLI) with the Klein \( z^* \)'s and without 1946-47; (2) with my \( z^* \)'s and without 1946-47; (3) with the Klein \( z^* \)'s and with 1946-47; (4) (CLI) with my \( z^* \)'s and with 1946-47. The results indicate that the anomalous CLI estimates of equation 4.0 are not due to the change in the list of predetermined variables \( z^* \), but are somehow due instead to the addition of 1946 and 1947 to the sample.\footnote{\( (KLI) \) \( W_i = .41 \ (pX - \varepsilon) + .17 \ (pX - \varepsilon)_{-1} + .17t + 5.04 \)
(2) \( W_i = .413 \ (pX - \varepsilon) + .175 \ (pX - \varepsilon)_{-1} + .17t + 5.05 \)
(3) \( W_i = 8.17 \ (pX - \varepsilon) - 7.56 \ (pX - \varepsilon)_{-1} - 2.68t - 9.42 \)}
12 SUMMARY AND CONCLUSION

The revised version of Klein's model, consisting of equations 1.0, 2.0, 3.4, 4.0 or 4.2, 5.1, 6.2, 7.0, 9.0, 10.0, 11.0, and the identities 12, 13, 14, and 18, has been subjected to several tests. Table 5 summarizes the results of tests pertaining to the structural equations. Table 3 presents the results of the naive model tests, which pertain to the equations of the reduced form.

Table 5

<table>
<thead>
<tr>
<th>Equation</th>
<th>SETI Test*</th>
<th>Smaller Correlation Test*</th>
<th>Characteristic Root Test*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P = .95$</td>
<td>95% level</td>
<td>95% level</td>
</tr>
<tr>
<td></td>
<td>$P = .75$</td>
<td>90% level</td>
<td>90% level</td>
</tr>
<tr>
<td>1.0</td>
<td>Investment</td>
<td>$\gamma = .99$</td>
<td>CLS</td>
</tr>
<tr>
<td>2.0</td>
<td>Inventory</td>
<td>$\gamma = .75$</td>
<td>neither</td>
</tr>
<tr>
<td>3.4</td>
<td>Production</td>
<td>$R$</td>
<td>$CLS$</td>
</tr>
<tr>
<td>4.0</td>
<td>Labor</td>
<td>$R$</td>
<td>$CLS$</td>
</tr>
<tr>
<td>4.2</td>
<td>Labor</td>
<td>$R$</td>
<td>$CLS$</td>
</tr>
<tr>
<td>5.1</td>
<td>Wage</td>
<td>R</td>
<td>neither</td>
</tr>
<tr>
<td>6.2</td>
<td>Consumption</td>
<td>$R$</td>
<td>both</td>
</tr>
<tr>
<td>7.0</td>
<td>Owned housing</td>
<td>R</td>
<td>both</td>
</tr>
<tr>
<td>9.0</td>
<td>Rent</td>
<td>$CLS$</td>
<td>R</td>
</tr>
<tr>
<td>10.0</td>
<td>Rental housing</td>
<td>$CLS$</td>
<td>R</td>
</tr>
<tr>
<td>11.0</td>
<td>Interest</td>
<td>neither</td>
<td>R</td>
</tr>
</tbody>
</table>

Source: Tables 2 and 4, and Section 11, parts (a) and (d).

* R means reject; a blank space means accept.

With the exception of the consumption function 6.2, all the equations estimated by the limited information method fit the post-sample year 1948 just as well as they fit the data of the sample period. This is shown by the SETI test.

The predictions for 1948 made by the equations of the reduced form are, on the average over all equations, no better (measured by whether their errors are smaller in absolute value) than predictions made by naive models which simply extrapolate either the value of each variable from the preceding year or the trend between the two preceding years (Table 3).  

Note 67 concluded:

(4) \begin{equation}
(\text{CLI}) W_t = -8.29 \ (pX - \bar{X}) + 8.95 \ (pX - \bar{X}) - x + 3.49t + 15.17 
\end{equation}

Note the similarity between 1 and 2 and (except for sign) between 3 and 4. (This sign difference is not an error in computation; it is due to a change in sign of a determinant entering the estimate of the parameter on which the estimates are normalized.)

*6 The equations are given in Sections 5 and 7, and more compactly in Table 2.

*6 And, if $P$ and $\gamma$ are both relaxed to 0.75, the production function 3.4 and the wage adjustment equation 5.1.

*9 In fact they are worse if the restricted least squares method is used instead of the ordinary least squares method.
However, the reduced form predictions are quite consistently better than the predictions of naive model I for variables that changed more than usual in 1948. Further, the equations of the reduced form may be preferable to naive models for predicting effects of exogenous changes even when both methods make equally large errors in the ordinary prediction of the magnitudes of economic variables, especially when the exogenous changes are unusually large.

The least squares method yields on the average smaller calculated disturbances for 1948 than do our asymptotically superior methods, for both structural and reduced form equations.\(^7\) This is seen by a simple pairwise comparison of calculated disturbances in Tables 2 and 3.

Four equations, as estimated by the limited information method, are rejected by the two-sided test for serial correlation of disturbances, at either the 95 or the 90 per cent level of significance.

Four equations are rejected at the 95 per cent significance level by the characteristic root test of the totality of a priori restrictions imposed on a given equation, and seven at the 90 per cent significance level.

Several avenues of future work suggest themselves on the basis of the experience of this paper.

1) Better use could be made of existing economic theory. That is, equations to be estimated should be consistent with the known properties of the equations of micro-economics. Also, a better theory of economic expectations and of behavior under uncertainty would be useful.

2) Studies of narrower sectors of the economy would probably be fruitful, because it is desirable whenever possible to refine our approximations by using variables and equations that apply to more homogeneous groups of firms or individuals. Furthermore, there are several industries and economic sectors for which data, as well as facts pertaining to the technical and institutional environment, are much more plentiful than for the economy as a whole.

3) Cross-section data, i.e., data pertaining to different parts of the economy as of a given point in time such as are obtained in surveys, are becoming increasingly available. It will be possible to combine time series and cross section studies to advantage.

4) One misfortune of the econometrician is that exogenous variables do not vary enough to give him a good idea of their respective influences. The war years are very valuable in this regard, because exogenous changes are ordinarily much larger than in peacetime. Therefore they might be included

\(^7\) This is something to be expected if there are no important changes of structure, and is not contrary to the claims made for limited information estimation; see Section 11, part (c).
in the sample, of course together with appropriate changes in certain parts of the model to allow it to accommodate the wartime government policies. 5) The use of quarterly data would multiply the effective sample size by approximately four,\(^7\) thus producing more accurate estimates, provided the problem of serial correlation can be solved (see next item).

6) The development of practical methods of estimation that do not require the assumption of zero serial correlation of disturbances would be useful. As already mentioned, Chernoff and Rubin have worked on this problem but as yet no attempt has been made to use their results.

7) Mathematical (or experimental)\(^8\) studies to determine the size of the small-sample bias in the estimation of structural parameters by the least-squares method and by the various maximum likelihood methods would be very helpful in deciding which procedure to use.

8) Studies might be made of the effect of estimating the parameters of a model by using data generated by a structure not belonging to the given model, i.e., the effect of estimating from the wrong model. This is a general problem which includes the case of estimating by the least squares method when to do so is not theoretically justified. If a “slightly incorrect” model always or often leads to absurd results, the type of econometrics presented in this paper will suffer a severe setback, because we know from the start that our models are at least slightly incorrect.

9) It would be interesting, though expensive, to estimate the parameters of a fairly large system of equations by the full information maximum likelihood method and analyze the results. But this would not be likely to be immediately useful in getting better estimates unless the sample size were much larger than 22.

Appendix A

Time Series

Until 1942 all time series are as given in Klein (11), pp. 141-3, except that those marked with an asterisk below have been revised as indicated

\(^7\) The effective sample size would be multiplied by \textit{exactly} 4, except for several small points: the fact that one degree of freedom goes into the estimation of each parameter; the possibility of adding four new parameters in order to allow for seasonal changes (this is done by introducing four new exogenous variables \(x_1, x_2, x_3, x_4\), such that in the \(i\)th quarter all are 0 except \(x_i\) which is 1, and estimating the parameter of each); etc.

\(^8\) Orcutt and Cochrane (18, 19) have used sampling experiments of a type that might be widely applied in getting information of any desired degree of statistical reliability about certain problems that seem to be secure against direct mathematical attack.
in Appendix B, and the $X$ series presented here reflects the correction of a computational error, which has been corrected in Klein (13) also.

$$
\begin{array}{cccccccc}
C & I & q & \Delta H & D_1 & q_1 & D_2 & D_3 & D^* \\
1941 & -1.748 & 1.303 & - .379 & .870 & 1.497 & .831 & .174 & 1.977 \\
1943 & 76.833 & -3.950 & 1.351 & -1.224 & .743 & 1.621 & .539 & .143 & 1.947 \\
1945 & 86.517 & 1.311 & 1.619 & 4.020 & 1.334 & 2.000 & .941 & .182 & 1.926 \\
1946 & 81.708 & 2.266 & 1.964 & .345 & 1.629 & 2.671 & 1.162 & .172 & 1.943 \\
1948 & 81.708 & 2.266 & 1.964 & .345 & 1.629 & 2.671 & 1.162 & .172 & 1.943 \\
1943 & 23.224 & 1.144 \\
1944 & 70.902 & 141.769 & 99.705 & 1.333 & 28.211 & 87.834 & 10.613 & .820 & 1.146 \\
1945 & 59.604 & 135.446 & 96.286 & 1.369 & 29.595 & 84.908 & 11.002 & .889 & 1.147 \\

\Delta F & v & N^* & i & \varepsilon_a & K & \varepsilon & X & H \\
1943 & 3.16 & \\
1944 & 3.05 & 112.299 & 27.361 \\
1945 & 571 & 100.0 & 31.3 & 2.87 & 98.787 & 7.9 & 105.142 & 26.936 \\
1946 & 1088 & 100.0 & 31.8 & 2.74 & 976 & 101.098 & 9.4 & 95.339 & 30.956 \\
1947 & 1381 & 100.0 & 32.7 & 2.86 & 853 & 103.364 & 10.0 & 87.304 & 31.301 \\
1948 & 1582 & 100.0 & 33.6 & 3.08 & 888 & 105.253 & 10.6 & 90.263 & 34.342 \\

M & N & N_p & w & X \\
1920 & 38.464 & 38.335 & 38.609 & 1.080 & 40.3 \\
1921 & 38.100 & 37.737 & 39.239 & 0.921 & 38.8 \\
1922 & 40.623 & 36.335 & 39.850 & 0.938 & 42.8 \\
1923 & 43.249 & 39.035 & 40.815 & 1.040 & 49.3 \\
1924 & 46.826 & 38.744 & 41.592 & 1.035 & 48.5 \\
1925 & 49.981 & 39.379 & 42.044 & 1.056 & 52.5 \\
1926 & 50.876 & 40.748 & 43.072 & 1.087 & 55.6 \\
1927 & 53.802 & 40.792 & 44.103 & 1.086 & 55.8 \\
1928 & 55.355 & 40.969 & 45.128 & 1.118 & 56.0 \\
1929 & 54.555 & 42.489 & 46.247 & 1.132 & 58.1 \\
1930 & 53.248 & 40.397 & 46.757 & 1.067 & 52.3 \\
1931 & 47.561 & 37.214 & 47.313 & 0.931 & 44.1 \\
1932 & 48.854 & 33.816 & 47.967 & 0.787 & 35.1 \\
1933 & 41.532 & 33.770 & 48.627 & 0.728 & 36.7 \\
1934 & 46.270 & 36.177 & 49.127 & 0.788 & 42.2 \\
1935 & 51.273 & 37.162 & 49.583 & 0.840 & 47.1 \\
1936 & 56.360 & 39.142 & 49.961 & 0.902 & 55.3 \\
1937 & 55.815 & 41.026 & 50.433 & 0.999 & 57.5 \\
1938 & 58.066 & 38.657 & 50.908 & 0.957 & 52.6 \\
1939 & 63.253 & 40.014 & 51.437 & 0.999 & 61.0 \\
1940 & 70.008 & 41.851 & 51.722 & 1.050 & 66.9 \\
1941 & 76.336 & 45.369 & 51.633 & 1.213 & 79.8 \\
1942 & 47.678 & 51.427 \\
1943 & 48.149 & 48.821 \\
1944 & 130.225 & 47.111 & 48.900 & 1.864 & 112.299 \\
1945 & 150.793 & 45.662 & 48.181 & 1.859 & 105.142 \\
1946 & 164.004 & 48.533 & 52.145 & 1.886 & 95.339 \\
1947 & 170.610 & 51.019 & 54.937 & 2.053 & 87.304 \\
1948 & 168.700 & 52.066 & 56.021 & 2.202 & 90.263 \\
\end{array}
$$
Appendix B

Sources of Data and Construction of Time Series

Construction of time series for 1942 and later, and for the few of Klein's figures for years before 1942 that were revised, is indicated below. My time series are intended to be as consistent as possible with Klein's, since they are extensions of Klein's. The variables denoted by numbers in parentheses correspond to those in the appendices to Klein (11, 13), with the exception of my numbers (13), (14), (15), and (38). The following abbreviations are used:

BAE: Bureau of Agricultural Economics
BLS: Bureau of Labor Statistics
C.C.M.: Construction and Construction Materials
F.R.B.: Federal Reserve Bulletin
M.L.R.: Monthly Labor Review
S.A.U.S.: Statistical Abstract of the United States
S.C.B.: Survey of Current Business

$C$: consumption, in billions of 1934 dollars.

$$C = \frac{(1) + (2)}{(3)}$$

(1) = consumer expenditures, Department of Commerce old series, *S.A.U.S.*, 1947, p. 273, for years through 1946. 1947-48 values were obtained from a regression (1939-46) of the old series on the new series (*S.C.B.*, July 1948, p. 16, Table 2; and July 1949, p. 10, Table 2).

(2) = imputed rents on owner-occupied residences, *S.C.B.*, July 1948, p. 24, Table 30; and July 1949, p. 23, Table 30.


$^1$Weights used were, respectively: 1944, .596 and .404; 1945, .586 and .414; 1946, .600 and .400; 1947, .590 and .410; 1948, .584 and .416.
\[ I = \frac{4}{5} + \frac{6}{7} - \frac{8}{9} - \frac{10}{7} \]


(5) = price index of business capital goods, 1934: 1.00, regression on Solomon Fabricant's index (*Capital Consumption and Adjustment*, pp. 178-9, and private correspondence) of a weighted average of the Aberthaw index (*S.A.U.S.*, 1948, p. 792, and *S.C.B.*, Feb. 1949, p. S-6), the American Appraisal Co. index (*S.A.U.S.*, 1948, p. 792, and *C.C.M.*, May 1949, p. 54), and the BLS index for metals and metal products (*S.A.U.S.*, 1948, p. 296, *M.L.R.*, March 1949, p. 381); weights \( \alpha \), \( \beta \), and \( \gamma \), respectively, are such that the weighted average is the same as Fabricant's index in 1934, in 1941, and on the average for 1934-41.

(6) = gross expenditures on farm service buildings and machinery; equal to expenditures on farm buildings excluding operators' dwellings, farm machinery excluding motor vehicles, farm trucks, and farm autos used in production (assumed to be 50 per cent of expenditures on autos in 1942-45 and 40 per cent thereafter), BAE, private correspondence.

(7) = price index of farm capital goods, 1934: 1.00, weighted average of price indexes for building materials for other than housing (*Agricultural Statistics*, 1947, p. 524, and BAE, private correspondence), farm machinery (same), and motor vehicles (BLS metals and metal products index; see (5) above). The weights are proportional to expenditures on each of the three categories of capital goods, respectively. (Because of an error, current dollar expenditures were used as weights instead of constant dollar expenditures, but as the resulting error in \( I \) is less than 1 per cent in all cases, and less than 0.1 per cent in most cases, no recomputation was made.)

(8) = depreciation charges on private producers' nonagricultural plant and equipment, regression (1929-43) of Mosak's nonagricultural depreciation (*Econometrica*, 13, 1945, p. 46) on the Department of Commerce depreciation series (*S.C.B.*, July 1947 Supplement, p. 20, Table 4; July 1948, p. 17, Table 4; and July 1949, p. 11, Table 4).

*The BAE index of motor vehicle prices was discontinued at the start of the war.*
(9) = price index underlying depreciation charges, 1934: 1.00, regression (1934-41) of Fabricant's depreciation price index (Fabricant, Capital Consumption and Adjustment, p. 183, and private correspondence) on (5).

(10) = depreciation charges on farm service buildings and machinery, BAE, private correspondence.

$q$: price index of private investment goods, 1934: 1.00.

\[ q = \frac{(5) \times (4) + (7) \times (6)}{(4) + (6)} \]

$\Delta H$: value of the change in inventories, in billions of 1934 dollars.

\[ \Delta H = \frac{(11)}{(12)} \]

(11) = value of change in inventories, Department of Commerce old series, S.A.U.S., 1947, p. 273, and S.C.B., May 1942, p. 12, for years through 1946. 1947-48 values were obtained from a regression (1939-46) of the old series on the new series (S.C.B., July 1948, p. 16, Table 2, and July 1949, p. 10, Table 2).

(12) = BLS wholesale price index of all commodities, 1934:1.00, F.R.B., March 1949, p. 297.

$D_1$: gross construction expenditures on permanent, owner-occupied, single family, nonfarm residences, in billions of 1934 dollars.

\[ D_1 = \frac{1.076 \times 1.126 [0.63 \times (13) \times (14) + .32 \times (15)]}{(16)} \]

(13) = ratio of 1-family permanent nonfarm residences started to total permanent nonfarm units started, H.L.S., 1947, p. 193; and M.L.R., February 1949, p. 179 (graph), and May 1949, p. 620.


(15) = private repairs and maintenance expenditures on nonfarm residences, C.C.M., May 1948, p. 15. (This figure is not available after 1944; hence total residential repairs and maintenance was multiplied by the ratio of nonfarm to total new residential construction to get an approximation; C.C.M., May 1949, pp. 6, 15.)

1.076 = ratio of average permit valuation of single-family urban units to all urban units in 1942, BLS Bulletin 786, The Construction Industry in the U. S., p. 21, Table 11.

\(^30.63 = \) fraction of single-family, nonfarm dwelling units constructed 1935-40 that were owner-occupied in 1940, Census of Housing, 1940, III, Part I, Table A-4 (quoted by Klein, p. 144).
1.126 = ratio of average rental value of owner-occupied single-family nonfarm residences (constructed 1935-40) to that of all single-family nonfarm residences (constructed 1935-40), *Census of Housing, 1940*, III, Part I, Table A-4 (Klein, p. 144).

.32 = ratio of owner-occupied single-family nonfarm units to total nonfarm units in 1940 (Klein, p. 144).


$q_1$: construction cost index, 1934: 1.00.

$q_1 = (16)$

$D_2$: gross construction expenditures on rented nonfarm residences, in billions of 1934 dollars.

$$D_2 = \frac{(17)}{(16)} - D_1$$

(17) = (14) + (15).

$D_3$: gross construction expenditures on farm residences, in billions of 1934 dollars.

$$D_3 = \frac{(18)}{(19)}$$

(18) = gross construction expenditures on farm residences, *C.C.M.*, May 1948, pp. 8, 15, and May 1949, pp. 8, 15.


$D'$: depreciation of all residences (farm and nonfarm), in billions of 1934 dollars (on the basis of 3 per cent per year).

$$D' = (67.6)(.97)^{t-1934}(.03) + \sum_{i=1934}^{t-1} (D_1 + D_2 + D_3)$$

$$\times (.985)(.97)^{t-1934}(.03) + (D_1 + D_2 + D_3)$$

for $t > 1934$

$67.6$ = estimated value, January 1, 1934, of the stock of residential dwellings in the U.S. (Klein, p. 145).

$G$: government expenditures for goods and services (not excluding government interest payments) plus net exports plus net investment of nonprofit institutions, in billions of 1934 dollars.

$$G = \frac{(20) - (21) - (22) + (22) + (24) + (25) - 0.1}{(23) + (16) + (12) + (16)}$$

(20) = government expenditures for goods and services, Department of Commerce old series, *S.A.U.S.*, 1947, p. 273, for years through 1946. 1947-48 values were obtained from a regression (1939-46).
of the old series on the new series (S.C.B., July 1948, p. 16, Table 2; and July 1949, p. 10, Table 2).

(21) = government interest payments, S.C.B., July 1948, p. 17, Table 4; and July 1949, p. 11, Table 4.

(22) = public construction expenditures (including work-relief construction), S.C.B., July 1948, p. 25, Table 31; and July 1949, p. 24, Table 31.

(23) = BLS wholesale price index of nonfarm products, 1934: 1.00, H.L.S., 1947, p. 126, and M.L.R., March 1949, p. 381.

(24) = net exports of goods and services and gold, equal to net foreign investment, S.C.B., July 1948, p. 16, Table 1; and July 1949, p. 10, Table 1.

(25) = gross construction expenditures by nonprofit institutions, S.C.B., July 1947 Supplement, p. 44, Table 31; July 1948, p. 25, Table 31; and July 1949, p. 24, Table 31.

0.1 = estimate of depreciation of nonprofit institutions' plant, based on a rate of approximately 3 per cent (Klein, p. 146).

\[ Y + T = C + I + \Delta H + D_1 + D_2 + D_3 - D'' + G \]

\[ Y: \text{disposable income, in billions of 1934 dollars.} \]

\[ Y = \frac{1}{(3)} \left[ (1) + (2) + (4) + (6) - \frac{(8)}{(9)} (5) - (10) + (11) + (17) \right. \]
\[ + (18) - (16)D'' + (20) + (24) + (25) - 0.1 \]
\[ - (26) - (27) - (28) + (29) \]

(26) = federal government receipts, S.C.B., July 1948, p. 17, Table 8; and July 1949, p. 12, Table 8.

(27) = state and local government receipts, same sources as for (26).

(28) = net corporate savings (undistributed corporate profits after taxes plus corporate inventory valuation adjustment, plus excess of wage accruals over disbursements, S.C.B., July 1948, p. 17, Table 4; and July 1949, p. 11, Table 4).

(29) = government transfer payments, same sources as for (28).

\[ p: \text{price index of output as a whole, 1934: 1.00.} \]

\[ p = \frac{1}{Y + T} \left[ (1) + (2) + (4) + (6) - \frac{(8)}{(9)} (5) - (10) + (11) + (17) \right. \]
\[ + (18) - (16)D'' + (20) + (24) + (25) - 0.1 - (21) \]

\[ W_2: \text{government wage-salary bill, in billions of current dollars.} \]

\[ W_2 = (31) \]
(31) = government wages and salaries, including work relief, Department of Commerce old series, S.A.U.S., 1947, p. 269, for years through 1946. 1947-48 values were obtained from a regression (1939-46) of the old series on the new series with adjustments for income in kind to armed forces (S.C.B., July 1948, p. 16, Table 1; and July 1949, p. 10, Table 1).

\[ W_1 = (30) - (31) \]

(30) = total employee compensation, including work relief, Department of Commerce old series, S.A.U.S., 1947, p. 269, for years through 1946. 1947-48 values were obtained from a regression of the old series on the new series (S.C.B., July 1948, p. 16, Table 1; and July 1949, p. 10, Table 1).

\[ R_1 = (32) \]

(32) = sum of owner-occupied and tenant-occupied nonfarm rents, S.C.B., July 1948, p. 24, Table 30; and July 1949, p. 23, Table 30.

\[ R_2 = (33) \]

(33) = farmhouse rentals, same sources as for (32).

\[ r = (34) \]


\[ \Delta F = (35) \]

(35) = increase in nonfarm families, thousands, S.A.U.S., 1948, p. 46, and Bureau of Census, Current Population Reports, Series P-20, No. 21, p. 9. As the number of families is not given as of the same date each year, adjustments were based on linear interpolation between dates given.

\[ v = \text{percentage of nonfarm housing units occupied at the end of the year, assumed equal to 100.} \]

\[ N^a = 31.3 + \sum_{i=1946}^{t} (38)_t \]

(38) = millions of nonfarm housing units finished during the year, M.L.R., March 1948, p. 368, for 1946 and 1947. This series has
been discontinued; for 1948 the number of nonfarm units started was used as an approximation (M.L.R., May 1949, p. 620).


\( i \): average corporate bond yield.

\( i = (40) \)


\( \varepsilon_R \): excess reserves, in millions of current dollars.

\( \varepsilon_R = (43) \)


\( K \): end of year stock of private producers’ plant and equipment, in billions of 1934 dollars.

\[
K = 107.8 + \sum_{t=1935}^{t} I, \quad t \leq 1945
\]

\[
K = 1.0 + 107.8 + \sum_{t=1935}^{t} I, \quad t \geq 1946
\]

107.8 = end of 1934 stock of private producers’ plant and equipment (Klein, p. 148).

1.0 = estimate of surplus property transferred to the private sector at the close of the war (Klein, p. 150).

\( H \): end of year stock of inventories, in billions of 1934 dollars.

\[
H = 21.8 + \sum_{t=1935}^{t} (\Delta H),
\]

21.8 = end of 1934 stock of inventories (Klein, p. 149).

\( \varepsilon \): excise taxes, in billions of current dollars.

\( \varepsilon = (45) \)

(45) = excise taxes, regression (1931-41) of Klein's excise series (p. 149) on the sum of federal excises plus state and local sales and social insurance taxes (S.C.B., July 1947 Supplement, p. 21, Table 8; July 1948, p. 17, Table 8; and July 1949, p. 12, Table 8).

\( X \): private output excluding housing services, in billions of 1934 dollars.

\[
X = Y + \frac{1}{p} (W_2 + R_1 + R_2)
\]

\( M \): end of year money supply, in billions of current dollars.

\( M = (46) \quad t \leq 1922 \)

\( M = (47) \quad t \geq 1923 \)
(46) = demand and time deposits adjusted plus currency outside banks, average of June 30 figures before and after, Federal Reserve Board, *Banking and Monetary Statistics*, p. 34.


$N$: labor input, in millions of full time equivalent man-years.

$$N = (48) \quad t \leq 1928$$

$$N = (49) \quad t \geq 1929$$

(49) = number of full time equivalent persons engaged in production in all private industries, excluding work relief, *S.C.B.*, July 1947 Supplement, p. 40, Table 28; July 1948, p. 23, Table 28; and July 1949, p. 22, Table 28.

(48) = regression (1929-38) of (49) on Kuznets' estimates of total persons engaged in private production (*National Income and Its Composition*, pp. 314-5, 346-7).

$N_L$: labor force, including work-relief employees but excluding other government employees, in millions of man-years.

$$N_L = (50) - (51)$$


(51) = government full time equivalent civilian employees excluding work-relief employees, *S.C.B.*, July 1947 Supplement, p. 36, Table 24; July 1948, p. 22, Table 24; and July 1949, p. 20, Table 24, for years after 1929. 1920-28 values were obtained from a regression (1929-38) of the above series on Kuznets' estimates of the same quantity (*National Income and Its Composition, 1919-1938*, pp. 314-5).

$w$: private money wage rate, in thousands of current dollars per man-year.

$$w = \frac{W_1}{N}$$

Appendix C

**TIME SERIES FOR 1946-1947**

During the discussion at the Conference on Business Cycles Research in November 1949, Lawrence Klein pointed out a discrepancy in the time
series for 1946 and 1947 which were used in this paper: the series for real net national product \( Y + T \) and for real private output \( X \) show decreases of about 10 per cent from 1946 to 1947, while during the same two years the series for private employment \( N \) rose about 5 per cent and the Federal Reserve Board index of industrial production rose 10 per cent. Since these four series are meant to measure magnitudes that have to move closely together (except that the agricultural sector is not represented in the Federal Reserve index), it is clear that something is wrong. It is difficult to see how the series for employment and industrial production could be seriously in error for this period, but the series for \( Y + T \) and \( X \) might be thrown off by either or both of two causes.

First, the series for \( Y + T \) and \( X \) are constructed by adding component series, each of which is first expressed in current prices, then deflated by an appropriate price index. It is very likely that the published price indexes (which were used in the paper) are too low for the years toward the end of the reign of price controls, including 1946, because of failure to take account of reductions in quality and service, black market activities, and the practice on the part of manufacturers of concentrating their output in their more expensive lines. This understatement has been estimated by the Technical Committee on the consumers' price index (also known as the Mitchell Committee) not to exceed about 4 per cent in any year (see the *Economic Report of the President*, January 1950, pp. 156 and 169), and by various others to be considerably larger. It can be expected to have disappeared by some time in 1947, because virtually all controls were lifted in November 1946, and many had been lifted or relaxed before then. Therefore it is a good surmise that while the published price indexes are too low in 1946, they again measure approximately what we want them to measure in 1947. If this is true, the deflated series for \( Y + T \) and \( X \) are too high in 1946, and therefore their apparent drop from 1946 to 1947 is partly or wholly illusory — there may even have been a rise, camouflaged by the understated 1946 price indexes. My guess would be that the entire discrepancy is not to be explained in this manner, however.

Second, as indicated in Appendix B, the time series used were extensions of Klein's own time series, based like his on the series released by the Department of Commerce before the publication in 1947 of its revised national income series. Some of the 1947-48 figures were obtained from regressions of the unrevised series on the corresponding revised series. It would have been sounder to adjust all the time series, including Klein's, to conform to the revised Department of Commerce series, or failing that, to obtain the 1947-48 extrapolations of the unrevised series by adjusting the revised series for changes in definition instead of using regressions.

Similar discrepancies, of comparable magnitude, are obtained for
1946-47 for the whole economy and for separate industries if the national income originating in the economy and in each of several industries, deflated by the corresponding wholesale price index, is compared industry-wise with the number of full time equivalent persons engaged in production or with the Federal Reserve index of industrial production. (They are clearly visible, even though the industrial classifications are not quite the same in the Federal Reserve index and national income accounts as in the wholesale price index.) Because these discrepancies are comparable in magnitude to the one pointed out by Klein, it seems likely that it is unnecessary to look to my regression technique for an explanation of the error in the relationship of the 1946 to the 1947 figures; it even seems likely that the regression technique made no significant contribution to that error (though no doubt it introduced others).

It remains to determine the effect of the discrepancies on the results of the paper. Of course the most reliable way would be to revise all the data and re-estimate all the equations. Here it is possible only to try to obtain a rough idea of the effect, by means of some approximation 'corrections' consisting of making changes in some of the 1946-47 time series so that they become consistent with the Federal Reserve index of industrial production, then re-estimating certain of the structural and reduced form equations by the ordinary least squares method. The detailed steps and results of this exploratory 'correction' procedure are explained below.

The time series for real private output $X$, disposable income $Y$, and consumption $C$ are accepted as correct for 1947, and are 'corrected' for 1946. Let unprimed symbols stand for the values underlying the original computations of the paper, and primed symbols for the 'corrected' values. Then,

$$X'_{1946} = \frac{170}{187} X_{1947}$$

where $\frac{170}{187}$ is the ratio of the 1946 to 1947 values of the Federal Reserve index of industrial production. (If employment were used as the correction standard instead, a less drastic reduction factor than $\frac{170}{187}$ would result; however, we use $\frac{170}{187}$ so as to be sure not to underestimate the effect of the discrepancy we are analyzing.) Also,

$$Y'_{1946} = \frac{X'_{1946}}{X_{1946}} Y_{1946}$$

$$C'_{1946} = \frac{X'_{1946}}{X_{1946}} C_{1946}$$
\[
\begin{bmatrix} \left( \frac{M}{p} \right)_{t} \end{bmatrix} = \begin{bmatrix} X'_{1946} \\ \vdots \end{bmatrix} \begin{bmatrix} \left( \frac{M}{p} \right)_{t-1} \end{bmatrix}, \quad t = 1946, 1947
\]

The purpose of these changes is to gear output \(X\) for 1946 to the Federal Reserve index (while accepting \(X\) for 1947), and then to make the same percentage change in the 1946 values of \(C\), \(Y\), and \(\left( \frac{M}{p} \right)_{t-1}\) as was made in the 1946 value of \(X\). The ‘corrected’ values are shown in Table C1.

Table C1

**Corrected Time Series**

<table>
<thead>
<tr>
<th></th>
<th>(X)</th>
<th>(C)</th>
<th>(Y)</th>
<th>(\left( \frac{M}{p} \right)_{t-1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1946</td>
<td>79.367</td>
<td>72.023</td>
<td>74.864</td>
<td>91.695</td>
</tr>
<tr>
<td>1947</td>
<td>....</td>
<td>....</td>
<td>....</td>
<td>87.462</td>
</tr>
</tbody>
</table>

Then the production functions (3.2) and (3.4), the consumption function (6.2), and the reduced form equations for \(C\) and \(Y\) are re-estimated by the ordinary least squares method, incorporating the above changes into the time series. The results of the structural re-estimation are shown as the \(CLS'\) estimates in Table C2, which reproduces the relevant \(CLS\) estimates from Table 2 for convenience in comparison.

Table C2

**Re-estimation of Certain Structural Equations**

<table>
<thead>
<tr>
<th>Eq.</th>
<th>(X)</th>
<th>(N)</th>
<th>(K)</th>
<th>(t)</th>
<th>(S)</th>
<th>Obs. Value 1948</th>
<th>Calc. Value 1948</th>
<th>Calc. Dist. 1948</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
</tr>
<tr>
<td>3.2</td>
<td>Production</td>
<td>(X)</td>
<td>(N)</td>
<td>(K)</td>
<td>(t)</td>
<td>(S)</td>
<td>Obs. Value 1948</td>
<td>Calc. Value 1948</td>
</tr>
<tr>
<td>(CLS)</td>
<td>1.25</td>
<td>-.296</td>
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<td>(N)</td>
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<td>(Y)</td>
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<td>(S)</td>
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The results of the re-estimation of the reduced form equations are shown opposite the primed variables in Table C3, which reproduces certain parts of Table 3 for convenience in comparison.
### Table C3

**Re-estimation of Certain Reduced Form Equations**

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From Table C2 it appears that the differences between the estimates obtained in this paper and the estimates that would be obtained if the time series discrepancies were corrected are not likely to be negligible, and that some of the structural equations would be likely to fit better in 1948 as a result of the corrections. From Table C3 it appears that the correction process would be likely to produce non-negligible changes in the predictions made by the reduced form. But Table C3 does not indicate that better predictions of the important variables C and Y would be obtained if the time series discrepancies were corrected.

The variables whose time series are most likely to be changed by a revision of the data, in a way important enough to influence the results, are C, I, q, ΔH, G, Y, T, p, and X. Those likely to be affected in an unimportant way (because they are small or stable) are D, q, D, D', D", r, K, and H. Those not likely to be affected at all (because they are independent of price indexes) are (pX, ε), W, W', R, R, ΔF, v, N, i, ε, ε, M, N, N, and w. Accordingly, equations 1.0, 2.0, 3.4, 4.2, and 6.2 are likely to be affected in a significant way because they are dominated by variables from the first of the aforementioned groups. Similarly, equations 4.0, 5.1, 7.0, 9.0, 10.0, and 11.0 are not likely to be affected significantly because they are not dominated by variables from the first group.

Naive model I as applied to 1948 is unaffected by the changes made here because it does not reach as far into the past as 1946. Naive model II is affected, however, and will be led by the changes to make uniformly higher predictions of deflated quantities (usually an improvement in performance) and a lower prediction of the general price index (also an improvement).

The upshot of the calculations based on these approximate 'corrections' is something like this: if the data were revised and the equations re-estimated, the estimates of the parameters would be changed, and the 1948 fit of some structural equations would probably be improved, but there is no evidence that the predictions of important variables by the reduced form would be improved.
Appendix D

CHOICE OF PREDETERMINED VARIABLES FOR ESTIMATION BY THE ABBREVIATED VARIANT OF THE LIMITED INFORMATION METHOD

The limited information estimates of any structural equation depend upon observations of a subset of the predetermined variables that are not in the equation being estimated but are in the system. The elements of this subset are called $z'''s$ and there must be at least as many as $H - 1$ of them if $H$ is the number of jointly dependent variables in the equation being estimated (see text, Sec. 1). Of course, there may be more than $H - 1$; if so, the estimates will be better. In our model the largest value of $H - 1$ for any equation is 4, for equation 3.0; if this is excepted, the largest value is 2, for each of several equations. Therefore the number of $z'''s$ required for any equation is 2 except in the case of equation 3.0, which requires 4.

Now there are 25 predetermined variables in the complete model, and no equation contains more than 4. Thus, for each equation there are at least 21 variables available for use as $z'''s$, and so there is an arbitrary choice of $z'''s$ to be made for each equation. If there were no costs in money and in degrees of freedom, one would always use all the available variables as $z'''s$. Because of these costs, a proper subset of the available variables has been used in each case, i.e., the abbreviated variant of the limited information method has been used.

The stochastic equations have been divided into four groups in such a way as to minimize the intersection of the set of jointly dependent variables in any group with the corresponding set for any other group; in fact every such intersection is empty. Then for any equation the set of $z'''s$ is the set of all predetermined variables in the group to which the equation belongs, minus the set of predetermined variables appearing in the equation (see the accompanying table).

<table>
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<th>Predetermined</th>
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<td>$pX - \xi$</td>
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<tr>
<td>II</td>
<td>(6.2), (7.0), (9.0)</td>
<td>$C$, $Y$, $D_t$, $r_t$, $q_t$</td>
<td>$(M/p)\Delta F$, $v$, $1/r_s$, $r_s$, $(q_t)$, $(q_t)\Delta F$</td>
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<tr>
<td>III</td>
<td>(10.0)</td>
<td>$(Y + X_t + Y_{-t})$, $\Delta r$</td>
<td>$\Delta F_{-t}$, $t$, $\Delta s$, $t$</td>
</tr>
<tr>
<td>IV</td>
<td>(11.0)</td>
<td>$\Delta t$</td>
<td>$t$, $\Delta s$, $\Delta t$</td>
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</table>

Klein's grouping of equations was quite similar. In particular for group I he used exactly the same predetermined variables as I did, except that in
place of $w_{-1}$ and $(N_{L} - N)_{-1}$, he used $H_{-2}$ and $X_{-1}$. This is mentioned here because of its possible bearing on certain anomalies in the CLT estimates of equations 1.0 and 4.0 of group I. The matter is discussed in the text in Section 11, part (f).

Appendix E

**ESTIMATION OF THE PARAMETERS OF THE REDUCED FORM**

This appendix is a note on the restricted least-squares method of estimating reduced form parameters, referred to in Section 4. We first describe the method assuming that a one-element subset of structural equations is chosen to provide the restrictions.

Suppose there is a model consisting of $G$ equations in $G$ jointly dependent variables $y$ and $K$ predetermined variables $z$. Suppose that one of its equations is

$$
\beta_1 y_1 + \ldots + \beta_H y_H + 0 + \ldots + 0 + \gamma_1 z_1 + \ldots + \gamma_{K^*} z_{K^*} + 0 + \ldots + 0 = u
$$

where $H < G$ and $K^* < K$. Consider $H$ equations of the reduced form,

$$
y_i = \sum_{k=1}^{K^*} \pi_{ik} z_k + \sum_{k=K^*+1}^{K^{**}} \pi_{ik} z_k + v_i \quad i = 1, \ldots, H
$$

where $K^{**}$ is the number of predetermined variables assumed to be known to be in the model but not in 1. Then $K^{**} = K - K'$. The parameters $\pi_{ik}$ can be estimated by least-squares. The least-squares estimates can be made more efficient by altering them to take account of the restrictions implied by the zeros in 1, as follows. It must be possible to get equation 1 from a linear combination of equations 2, in fact, from that combination obtained by taking $\beta_i$ times the $i$th equation of 2, $i = 1, \ldots, H$, and summing the results. This means that there are $K^{**}$ equations, one for each $z_k$ excluded from 1, thus:

$$
\sum_{i=1}^{H} \beta_i \pi_{ik} = 0 \quad k = K' + 1, \ldots, K^* + K^{**}
$$

Now if $K^{**} > H - 1$, i.e., if 1 is overidentified, 3 is overdetermined. Hence if 3 is to hold, and it must, a restriction is implied on the matrix of the $\pi_{ik}, i = 1, \ldots, H, k = K' + 1, \ldots, K^* + K^{**}$, keeping its rank down to $H - 1$. This restriction may be applied to the matrix of least-squares estimates of the $\pi_{ik}$, to make them conform to the restrictions implied by the zeros in 1. The computation is not difficult, once the limited information estimates for 1 are obtained.

Similarly, if there are other structural equations besides 1 which also contain some of the jointly dependent variables $y_1, \ldots, y_H$, say $y_1$, the estimates of the parameters of the reduced-form equation for $y_1$ can be
made to conform simultaneously to the restrictions implied in the form of two, three, \ldots, or all these other structural equations as well. This further increases the efficiency of the estimates, but makes them more difficult to compute.

Appendix F

**Calculated Disturbances for CLI Limited Information Estimates of Equations**

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Appendix G

**References**


