The Rationale of Money Demand and of Money Illusion,¹ JACOB MARSCAK, Cowles Commission for Research in Economics and The University of Chicago.

Consider the following propositions: (1) Each individual maximizes a utility function that depends only on the amounts of various goods he will consume during a period that begins after the marketing date and during which no further exchanges take place; (2) The numéraire (i.e., the thing whose price is fixed at unity) is neither a consumption nor a production good; (3) At least one individual has positive money stock. Conditions (1) defines a “static” or “uniperiod” model; (2) defines paper money; (3) excludes the regimes of “money-of-account.”

Jointly, these three conditions imply: (4) Prices of consumption goods that would clear the (perfect) market are infinite. That is, market equilibrium is not compatible with (1), (2), (3) taken jointly. This was proved mathematically in Patinkin’s first article and verbally by Phipps. It is difficult to understand why either of them could think they disagree.

Now replace (3) by: (3′) All money stocks are zero. (1), (2), (3′) jointly imply: (4′) Prices (also called absolute prices) of consumption

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goods are indeterminate; their ratios (so-called relative prices) are determine and finite. Leontief’s paper consists, in effect, in describing such a system: the regime of money-of-account.

Since in the system (1), (2), (3'), (4') money stocks are zero, this system cannot be supplemented by the so-called equation of exchange making positive absolute prices proportional to the total money stock. This contradiction is present in what Hickman described as the “classical” combination of a theory of formation of relative prices (determined in the “commodity-sector” where utilities are maximized and markets cleared) with the equation of exchange (or “monetary sector”), in a regime of paper money.

To reconcile finite and determinate absolute prices of consumers’ goods with positive stocks of paper money, one can drop the “static” condition (1). Replace it by: (1') Each individual maximizes a utility function which depends not only on present but also on future consumption flows; he will be able to exchange stocks of goods and of money at future marketing dates. If some but not all individuals expect the prices of all goods to fall, these individuals will demand positive money stocks; at the same time the prices of goods will be determinate, positive, and finite.

This implies sharp alternations between an individual’s bearish “flight into money” and bullish “flight into (certain) goods.” More realistic, smooth fluctuations of stocks are obtained if we introduce market imperfection, including transaction costs. If barter is excluded, one can define “illiquidity” of a good as the slope of the marginal money revenue and marginal money outlay curve of the individual considered, respectively, as a seller and a buyer of this good. The individual demand for stocks of various goods and of money, given expected shifts in that curve, will depend on their degrees of illiquidity; for speculative purchase to be followed by resale, a relatively “liquid” good (and money in particular) is preferred.

It is neither necessary nor sufficient to introduce uncertainty in order to explain positive stocks of money and the smooth fluctuation of these stocks.

Any system that admits positive demand for paper money stocks has also to admit that a change in the absolute prices of goods (the cheapening or appreciation of paper money in terms of goods) induces the individual to change his holdings of money relative to his holdings and consumption of goods. Hence, there is “money illusion” in all systems except that of money-of-account.
THE RATIONALE OF THE DEMAND FOR MONEY
AND OF "MONEY ILLUSION" *

by Jacob Marschak, Chicago

§ 1. The problem outlined. — § 2. A static model of pure exchange economy. —

§ 1. — THE PROBLEM OUTLINED

1. A. Unlike precious metals, paper money is neither a consumption nor a production good. The classical theory of market equilibrium was developed at a time when metallic money prevailed. It fails to relate the desired quantity of paper money and the exchange ratios between paper money and various commodities (the "absolute" prices of commodities) to assumptions of rational (utility-maximizing) behavior. As a makeshift, an "equation of exchange" was used by later authors (1). It appealed to the alleged existence of an "institutional" constant (velocity of circulation) superimposed upon the assumption of rational behavior of individuals. More recently, this makeshift device was generalized, and slightly rationalized, into Keynes' "liquidity preference" equation (2). Like the rest of macroeconomics, this equation is still in need of being related to assumptions of rational behavior. Should the "liquidity preference equation" and other Keynesian equations have purely empirical claims, these would be hard to establish: the observed time series of the relevant variables (quantity of money, interest rate, price level, consumption, national income and, possibly, its distribution) are no doubt consistent with a large number of equation-systems other than the Keynesian one (3).

1. B. Patinkin [4] has proved — in a manner to be somewhat modified in our Section 2 below — the inconsistency of traditional microeconomics with positive quantities of paper money. After

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(1) E. g., Divisia [1].

(2) See Hicks [1].

(3) For an attempt to derive a macroeconomic system from the assumption that individuals maximize their utilities or profits see Klein [1], Technical Appendix pp. 192-199.
having contributed this important criticism, he has not attempted to give a better microeconomics improved upon Walras and Pareto.

Instead, Patinkin [2] followed (1) the more superficial approach of Cassel whose aggregate demand and supply equations emerge from nowhere. Had these equations been derived from the maximization of each individual’s utility subject to the limitation of his and the society’s resources these limitations would result in certain restrictions upon the macroeconomic system. Unless one derives these restrictions precisely, the consistency and determinacy of the system cannot be discussed.

1. G. Questions like the following arise in a paper money system:

Why can a unit of a thing that cannot be used by anyone either as a consumption or as a production good, acquire for the individual more than a zero amount of any useful thing? Yet, does not exactly this happen when goods are bought for paper money at prices that are less than infinite? Why is the desired stock of such money positive and not zero? In short: how can positive stocks of paper money be consistent with finite prices of physical goods? What explains the demand for paper money?

1. D. A related question is that of a money illusion. If the prices of all consumption goods (and of labor, the negative of the consumption good leisure) are multiplied by a constant, should this affect the demands for various goods on the part of a rational, i.e., a utility-maximizing consumer? Do his decisions depend on the prices of consumption goods or on the ratios of these prices? It is often stated that the latter must be the case; to respond to an equiproportional change in the prices of all consumption goods (including leisure) is irrational, a money illusion. Note that the question itself is meaningful only in a regime of paper money. If money were a consumption good (gold, cattle) it would be impossible to change the prices of all physical goods in the same proportion since the price of money must, by definition, remain equal to one.

1. E. The following question is usually associated with, but is not identical with, that of a money illusion. Under a regime of paper money, does the market consisting of utility-maximizing individuals determine the absolute prices of physical goods, or only their ratios? In the latter case, the observed fact that absolute prices have determinate values would have to be explained by some relationship outside of the market — e.g., by the government’s fixing the price of one of the physical goods. Suppose a certain theory seemed internally consistent under the assumption that absolute prices were determined in the market. If this assumption is disproved, the theory reveals itself as internally inconsistent; because if, before the assumption was disproved, the theory was internally consistent and therefore...
contained as many independent equations as it had variables, now the equations will outnumber the variables by one.

1. F. If there is no money illusion, i.e., if price-ratios and not absolute prices govern the single individual, there is at least an intuitive presumption (subject to a mathematical confirmation or rejection) that the market determines price-ratios rather than absolute prices. Accordingly, some theories, to remain consistent, have to admit a money illusion (possibly as a human habit rather than a utility maximizing behavior) on the part of at least one group of decision makers — be it the workers or their unions (as explicitly assumed by Keynes (1), or be it the consumers (2)), the entrepreneurs, the banks, or the fiscal authorities.

1. G. In what follows, Section 2 discusses the inconsistency of paper money with positive money stocks in a static model. Sections 3-4 are devoted to the microeconomic determination of the demand for paper money, and of the ratios of exchange of paper money against goods (so-called absolute prices). The essential steps beyond Walras-Pareto would be the following:

(a) One has to remove the assumption that the individual is concerned with only one marketing date. It will be shown that in the resulting more general, a multi-period or dynamic model, stocks appear in an essential way (i.e., they are not necessarily proportional to flows);

(b) One has to introduce market imperfections, transaction costs, and the property of money as the legal tender;

(c) One has to introduce uncertainty.

1. H. Of these three steps, step (a), the formulation of a simple dynamic model is made in Section 3. This step is sufficient to explain the occurrence of positive stocks of money, but is not sufficient to explain why the money stock of an individual takes values other than either zero or the money equivalent of all the resources. Step (b), the introduction of market imperfection and the legal tender is made in Section 4. This step is sufficient for an explanation of continuous money stocks, even (contrary to the prevailing opinion) without introducing uncertainty. Accordingly, step (c), introduction of probabilities would merely add an (important) touch of reality.

1. I. A further step towards reality would be to introduce lending (and borrowing) or, more generally, claims (and liabilities) of all kinds. There are analogies, from the point of view of a single individual, between claims and physical assets. Therefore it would be convenient to introduce the more familiar phenomenon of production before treating claims.

1. J. Further, one would have to remove the assumption that the market is cleared instantaneously, and to develop the rationale

(1) The importance of this assumption was first pointed out by Leontief [1].
(2) For an empirical attempt see Marschak [2].
of the actions of individuals with uncertain information that tend, in the course of a time-consuming process, to diminish the difference between demand and supply.

1. K. These steps would develop a rational theory of the market, itself a necessary basis for a rational theory of the relation between economic aggregates, and of their movement through time. The currently popular set of aggregative equations, such as the consumption function, investment function, equation of exchange (or of liquidity preference), etc., does seem rather arbitrary, and lacking a theoretical foundation (6).

In the present paper, however, we confine ourselves to steps (a) and (b) listed in 1. G. above.

§ 2. — A Static Model of Pure Exchange Economy

2. A. We shall call a « good » or « commodity » anything that can be demanded in nonzero quantities. Let there be \( A \) individuals and \( N \) goods. The \( a \)-th individual \( (a = 1, ..., A) \) enters the market with the initial quantity \( x_{na} \) of the \( n \)-th good \( (n = 1, ..., N) \). He leaves it with a desired quantity \( \bar{x}_{na} \). The difference \( x_{na} - \bar{x}_{na} \) is his demand for this good; this expression can also be called his « negative supply », or the « excess of his demand over his supply », or his « net demand » for the \( n \)-th good. The sum \( \sum_{a=1}^{A} (x_{na} - \bar{x}_{na}) \) is best called « aggregate net demand ».

2. B. We shall call a « consumption good » a good that enters the utility function of at least one individual. Dollar bills, factories, and bonds are goods but not consumption goods; (but factories and bonds will not be considered in a pure exchange economy).

2. C. A « flow » is the rate of use of a « stock », per unit of time. We shall formulate a static model in terms of « flows ». It will be presently seen (in 2. E.) that its reformulation in terms of « stocks » does not add anything essential. The same will not be true of a dynamic model (Sections 3 and 4).

2. D. We shall first assume all goods to be consumers goods, and let the \( a \)-th individual maximize his utility function of the \( N \) flows,

\[
\max \left( x_{1a}, ..., x_{Na} \right),
\]

subject to his « budget restriction »

\[
\sum_{n=1}^{N} (\bar{x}_{na} - x_{na}) p_n = 0, \quad a = 1, ..., A.
\]

(6) There is truth in G. Stigler's [1], remark that « Most discussions of dynamic economies... would better be entitled 'What I know about Differential and Difference Equations... The economics... is appallingly absent ».
Using a Lagrange multiplier $\lambda$, we differentiate with respect to each $x_{na}$ the expression
\[ u^* (x_{la}, ..., x_{Na}) + \lambda \sum_{n=1}^{N} (\bar{x}_{na} - x_{na}) \bar{p}_n, \]
and equate each partial derivative to zero. We obtain, denoting by $u_{na}$ ("marginal utility") the partial derivative of $u^*$ with respect to its $n$-th argument,
\[ u_{na}/u_{Na} = \bar{p}_a/\bar{p}_N ; n = 1, ..., N - 1 ; a = 1, ..., A. \]
These $N - 1$ equations together with (2:2) determine, for any given set of the $N - 1$ price ratios $\bar{p}_a/\bar{p}_N, ..., \bar{p}_{N-1}/\bar{p}_N$, the $N$ desired flows $x_{la}, ..., x_{Na}$. Thus the individual’s demand $(x_{na} - \bar{x}_{na})$ for the $n$-th good depends on the $N - 1$ price ratios. Bargaining goes on (i.e., prices and exchanged quantities are subjected to trial-and-error adjustments, not further analyzed in this theory) till the market is cleared, i.e., till the aggregate demand for each commodity balances its aggregate supply; its aggregate net demand vanishes:
\[ \sum_{a=1}^{A} (\bar{x}_{na} - x_{na}) = 0, n = 1, ..., N. \]
The system (2:2-4) contains $N - 1 + NA$ independent equations since equations (2:2) together with the first $N - 1$ equations in (2:4) yield the $N$-th equation in (2:4). This number of independent equations equals the number of the following variables: $N - 1$ price ratios plus $NA$ demands. If, now, the $N$-th consumption good is used as a numéraire, an extra equation is added:
\[ P_N = 1, \]
permitting to determine the $N$ absolute prices instead of the $N - 1$ price ratios.

2. E. The well-known model just given (1) has certain tacitly implied properties of great importance. Marketing takes place on a single date. The individual’s satisfaction is not affected by flows other than those determined on that date; and no distinction is made by the individual between the different post-market dates at which consumption may take place. Suppose such an individual plans for a period of $T$ time units (a "horizon", possibly his whole life). Suppose he enters the market with $\bar{x}_{na}$ initial flow units of a certain kind (future daily housing or heating services of a house or of a pile of coal, future daily labor services or annual crops) and leaves it with $x_{na}$ desired flow units. We can then also say, under the assumption of the static model, that he enters the market with $\bar{y}_{na}$ stock units (rooms, coal calories, expected man-years of labor or bushels of grain) and leaves the market with $y_{na}$ stock units, where

(1) For example, Pareto [1], Appendix, 63; Wicksell [1], pp. 66-67.
Because of this proportionality, a static model can be said to admit stocks « in a non-essential way only ».

2. F. Suppose the N-th good is paper money. Equations (2.2), (2.4), (2.5) still hold. But the utility function (2.1) is replaced by

\[ u^a(x_{1a}, ..., x_{N-1a}) \]

hence

\[ u_{Na} = 0. \]

Return for a moment to a system with non-paper money. Assume that there is no universal satiety \( \kappa \); that is, \( u_{na} \neq 0 \) for at least one individual \( a \) and one consumption good \( n \neq N \). Then, by (2.3), (2.5), its absolute price,

\[ p_a = \frac{u_{na}}{u_{Na}}, n = 1, ..., N, \]

approaches infinity as \( u_{Na} \) approaches zero. It follows that in a static model with paper money, prices of consumption goods are bid up indefinitely. No equilibrium can be reached.

2. G. A more complete solution is arrived at (and was in essence, obtained by Patinkin [1]) if one introduces new restrictions. Returning first again to the system in which the numéraire is a consumption good, observe that in a static model no debts, or servicing of debts, can exist. Hence all stocks and flows are non-negative:

\[ x_{na} \geq 0, n = 1, ..., N. \]

To maximize (2.1) subject to (2.2) and (2.10') rewrite the latter restriction thus:

\[ x_{na} - (r_{na})^2 = 0, n = 1, ..., N, \]

where the \( r \) is known to be real. The \( a \)-th individual maximizes with respect to \( x_{na}, r_{na} (n = 1, ..., N) \) the expression

\[ v + \lambda \sum_n (x_{na} - x_n) p_a + \sum_n u_{na} (x_{na} - (r_{na})^2), \]

where the prices \( p \) are given, and the \( \lambda, u_n (n = 1, ..., N) \) are Lagrange multipliers. We obtain

\[ u_{na} + u_{na} = \lambda p_a, \]

where

\[ \lambda p_a = u_{na} + 0 \sqrt{x_{na}}, n = 1, ..., N. \]

Hence, for any two goods — say \( n, N —

\[ \frac{p_n}{p_N} = \frac{u_{na} + 0 \sqrt{x_{na}}}{u_{Na} + 0 \sqrt{x_{Na}}}. \]
This is a more general equation than the usual marginal utility theorem (2.3). It reduces to that theorem if the optimal flows $x_{na}$, $x_{Na}$ of the two goods are known to be non-zero.

2. H. If $N$ is paper money, and the optimal flow of $n$ is non-zero, then $m_{Na} = 0$, $\rho^N = 1$, $x_{na} \neq 0$; therefore (2.14) becomes

$$\rho_n = \frac{m_{na}}{(o_i)\sqrt{x_{Na}}}.$$  

(2.15)

If $x_{Na} \neq 0$ we again obtain — as in (2.9) — an infinite price for any consumption good $n$ [unless the flow of that consumption good is zero; see (2.14)]. But, in addition, we have to consider the possibility that $x_{Na} = 0$, i.e., that the optimal flow [and hence — by (2.6) — the optimal stock] of money is zero for all individuals; then, by (2.4)

$$\sum_{a=1}^A \bar{x}_{Na} = 0,$$

(2.16)

and since $\bar{x}_{Na} \geq 0$ for every $a$, (2.16) implies that $\bar{x}_{Na} = 0$ for every $a$. Hence, paper money occurs not physically but only as a *money of account*. The denominator on the right side of (2.15), and therefore the whole expression, becomes indeterminate, permitting finite though indeterminate prices for consumption goods. Under this regime, every consumption good with a positive optimal flow can be assigned in the equilibrium any finite absolute price compatible with the requirement that the ratio between the prices of any two such consumption goods — say $\rho_n/\rho_m$ — be equal to the ratio of their marginal utilities: see (2.14), replacing $N$ by $m$, with $x_n > 0$, $x_m > 0$. But if *money of account* is replaced by paper money with positive physical stocks, finite equilibrium prices of consumption goods with positive flows become impossible (1).

2. I. This agrees with common sense. Individuals endowed with physical stocks of useless money will try to dispose of it: its price relative to that of useful things will tend to zero (2); or, in other words, the price of useful things in terms of money will tend to infinity. If, on the other hand, no physical stocks of money exist, no such bidding of money against consumption goods takes place. Their prices may well be finite; but there is no reason for one rather than another set of (finite) prices to establish themselves in the market, provided that the prices of different consumption goods are proportional to their marginal utilities for all individuals. If the prices 2, 3, 5, ... (say) satisfy this latter condition, the prices 4, 6, 10, ... or 6, 9, 15, ... will also satisfy it, and any of these price-sets will be an equilibrium price-set under the regime of *money of account*.

2. J. Economic literature contains occasionally a distinction

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(1) I believe the model discussed by Lautiak [2] in his criticism of Patinkin is, in fact, an *money-of-account* model.

(2) This common sense is well stated in a paper by Phipps [1] who disagrees with Patinkin on the form but not, I think, on the substance.
between the "commodity-sector" of the economy — the system of equations (2.2)-(2.4) derived from utility-maximization under limited resources — and the "monetary sector" which is not derived from utility-maximization and consists of an "equation of exchange". It is then stated for a paper-money economy that while the commodity-sector determines only the relative prices of consumption goods, their absolute prices are fixed by the "equation of exchange", i.e., a relation between a certain average of prices and the stock of money (1).

However, such an equation is meaningless in a static model of a paper-money economy since, as we have seen, such a model admits finite commodity prices only when the money stock is zero. In the subsequent sections we shall see that, in a non-static model, each individual's optimal stock of money (and of other goods) is indeed related to prices. This relation may give the rationale of any "equation of exchange". The latter would follow from, and will not be superimposed upon, the equations derived from the maximization of utility under given constraints.

2. K. The static model is the usual model of the general equilibrium theory. We can now answer, with respect to this model, the questions asked in 1. C. The coexistence of paper money and determine and finite prices of consumption goods is not consistent with this model. This answer rules out the question whether in such a model the equilibrium demand for a consumption good is or is not homogenous of zero degree in absolute prices: the model itself excludes equilibrium. (The question is similar to this one: If a rectangle has five angles, is their sum 360° or 450°? )

2. L. However, the question of "money illusion" is not meaningless in a regime of "money-of-account". This regime can be actually described by the equations of Section 2. D., in N consumption goods if we replace (2.5) by

\[ p_N = c, \]

where \( c \) is a constant arbitrarily fixed by some authority. \( c \) is a certain number of units of money-of-account. Money-of-account is not itself a "good" as defined in 2. A. However, if one insists, we can number money-of-account as the \( N + 1 \)-th "good" but then enter it in none but the following two equations, that will replace (2.17):

\[ p_N/p_{N+1} = c ; \quad p_{N+1} = 1. \]

We can now make, with reference to a "money-of-account" regime, two distinct statements, each of which is sometimes called the "absence of money illusion" (they answer, respectively, the questions put in 1. D. and 1. E. above).

(1) For a succinct statement of this point of view see Hickman [1].
i. By (2.1)-(2.3), the optimal flows chosen by an individual depend on his utility function, initial flows, and the ratios between consumption goods' prices; they do not depend on c;

ii. By (2.1)-(2.4), the utility functions and initial flows of all individuals determine the price-ratios but leave the prices undetermined.

Later, in 3. T.-3. U., similar questions about «money illusion» will again arise meaningfully; and, again, in 4. N.-4. O. But the answers will be different.

§ 3. - A Simple Dynamic Model (1)

3. A. We shall now introduce the expectation of changing — viz., falling — prices as a simple but far from satisfactory explanation for the desire to hold paper money. It will be seen that such a model has important unrealistic features, not due to the paper money's lack of immediate utility but to other properties of the model. Modifications will be introduced in Section 4 that will make the model more realistic.

3. B. Suppose the individual plans for a finite sequence of time intervals devoted alternately to marketing and to consumption. The marketing intervals are negligible in length (2). The length of each consumption interval is used as time unit. As in 2. E., denote flows by x and stocks retained by y. Stocks sold will be denoted by z. Considering now the plan of the a-th individual regarding n-th consumption good (possibly leisure), we can omit the subscripts a, n for brevity; we shall indicate time by a superscript. The individual plans at time o for a succession of marketing dates (interspersed with consumption periods) ending at time T, the «horizon». See (3:1).

y is the «initial stock», a non-negative quantity possessed by the individual before the marketing date o. On this date, exchanges will be made. They will result in the individual's retaining a non-negative optimal stock y^o which may be larger or smaller than y. Thus the amount sold, z = y – y^o may be negative, zero, or positive. A non-negative optimal quantity x^o, not larger than the retained stock y^o, will be consumed during the interval (0,1). Hence, a non-negative stock y^o x^o will be brought to market at date 1. And so on.

(1) The author hesitates between calling thus a «dynamic» or a «multi-period» model. In the best known dynamic economic models, expectations are not given (as they are in our model) but are themselves functions of current observations made by the individual — see, for example, Tinbergen [1], pp. 103-104; Hurwicz [2], p. 120. However, in recent writings on planning — e.g., by Dantzig [3], p. 205 — a «dynamic» plan contains all expectation constants as given at the time of planning, like in the present model (see the price column of our (3:1). The term «multi-period» model would not quite meet the need, as our model may contain just one period of consumption (but two marketing dates).

(2) That is, we do not study here the time-consuming adjustment in the market: see 1. J. above.
For at least one good, there are assumed to exist at least two dates at which its prices are different.

The initial stock \( y_n \) (we re-introduce the subscripts now) is given to the individual. He tries to retain, sell (or buy) and consume, respectively the following optimal quantities of \( n = 1, ..., N \):

\[
\begin{align*}
3.2 & \quad \begin{cases}
    \text{stocks retained} : & y_{na}^0, y_{na}^1, ..., y_{na}^{T-1}, y_{na}^T \\
    \text{stocks sold} : & z_{na}^0, z_{na}^1, ..., z_{na}^{T-1}, z_{na}^T \\
\end{cases} \\
3.3 & \quad \text{consumption flows} : x_{na}^0, x_{na}^1, ..., x_{na}^{T-1}.
\end{align*}
\]

3. C. The individual's satisfaction depends on the consumption flows (3:3) in each of the \( T \) considered periods; but it also depends on the final stocks \( y_{na}^T \). To be sure, he is indifferent to the particular way in which he or his heirs will arrange their consumption in the future following the instant \( T \). But he is not indifferent as to the resources evaluated at the prices of instant \( T \) — with which he or his heir will start that future. The second and third column headings of the Plan (3:1) have to be interpreted, with respect to the last line, as follows: `Stocks Brought (Retained)` means `Stocks Brought (Retained)` if Trading would take Place \( \nu \). Accordingly, we include the quantities \( y_{na}^T \) in the argument of the utility function of the \( a \)-th individual, which we write thus:

\[
\begin{align*}
3.4 & \quad u^a(x_{1a}^0, ..., x_{Na}^0, ..., x_{1a}^{T-1}, ..., x_{Na}^{T-1}, y_{1a}, ..., y_{Na}).
\end{align*}
\]

We shall write for every \( n \): \( \frac{\partial u}{\partial x_{na}^t} = u_{na}^t (t = 0, ..., T-1); \frac{\partial u^a}{\partial y_{na}^T} = u_{na}^T \).

3. D. The Plan (3:1) must satisfy the following conditions — analogous to (2:2) — each stating that stocks brought to the market are exchanged, at certain prices, for stocks with which the next consumption period is started: for every \( a \),
\[ (3:5) \quad \sum_{n=0}^{N} z_{na}^t \rho_n^t = 0, \quad t = 0, \ldots, T; \]

where for every \( a \) and \( n \), the quantities \( x, y \) and \( z \) are connected by the identities given in the column \( \rho \) stocks sold \( \rho \) of the Plan (3:1)

\[ (3:6) \quad \begin{align*}
  z_{na}^0 &= \bar{y}_{na} - \bar{y}_{na}^0, \\
  z_{na}^t &= \bar{y}_{na}^t - \bar{x}_{na}^{t-1} - \bar{v}_n^t, & t &= 1, \ldots, T.
\end{align*} \]

Note that the prices \( \rho_n^t \) settled at instant \( 0 \) are the same for all individuals; but the expected prices \( \rho_n^0, \quad t > 0, \) will, in general, vary from one man to another. The expected prices need not clear the market; but the prices of instant \( 0 \) must be such as to have the market cleared [analogous to (2.4)]:

\[ (3:6') \quad \sum_{a=1}^{A} z_{na}^0 = 0, \quad n = 1, \ldots, N. \]

3. E. We rule out negative consumption flows, by entering not labor but leisure. For the present model we also rule out liabilities, i.e., negative stocks (see I.I. above). The stocks and flows obey therefore the following inequalities, already stated in our comments following the Plan (3.1):

\[ (3:7') \quad y_{na}^t - x_{na}^t \geq 0; \quad x_{na}^t \geq 0; \quad y_{na}^T \geq 0; \quad n = 1, \ldots, N; \quad t = 0, \ldots, T - 1. \]

They can be rewritten thus:

\[ (3:7) \quad \begin{align*}
  x_{na}^t - (s_{na}^t)^r &= 0, \quad t = 0, \ldots, T - 1, \\
  y_{na}^t - x_{na}^t - (s_{na}^t)^s &= 0, \quad n = 1, \ldots, N, \\
  y_{na}^T - (s_{na}^T)^s &= 0.
\end{align*} \]

where the \( r, s \), are known to be real.

3. F. Each of the optimal stocks and flows listed in (3:2), (3:3) is determined by the \( a \)-th individual as a function of the following given: his \( N \) initial stocks \( y_{na} \), the prices \( \rho_n^0 \) that clear the market at date \( 0 \), and the expected prices \( \rho_n^t, \quad t = 1, \ldots, T \). For each set of these given, optimal stocks and flows are obtained by maximizing (3:4) subject to the conditions (3:5), (3:6), (3:7), (3:8), (3:9). Omit the index \( a \) (indicating the individual) for brevity and maximize with respect to \( x_{n}^t, y_{n}^t, z_{n}^t, x_{n}^{T}, y_{n}^{T}, z_{n}^{T}, \Sigma_{n}^t (l = 0, \ldots, T - 1; \quad n = 1, \ldots, N) \), in the order stated, the following expression:

\[ (3:10') \quad u + \sum_{t=0}^{T} \sum_{n} x_{n}^t \rho_n^t + \sum_{n} y_{n}^t \bar{y}_{n} - y_{n}^t - \bar{z}_{n}^0 + \sum_{t=1}^{T} \sum_{n} x_{n}^t \bar{y}_{n}^{t-1} - x_{n}^{t-1} - \bar{y}_{n}^{t-1} - \bar{z}_{n}^t + \sum_{n} \sum_{t=0}^{T-1} \rho_n^t (x_{n}^t - (r_{n}^t)^s) + \sum_{n} \sum_{t=0}^{T} y_{n}^t \bar{y}_{n}^t - x_{n}^t - (s_{n}^t)^s, \]
where the $x, \lambda, \mu, v$, with appropriate sub- and superscripts are Lagrange multipliers, and summation $\Sigma$ is from $i$ through $N$. Equating the partial derivatives to zero, we obtain:

$$0 = u_n^i - x_n^i + \mu_n^i - v_n^i = -x_n^i + x_n^{i+1} + v_n^i =$$
$$= u_n^T - x_n^T + \mu_n^T - v_n^T = \lambda^T p_n^T - x_n^T =$$
$$= v_n^i s_n^i = u_n^T r_n^T - v_n^i s_n^i; \quad t = 0, ..., T - 1.$$

Rearranging,

$$u_n^i + \mu_n^i = \lambda^t p_n^i = \lambda^{i+1} p_n^{i+1} + v_n^i = x_n^i;$$
$$u_n^T + \mu_n^T = \lambda^T p_n^T = x_n^T;$$
$$v_n^i s_n^i = u_n^T r_n^T - v_n^i s_n^i = 0; \quad t = 0, ..., T - 1.$$

With the help of (3.7)-(3.9) we can eliminate the $x, r, s$ and obtain, for every $n \ (1, ..., N)$,

$$\begin{cases} 
\mu_n^i + \mu_n^T = \lambda^t p_n^i = \lambda^{i+1} p_n^{i+1} + v_n^i, \quad t = 0, ..., T - 1; \\
\mu_n^i = 0 \quad \forall x_n^i, \quad \forall n; \\
u_n^T + \mu_n^T = \lambda^T p_n^T, \quad \forall T.
\end{cases}$$

Eliminating the $T + 1 \lambda$'s from these $N (4 T + 2)$ equations, we obtain the following equations for the $a$-th individual (we re-introduce the $a$-subscript in order to count, in 3, G, below, the unknowns and equations of the whole market),

(3.11) $\mu_n^a = 0 \quad \forall y_n^a, \quad \forall n = 0 \quad \forall y_n^T, \quad \forall n = 0 \quad \forall y_n^T - x_n^a.$

(3.12) $\mu_n^{i+1} + \mu_n^T = u_n^{i+1}, \quad \forall x_n^{i+1}, \quad \forall n = 1, ..., N; \quad t = 0, ..., T - 1;$$

(3.13) $\mu_n^a = (u_n^a + \mu_n^a)/(u_n^a + \mu_n^a), \quad n = 1, ..., N - 1, t = 0, ..., T.$

Equations (3.11) imply, jointly with (3.7'), for $t = 0, ..., T - 1$:

(3.14) if $x_n^a > 0, \mu_n^a = 0$; therefore if $\mu_n^a \neq 0, x_n^a = 0$;

if $y_n^a > 0, \mu_n^a = 0$; therefore if $\mu_n^a \neq 0, y_n^a = 0$;

if $y_n^a > x_n^a, \mu_n^a = 0$; therefore if $\mu_n^a \neq 0, y_n^a = x_n^a.$

3. G. Analogous to the procedure in 2. D., we consider first the $a$-th individual, with all prices given to him. We have the following:
Equations

(3.5) \[ T + 1 \]

(3.6) \[ (T + 1) N \]

(3.11) \[ N (t + 1) \]

(3.12) \[ N T \]

(3.13) \[ (N - 1) (T + 1) \]

Total: \[ N (4T + 3) \]

Unknowns (where \( t = 0, ..., T - 1 \))

\[ x^t_n : NT \]

\[ y^t_n, y^t_{Na} : N (T + 1) \]

\[ x^t_{na}, z^t_{na} : N (T + 1) \]

\[ v^t_{na}, v^t_{Na} : N (T + 1) \]

\[ v^t_{na} : NT \]

\[ N (4T + 3) \]

Thus \( x^t_n \) is determined for each individual \( a (= 1, ..., A) \) and each good \( n (= 1, ..., N) \) as a function of all prices for all marketing dates. The clearing-of-the-market condition (3.6') for the date 0 provides \( N \) further equations; of these, however, one is redundant because of (3.5). The remaining \( N - 1 \) equations of (3.6') determine the set of \( N - 1 \) price-ratios \( \beta^t_n / \beta^t_{Na} \) \((n = 1, ..., N - 1) \) for the date 0; this set is the same for all individuals. As already stated in 3. D, the subsequent (expected) price ratios \( \beta^t_{na} / \beta^t_{Na}, t > 0 \), are not necessarily common to all individuals and need not clear the market. These expected price-ratios, the initial stocks \( \bar{x}_{na} \), and the utility functions \( u^a \) constitute the initial conditions (the given) for the market as a whole.

3. H. Equations (3.11)-(3.13) repeat, for each period, the generalized marginal utility proposition (2.14) of the static model. But, because (3.12) connects each pair of successive dates we can now study, in addition, the effect of expected price changes. We shall assume all prices positive and finite, and show that in a dynamic model such prices are consistent with both maximized utilities and positive stocks of paper money.

3. I. Note that the expression \( y^t_n - x^t_n \) in (3.11) is the stock brought to market at date \( t + 1 \); see (3.1). When \( y^t_n - x^t_n > 0 \), the stock retained at date \( t \) exceeds the consumption needs of the ensuing period \( (t, t + 1) \). When \( y^t_n = x^t_n \) we say that the stock is "unloaded." In this case the individual retains only as much of the \( n \)-th good as is needed for his immediate consumption during the period \( (t, t + 1) \); the possible needs of later periods will be satisfied by later repurchases.

3. J. We have, by (3.12)-(3.13),

\[ \frac{\beta^t_{na}}{\beta^t_{Na}} = \frac{\mu^t_{na} + \nu^t_{na} - y^t_{Na}}{\mu^t_{Na} + \nu^t_{Na} - y^t_{Na}}, \]

\[ n = 1, ..., N - 1; t = 0, ..., T - 1. \]

Suppose that, at date \( t \), neither \( n \) nor \( N \) is "unloaded"; that is, \( y^t_{na} > x^t_{na} \) and \( y^t_{Na} > x^t_{Na} \). Then by (3.13), (3.12), (3.15), \( \beta^t_{na} / \beta^t_{Na} = \) \( \beta^{t+1}_{na} / \beta^{t+1}_{Na} \). Therefore if \( \beta^t_{na} / \beta^t_{Na} \neq \beta^{t+1}_{na} / \beta^{t+1}_{Na} \) then \( y^t_{na} = x^t_{na} \) or (1) \( y^t_{Na} = x^t_{Na} \). That is, if the individual expects the prices of two goods

(1) We use "or" in the sense of "and/or" as distinct from "either-or."
to change in different proportions, then he unloads the stock of at least one of them.

3. **K.** The individual never unloads the stocks of all goods (excepting the trivial case when all sales are zero from the next market date on). For, if not all sales are zero, then, since all prices are positive, (3:5) implies that some sales must be positive and some negative. But this would be contradicted if \( y_{m}^{t-1} = x_{m}^{t-1} (t = 1, ..., T) \) for all \( m \), since then by (3:6), (3:7), \( \gamma_{m}^{t} = -\dot{y}_{m}^{t} \leq 0 \) for all \( n \). Hence the stocks of at least one good are not unloaded.

3. **L.** We shall now show that if no two prices are expected to change in the same proportion from time \( t \) to \( t + 1 \), then the stocks of all goods are unloaded at the time \( t \) in favor of one. For, if for any pair of goods — say \( m, n = 1, ..., N \) — the prices are expected to change in different proportions, then by 3. **J.,** at least one of these two goods is unloaded. Hence, there can be at most one good, say \( g \), such that its stocks are not unloaded. But since we have just seen in 3. **K.,** that there must be at least one such good, it follows that all stocks are unloaded at the time \( t \) in favor of a single good. This good thus "absorbs all resources" of the individual as of date \( t \), apart from those needed for immediate consumption (4); provided all prices are expected to change in different proportions.

3. **M.** This agrees with the common sense of speculation. Suppose that I am sure, at time \( t \), that all prices will change during the next period in different proportions, and that the \( g \)-th good will rise in a larger proportion than all other goods. In this case, my best plan, for the marketing date \( t \), is to reduce my possessings of all these goods down to the amounts required for consumption during the ensuing interval \((t, t + 1)\) by exchanging against stocks of the \( g \)-th good; and to resell, at date \( t + 1 \), as much of this good as will be needed, at the then prevailing prices, to acquire other goods, for consumption and possibly (depending on the prices expected for time \( t + 2 \)) for speculation.

3. **N.** Note, however, that this "common sense" argument does not explain (as do the equations) what determines the amount of consumption.

3. **O.** In general, the set of all goods can be split into two or more subsets, each consisting of goods whose prices are expected to change in the same proportion (a subset may have one or more elements). The stocks in all the subsets but one will be unloaded in favor of the remaining subset. We forego the formal proof of two intuitively obvious propositions: 1) that the favored subset consists of goods whose prices are expected to rise in a higher proportion than the prices of all other goods; and 2) that the individual has no preference as between the possible distributions of stocks (not needed for consumption) between the various goods forming the favored subset.

---

(4) Or, to put it in a different way: this single good absorbs all resources of the individual as of date \( t + 1 \).
3. P. Thus, the dynamic model defined in 3. B. implies that all resources of an individual (apart from those providing for immediate consumption) are invested, at any time, in stocks of the one good (or of the several goods) promising the highest proportionate rise of price. The shift of all resources from one kind of stocks to another depends on changes in relative prices, however slight these changes may be. This instability can be regarded as an idealization of phenomena such as the alternating «flight into commodities» or «flight into money». These terms were coined, I believe, in Germany during the inflation of the 20-ies and the deflation of the 30-ies. These phenomena do not presuppose paper money: «flight into commodities» occurred also in times of «gold inflation».

3. Q. If the N-th good is a numéraire, \( p^t_{Na} = 1 \) for every individual, \( a \), and every market date, \( t = 0, ..., T \). If, in addition, this numéraire is paper money, the partial derivative of utility with respect to money flow in every consumption period, \( \partial u^a_t / \partial x^t_{Na} = u^t_{Na} = 0, t = 0, ..., T - 1 \); also, its partial derivative with respect to the final money stock, \( \partial u^a_T / \partial x^T_{Na} = u^T_{Na} = 0 \); the individual’s satisfaction deriving from the prospect of ending with (or leaving to his heirs) a sum of paper money is entirely due to this paper money being convertible, at the prices of the horizon date \( T \), into consumption goods. Thus (3.13), (3.15) become, respectively, for every consumption good \( n = 1, ..., N - 1 \),

\[
(3.16) \quad p^t_{na} = (u^t_{na} + u^t_{na}) / \mu^t_{Na}, t = 0, ..., T
\]

\[
(3.17) \quad p^{t+1}_{na} = (u^{t+1}_{na} + u^{t+1}_{na} - u^t_{Na}) / (\mu^t_{Na} - \mu^t_{Na}), t = 0, ..., T - 1.
\]

The implications of (3.16) simply extend those of the static model to all flows of paper money and to its final stocks. Since there exists at least one consumption good \( n \) (< \( N \)) such that, for \( t = 0, ..., T - 1, u^t_{na} > 0, x^t_{na} > 0 < p^t_{na} < \infty \), therefore, by (3.14), \( u^t_{na} = 0 \), and \( \mu^t_{Na} = \rho^t_{Na} \); hence the paper money flow \( x^t_{Na} = 0 \). Similarly, the existence of a consumption good with a positive final stock \( y^T_{na} > 0 \) that affects the individual's satisfaction \( (u^T_{na} > 0) \) implies that the final paper money stock \( y^T_{Na} = 0 \). However, (3.17) implies that the stocks of paper money at previous dates, \( t = 0, ..., T - 1 \), need not all vanish. This distinguishes the dynamic model from the static model, in which all stocks were final (and proportional to corresponding flows).

3. R. By applying to (3.16), (3.17) the same reasoning as in 3. J.-3. O. (with \( p^t_{Na} = 1 = p^T_{Na}, x^t_{Na} = 0 = y^T_{Na}, t = 0, ..., T - 1 \), we

\[(f) \quad \text{As pointed out in 3.} \ D, \text{ the plans of individuals need not be mutually consistent except for the condition that the market of date } 0 \text{ be cleared. Thus, } y^T_{Na} = 0 \text{ for every } a \text{ does not contradict the fact that the initial aggregate money stock is positive.}\]
find that positive stocks of paper money are consistent with finite prices of all consumption goods for every \( t \) if, and only if, \( \beta_{nA}^t \leq \beta_{nA}^t \) for all consumption goods \( n < N \) such that \( \beta_{nA}^t > 0 \). Whenever this condition is satisfied — i.e., whenever the (absolute) price of no consumption good is expected to rise at the next market date — the individual invests all his resources (apart from immediate consumption needs) in paper money or in other goods whose price expects to remain unchanged. Whenever the price of at least one consumption good is expected to rise, the individual reduces his paper money stock to zero. Thus, the paper money stocks he brings to the market jump in an unstable and discontinuous fashion, sometimes vanishing and sometimes equaling the money equivalent of all his resources.

3. S. All stocks to be retained by the individual at the market dates \( t > 0 \) are planned stocks. No clearing-of-the-market condition exists for these dates. It is different with the date \( t = 0 \). For any good (including \( N \), paper money), the equilibrium stocks \( y_{nA}^0 \) \( (a = 1, \ldots, A) \) are actually retained by the individuals and must obey the condition [see (3:6'), (3:6)]

\[
\sum_{a=1}^{A} y_{nA}^0 = \sum_{a=1}^{A} y_{nA}, \quad n = 1, \ldots, N.
\]

Suppose that, at time 0, every individual expects the price of every consumption good to rise by time 1: \( \beta_{nA}^t \beta_{nA}^t \) \( a = 1, \ldots, A; n < N \). Then, as we have just seen in 3. R., \( y_{nA}^0 = 0 \) for every \( a \). Hence by (3:18), the total money stock \( \sum_{a=1}^{A} y_{nA}^0 = 0 \), and since \( y_{nA}^0 \geq 0 \), we have \( y_{nA} = 0 \) for every \( a \); moreover, since \( x_{nA}^t = 0 \) for every \( a \) and \( t \), it is seen from the (3:6), (3:7) that the planned money stocks \( y_{nA}^t \) must also vanish. We conclude that unless there is disagreement between at least two individuals about the direction of the price change of at least one commodity between the dates 0 and 1, the actual or planned possession of positive stocks of paper money is not consistent, under conditions of our model, with finite prices and maximized utilities.

3. T. We can now resume the questions of "money illusion" raised in Section 1 and already discussed in 2. K.-2. L. for classical static models. This time we have a model in which positive stocks of money are consistent with finite determinate prices.

The budget restriction equations (3:5) are not, in general, homogeneous in the prices of consumption goods. They can be rewritten thus:

\[
\sum_{n=1}^{N-1} \frac{y_{nA}^t}{\beta_{nA}^t} + \frac{y_{nA}^t}{\beta_{nA}^t} = 0,
\]

for every \( a \) and \( t \). As we have seen in 3. Q., the planned final paper money stock \( y_{nA}^F = 0 \) for every \( a \); and since the initial stock \( y_{nA}^0 \) for at least one individual, this individual must have | because of
(3.6), and because all \( x_{Na} = 0 \), \( z_{Na} \neq 0 \) for at least one date, say \( t = t' \). Hence (3.19) is, for \( t = t' \), non-homogenous in the prices \( \rho_{na}^0 \).

If all prices are multiplied by \( k > 0 \), the \( z_{na}^0 \) will change, depending on \( k \).

We are particularly interested in the effect of an equipropotional change in the current prices, \( \rho_1^0, ..., \rho_{N-1}^0 \). If \( z_{Na}^0 \) is non-zero, we have the case just discussed, with \( t' = 0 \). But suppose \( z_{Na}^0 = 0 \). If only the current, but not also the expected prices change in the same proportion, then the ratio \( \rho_{na}^1/\rho_{na}^0, n < N \), changes; and the values of the \( x, y, z \) that satisfy the equations (3.16), (3.17) for \( t = 0 \), will, in general, have to change. If, on the other hand, not only the current but also all expected prices are multiplied by \( k \), then let \( t = t' \) be a marketing date for which \( z_{Na}^0 \neq 0 \). We have seen that such a date must exist, and that the \( z_{na}^0 \) depend on \( k \). We shall not pursue the matter further for the particular (and unrealistic) model of this section. Suffice it to conclude that, in general, at least some of the individual’s decisions must be affected by an equipropotional change of current prices, even if accompanied by a similar change in expected prices.

Moreover, unless no individual changes his money stock at time \( 0 \), the budget restriction (3.16) is non-homogeneous in current prices for at least one individual, and hence, the market determines absolute prices and not only price ratios.

3. U. To sum up Section 3: the assumptions of the dynamic model of this section are sufficient to explain, by a consistent system of equations, the existence of positive paper money stocks desired by individuals, the rationale of the ‘money illusion’ of individuals, and the determination of absolute prices. But the model is highly unrealistic as it implies that stocks of money, as well as of consumption goods, fluctuate in a discontinuous fashion under the impact of ever so slight expected price changes.

§ 4. – Introducing Illiquidity

4. A. In the dynamic model of Section 3, no perfect knowledge of the future was assumed. It was merely assumed that the \( a \)-th individual \( (a = 1, ..., A) \) thinks that he has perfect knowledge of future prices beginning at date 1 and including the horizon date \( T \). At the marketing date 0, the set of current equilibrium prices \( \rho_1^0, ..., \rho_{N-1}^0 \) is established, common to all individuals. This set depends on the, generally differing, expectations of future prices by different individuals, and on their utility functions and initial stocks. The price-set \( \rho_{1a}, ..., \rho_{N-1a} \), that the \( a \)-th individual \( (a = 1, ..., A) \) has expected for the next marketing date 1 will, in general, not coincide with the price set which will clear the market when that date arrives (and which will be common to all individuals). Even if the individuals do
not then revise their price expectations for the marketing dates $2, 3, \ldots, T$ in the light of some new experience, they have at any rate to revise the price-set for the date $1$, lest the demands and supplies remain unequalized. Similarly, when the marketing date $2$ arrives, the $a$-th individual will, in general, have to revise his price-set for that date. And so on.

4. B. However, the model of Section 3 assumed a *perfect market*, in the following sense: The $a$-th individual ($a = 1, \ldots, A$) considers the prices $p_{1a}^0, p_{2a}^1, \ldots, p_{na}^T$ ($n = 1, \ldots, N$) as given to him, independent of his actions.

We now weaken this condition. Note that in the previous model the «budget restrictions» (3:5) of the $a$-th individual are the only restrictions upon his utility maximization that involve prices:

$$
\sum_{n=1}^{N} x_{na}^t p_{na} = 0; \quad t = 0, \ldots, T.
$$

Since the $z$'s are related by (3:6) to the stocks brought to the market, the budget restrictions determine, for each commodity and marketing date, the set of quantities that are within $a$'s reach, given his initial stocks and the expectations of prices. In a more general case this set of quantities might be determined by equations

$$
B_a^{(x)} (x_{1a}, \ldots, x_{Na}) = 0; \quad t = 0, \ldots, T,
$$

where for each date $t$, the function $B_a^{(x)}$ describes the «market conditions» expected by the individual. While the condition (4:1) can be represented, in the space of the $z$'s, by a hyperplane through the origin, the condition (4:2) — if it does not degenerate into (4:1), which is its limiting case — is represented by a hypersurface which contains the origin and which is convex if viewed from the negative orthant. This property reflects the existence of a «cost of transaction», in a model which (thus maintaining some limitations of Section 3) is not concerned with specific transactions between single pairs (or triplets, etc.) of individuals, that make up the marketing process. It considers only the net acquisitions or sacrifices (the $z$'s) of each individual, resulting from the bargaining in the market. The convexity of (4:2) expresses then the fact that, whenever — as the net result of marketing — the individual has exchanged one commodity against another (say $1$ against $2$), he had to sacrifice, in addition, positive amounts of at least one of the $N$ commodities: the so-called transaction cost (in money paid to advertising agents or brokers, or in one's own leisure, etc.). The commodity thus sacrificed may itself be commodity $1$ or $2$. Thus on Graph I, $N = 2$, and the straight line $AA$ represents the budget restriction in a perfect market, (4:1). The broken line $BB$ (convex if viewed from the negative quadrant) represents the budget restrictions (4:2) when, per unit of commodity $1$ given up or acquired, a constant amount of commodity $2$ is acquired or given up — the same as in the case $AA$; but when, in addition, a constant transaction cost is incurred,
by always giving up a further amount of commodity 2. We can measure the total amount of commodity 2, paid (received) by the individual to (from) one or more other individuals per unit of commodity 1, as the slope of the line drawn from the origin to the point \((z_1, z_2)\) on the curve. If this slope, the price including transaction cost, increases (decreases) with the amount of commodity 1 acquired (given up),

Graph I. - Curves Describing Alternative Market Conditions.

the budget restriction (4:2) is represented by a curve such as CC; the individual is a monopolistic buyer (monopolistic buyer) of commodity 2, relative to commodity 1 (\(^1\)).

4. C. For \(N > 2\) the economic interpretation of properties of the functions \(B^t_a\), if treated in full generality, becomes more complicated than is justified by the limited purposes of this article (\(^2\)). We shall give the general budget restriction (4:2) a special form: for every \(a\) and \(t\),

\[
\sum_{n=1}^{N-1} R^t_{na} (z_{na}) = 0; \quad R_{Na} (z_{Na}) = z_{Na}.
\]

\(^1\) Cf. Marschak [1], p. 318 seq. for an earlier attempt on these lines.

\(^2\) It is possible, but outside of our scope here — see 1. I. above — to interpret (4:2) so as to include production conditions along with market conditions. Note also that brokers' services mentioned above are production goods, and these are again outside of the scope of the paper.
The quantity \( R_n^{a} (\xi_n^{a}) \), for \( n < N \), is the revenue received (and its negative is the outlay paid) for the \( n \)-th commodity, in exchange for the \( N \)-th commodity. (4:3) is satisfied if every commodity, when exchanged, is exchanged only against the \( N \)-th commodity. The latter is the "universal means of payment". A condition sufficient for the validity of (4:3) is, that no barter is permitted to take place. We can think of this as an institutional restriction. Thus, we shall

\[ \text{Graph II. - Demand Curves for a Consumption Good (when) and for Cash (NN).} \]

not try here to explain the absence (or any existing extent) of barter in a society as a consequence of utility maximization by its members, combined with certain conditions — physical or institutional, but not including a prohibition of barter — which make certain transactions more costly than others (*)

4. D. It is convenient, and possibly clarifying, to define now the concepts of a (variable) price and of marginal revenue. On Graph II we have plotted the prices (omitting subscript \( a \) and superscript \( f \)).

(*) If one limits oneself — as we have done in this paper — to consumption goods, the most important case which we are excluding by our institutional assumption of an economy without barter is probably that of exchanging food and lodging for domestic services. More modern and more important examples involve claims and production goods: the exchange of shares and real properties between industrial corporations. Thus the question — raised but not answered by Wicksell [r], p. 67 — as to how the extent of barter is determined by market equations, apart from legal restrictions, remains an important one.
\[ (4.4) \quad \hat{p}_n = \frac{R_n(z_n)}{z_n}; \quad z_n \geq 0; \quad n = 1, \ldots, N; \quad \text{hence by (4.3)} \]
\[ (4.5) \quad \hat{p}_N = -1. \]

The broken line, \( wbsn \), is implied by the same assumption as the broken line \( BB \) on Graph I. The ordinates \( ob \) and \( os \) are, respectively, the buying and the selling price of the commodity \( u(< N) \) after payment of transaction cost constant per unit of transaction. To avoid,

Graph III. - Demand Curve (PP) and Marginal Revenue Curve (MM) for a Consumption Good; Demand and Marginal Revenue Curve for Cash (NN).

in our pure exchange model, the discussion of production goods such as advertising services or brokers' services we can interpret here the positive difference between buying and selling prices by the fact that if one has to sell in a hurry, he cannot recoup the price he had paid. The line \( NN \) represents equation (4.5): \( N \) is money.

4. E. It was shown in an earlier paper \(^1\) that the positive excess of the buying over the selling price of a commodity — as on Graph II — affects unfavorably its optimal stock held by a profit-maximizing firm. The present paper's problem is analogous. Since the mathematical analysis is tedious owing to the discontinuity of this price function, we shall use, instead, in the present paper, an approximation that is continuous and differentiable everywhere: the curve \( PP \) on Graph III. The possibly strong difference between the conditions of a buyer and those of a seller can be reproduced by making the negative slope in the neighborhood of \( z_n = 0, \) as large in absolute value as we like.

\(^1\) Marschak [3].
yet maintaining this slope finite. At the same time a non-positive (rather than zero) slope is maintained for all finite values of \( z_n \), thus generalizing constant unit transaction costs into possibly increasing ones. In general, the curve reproduces imperfect competition: its segment for all \( z_n > 0 \) is the "demand curve to the monopolist"; its segment for all \( z_n < 0 \) is the "demand curve of the monopolist". [On Graph I, the corresponding curve for the budget restriction (4:2) is \( CC \). The slope of \( PP \) can be made so near zero — i.e., the market can be regarded as nearly perfect — as we like, provided \( |z_n| \) is sufficiently large.

4. F. Marginal revenue is defined by

\[
M_* = M_*(z_n) = dR_*(z_n)/dz_n, \quad z_n \geq 0, \quad n = 1, ..., N; \quad \text{hence by (4:3)}
\]

\[
M_N = 1.
\]

A function \( M_*(z_n) \), for \( n < N \) is represented on Graph III by the curve \( MM \).

Since \( dp_*/dz_n \leq 0, \quad M_* = p_* + z_n \cdot dp_*/dz_n = p_*(1 + z_n \cdot dp_*/p_* \cdot dz_n) \leq p_* \)

for \( z_n \) positive; \( M_* \geq p_* \) for \( z_n \) negative. The absolute value of the (negative) elasticity \( z_n \cdot dp_*/p_* \cdot dz_n \) or simply the slope \( dp_*/dz_n \) has been used as a measure of market imperfection, with respect to the individual monopolist or monopolist. We notice that, for the universal means of payment, this measure vanishes; while it can be said — at least in the neighborhood of \( z_n = 0 \) — to be particularly large for commodities involving a high cost of selling. In this sense, we can identify the degree of market imperfection especially in the vicinity of \( z_n = 0 \) with the degree of "illiquidity", which is zero for money and is larger than zero (and depends on the quantity sold) for goods that are difficult to sell. Line \( NN \) on Graph III represents both equations (4:5) and (4:7). But remember that this interpretation is possible because of a simplifying but asymmetrical institutional assumption (made in 4. C.) which excluded barter, i.e., introduced an universal means of payment. Without this assumption (i.e., with money defined merely as a numéraire by \( P_N = 1 \)) a numerically high elasticity \( z_n \cdot dp_*/p_* \cdot dz_n \) might be interpreted equally well as a market imperfection for the commodity \( n \) or for the commodity \( N \) (the numéraire) or both; in fact, the elementary theory of imperfect markets would cease to be serviceable, and we would need the more general approach outlined in 4. B., above.

4. G. The givens of the perfect market model of Section 3 were, for the \( a \)-th individual: his \( N \) initial stocks \( y_{na} \), his utility function \( u_*(\cdot) \), the \( N \) prices of zero-instant \( p_n^0 \) (common to all individuals), and the \( NT \) expected prices \( p_{na}^{\tau}, ..., p_{na}^0 \). In the imperfect market model, the set of prices \( p_{na}^0, p_{na}^1, ..., p_{na}^\tau \), for a given \( a \), is replaced by as many revenue functions \( R_{na}^0, ..., R_{na}^\tau \).

For the market as a whole, the perfect model considered as the givens: the \( AN \) initial stocks, the \( A \) utility functions, and the \( ANT \)
expected prices. The imperfect market model considers as the givens; the AN initial stocks, the A utility functions, and the AN (\(T + 1\)) revenue functions. *The latter do not form an independent set.* Their forms and parameters are restricted by the condition (3.6’) that the market be cleared. This is another way of saying that, in the market, one man’s strength is another man’s weakness. For \(A < 5\), the Theory of Games has begun to discuss these restrictions. We need not try here to develop this discussion.

4. H. We have replaced the expression in (4.1) \(\sum_{n=1}^{N} x_{na}^t p_{na}^t\), linear in the \(x\), by a non-linear expression, \(\sum_{n=1}^{N} R_{na}^t (x_{na}^t)\). Both mean the sum of (positive and negative) revenues of the individual at the marketing date \(t\), expressed in terms of a single commodity, the numéraire. In the present model, these revenues are not only expressed mathematically but also collected (or paid out, if negative) in units of the numéraire which has become the universal means of payment, barter being excluded. Substituting appropriately in (3.5), (3.10) constant prices \(p_{na}^t (t = 0, ..., T)\) must be replaced by marginal revenues \(M_{na}^t (x_{na}^t)\). Then, since \(M_{N_{Na}}^t (x_{Na}^t) = 1\) for every \(t\), (3.13), (3.15) become for \(n = i, ..., N - 1\)

\[
M_{na}^t (x_{na}^t) = (u_{na}^t + \mu_{na}^t)/(u_{Na}^t + \nu_{Na}^t), \quad t = 0, ..., T
\]

\[
M_{na}^{t+1} (x_{na}^{t+1}) = (u_{na}^{t+1} + \mu_{na}^{t+1} - \nu_{na}^{t+1})/(u_{Na}^{t+1} + \mu_{Na}^{t+1} - \nu_{Na}^{t+1}), \quad t = 0, ..., T - 1.
\]

4. I. From (4.8) it follows that if two commodities \(m, n (= i, ..., N)\) have positive consumption flows, and hence \(\mu_{ma} = \mu_{na} = 0\), then their marginal utilities are proportional to their marginal revenues: a “generalized law of marginal utilities” [compare here (2.14)].

4. J. If the universal means of payment is paper money, \(u_{Na}^t = \nu_{Na}^t = 0\), \(t = 0, ..., T\). There exists at least one consumption good \(n \leq N\) such that \(M_{na}^t > 0\), \(\mu_{na}^t > 0\), and \(x_{na}^t > 0\), and hence \(\mu_{na}^t\) = 0. Therefore \(\mu_{Na}^t > 0\), \(x_{Na}^t = 0 = y_{Na}^T\), \(t = 0, ..., T - 1\). That is, again, as in 3. Q., there cannot be positive flows or positive final stocks of paper money (1).

4. K. On the other hand, positive stocks of money for \(t < T\) are possible, and we shall show that, different from those of Section 3, they need not jump between zero and the absorption of all resources, but may change continuously. We shall show it for a general case of paper or non-paper money. It will suffice to prove that, in the present model, it is possible to have changing and finite prices for all consu-

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(1) The positive quantity \(\mu_{Na}^t = u_{na}^t / M_{na}^t, n = i, ..., N - 1\), can be called “the marginal utility of paper money” for the \(a\)-th individual.
tion goods without the « unloading » of stocks occurring for any goods, including money. That is, the stocks of every good may exceed current consumption needs (and, hence, the stocks of the « good », paper money, may be always positive). Suppose that for every $n = 1, ..., N$, and $t = 0, ..., T - 1$, $y_{na}^t > x_{na}^t$. Then by (3:14) $y_{na}^T = 0$. Therefore by (4:9), (4:8).

(4:10) $M_{na}^t (z_{na}) = M_{na}^{t+1} (z_{na}^{t+1})$, $n = 1, ..., N$, $t = 0, ..., T - 1$; hence

(4:11) $M_{na}^0 (z_{na}) = M_{na}^1 (z_{na}) = ... = M_{na}^T (z_{na})$, for every $n$.

If markets were perfect, with no transaction costs, marginal revenues would be identical with prices, and (4:11) would imply the expectation of unchanging prices, i.e., would bring us back to the static model of Section 2. Under conditions of the present model, however, marginal revenues are not equal to prices; and the rule (4:11), telling the individual to equalize over time the marginal revenue for each good is consistent with expectations of changing prices, provided the marginal revenue functions are expected to change. These expectations have thus been shown to be consistent with stocks of single goods that can exceed current consumption needs (can exceed zero, in case the good is paper money) yet do not absorb all the resources of the individual.

4. I. The result (4:11) gives a hint as to how one might approach the general problem of the effect of transaction cost and market imperfection upon the size of investment in a commodity; although, because of the limited scope of this paper, we can deal only with investments for speculation and future consumption, neglecting production and lending. Moreover, the result (4:11) was obtained under two simplifying assumptions, made in 4. C. and 4. E. respectively: that barter is excluded, and that (Graph III), at any time $t$, the demand curves for the $n$-th good and the $a$-th individual in his capacity as either a buyer or a seller can be approximated by a single continuous and differentiable curve, $p_{na}^t (z_{na})$. Accordingly, there is a continuous and differentiable curve of marginal revenues and outlays, $M_{na}^t (z_{na})$, for the whole range of possible sales or purchases. Dropping the subscripts $n$, $a$, for brevity, expand

(4:12) $M^t (z^t) = a^t + b^t z^t + ...$,

where $a^t$, $b^t$ are, respectively, the values of $M^t (z^t)$ and $dM^t / dz^t$ approached as $z^t$ approaches $0$ from either the positive or the negative direction. Assume that transaction cost (or market imperfection in the neighborhood of $z = 0$) exists; then $b^t < 0$. To fix the ideas, put $t = 0$ or 1. Then by (4:11), (4:12)

(4:13) $a^0 + b^0 z^0 + ... = a^1 + b^1 z^1 + ...$.

Consider now the following special case: the givens of the problem are such that the individual's best plan involves buying a certain positive
amount \( z \) of the commodity at time \( 0 \), and selling the same amount at time \( 1 \):

\[
o < z = -z^0 = z^1.
\]

Substituting for \( z^0, z^1 \) in (4:13) and neglecting higher terms,

\[
z = (\alpha^0 - \alpha^1)/(\beta^0 + \beta^1).
\]

Or, since the \( \beta \)'s are negative,

\[
(4:14) \quad z = \frac{\alpha^1 - \alpha^0}{|\beta^0 + \beta^1|}.
\]

Since \( z > 0 \), we conclude:

(1): \( \alpha^1 > \alpha^0 \), i.e., speculative buying of a given commodity implies an expected upward shift in its marginal revenue curve. This corresponds to the expected price rise of the perfect market case treated in Section 3, but with two important differences: In the perfect market case, speculative buying had to imply (a) that for no other commodity a stronger proportionate price rise was expected; and (b) that the stocks of all goods with a weaker expected proportionate price rise, were unloaded. Transaction costs and market imperfection remove such implications and make it rational to speculate simultaneously in commodities whose marginal revenues, for given quantities, will rise in different proportions.

(2): The larger the absolute value \( |\beta^0 + \beta^1| \) for a given commodity, the smaller is its amount \( z \) that the individual invests in speculation; and note that the \( \beta \)'s — the slopes of the marginal revenue curves near \( z = 0 \) — can characterize both the transaction costs, and the market imperfections for small sales and purchases. (It is also easily seen that \( \beta \) is proportional to the usual measure of market imperfection, \( dp/dz \), apart from a second order term. We omit here a discussion of corresponding elasticity measures, given in 4.F, above, which would be necessary for inter-commodity comparisons). Thus an increase in transaction costs for a given commodity would drive the speculator out of it, and into stocks of commodities with more nearly horizontal marginal revenue curves — especially but not exclusively into stocks of the universal means of payment (whose marginal revenue curve is exactly horizontal).

These results are illustrated on Graph IV, where for simplicity the slopes \( \beta^0 = \beta^1 = \beta \) (say). On each of the two diagrams, the lines \( M^0 \) and \( M^1 \) are portions of the marginal revenue (or marginal outlay) curve, at the dates \( 0 \) and \( 1 \) respectively. Their intercepts \( \alpha^0, \alpha^1 \) are the same for the corresponding curves on the right and the left diagram; and \( \alpha^0 > \alpha^1 \). On each diagram we find a quantity \( AB = BC \) such that the marginal revenues on the two successive dates are equal: \( AK = CL \). The quantity \( AB = BC (= z, \text{say}) \) is the amount bought and sold in speculation. We see that \( z \) is larger on the left-hand diagram — the one where the absolute value of the slope \( \beta \)
is smaller, i.e., where the transaction cost (or market imperfection) is smaller.

4. **M.** In reviewing now the equations of our dynamic model

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**Graph IV.** - Unfavorable Effect of Illiquidity (slope \( \beta \)) of a Good upon its Amount (\( AB = BC \)) Bought and Re-Sold.

\( OZ \): Axis of Quantities; \( OM \): Axis of Marginal Revenues.

\( M_o \) and \( M_1 \): Marginal Revenue Curves at Times 0 and 1, respectively.

Top: A Less Liquid Good; Bottom: A More Liquid Good.

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— with paper money, market imperfection and no barter — we shall make use of demand functions of the individuals (expressing the price
as a function of the demand of a buyer or to a seller $f^t_{na} = f^t_{na}(z^t_{na})$, as these functions are most familiar from economic literature. The functions $f^t_{na}$ are related to the revenue functions $R^t_{na}$ and the marginal revenue functions $M^t_{na}$ by the identities — [see (4.4), (4.6)] —

\begin{align}
R^t_{na}(z^t_{na}) &= z^t_{na} f^t_{na}(z^t_{na}) \\
M^t_{na}(z^t_{na}) &= \frac{dR^t_{na}}{dz^t_{na}},
\end{align}

for every $a, t$, and $n < N$. Moreover, for every $a, t — [see (4.5), (4.7)] —

\begin{align}
f^t_{Na}(z^t_{Na}) &= M^t_{Na}(z^t_{Na}) = 1.
\end{align}

For each individual $a$, and every $t$, the budget restriction is [see (4.3)]

\begin{align}
\sum_{n=1}^{N-1} f^t_{na}(z^t_{na}) + z^t_{Na} &= 0.
\end{align}

Discarding, for the purposes of the present paper, the occurrence of consumption goods with zero-flows (such cases lead to a tedious study of inequalities and « corner solutions » similar to that of Section 3), and remembering that $\mu^t_{Na} = 0$, $\mu^t_{Na} > 0$ for every $a, t$ (see 4. J.), we obtain, by (4.8), for every $a, t$

\begin{align}
\frac{M^t_{1a}(z^t_{na})}{\mu^t_{1a}(X_a)} &= \cdots = \frac{M^t_{N-1,a}(z^t_{N-1,a})}{\mu^t_{N-1,a}(X_a)} = \frac{\mu^t_{Na}}{\mu^t_{Na}} > 0,
\end{align}

where $X_a$ is the set of all consumption flows and final stocks of the individual. If we do not discard the possibility that stocks can be « unloaded » we have by (4.8), (4.9), (3.14),

\begin{align}
M^{t+1}_{na}(z^{t+1}_{na})/M^t_{na}(z^t_{na}) = (I - \frac{\mu^t_{Na}}{\mu^t_{Na}})/(I - \frac{\mu^t_{Na}}{\mu^t_{Na}}),
\end{align}

for every $a, t < T$, and $n < N$; where

\begin{align}
\nu^t_{na} &= \sqrt{y^t_{na} - x^t_{na}}, \quad n < N, \\
\nu^t_{Na} &= \sqrt{y^t_{Na} - x^t_{Na}};
\end{align}

the identities connecting the $y$ with the $x$ and $z$ are given in (3.5). Finally, there is the clearing-of-the-market requirement for date $0$

\begin{align}
\sum_{a=1}^{N} z^0_{na} &= 0, \quad n = 1, \ldots, N;
\end{align}

this involves a restriction upon the current demand functions $f^0_{na}$ (see 4. G.).

4. N. We may now ask whether, in the generalized model of this section, the individual has « money illusion » will the quantities of stocks sold and retained by the individual, and his consumption
flows, remain unchanged if either all current prices or all current as well as all expected prices change in the same proportion? Note that now it is not the expected prices $p^t_{\text{na}}, t > 0$ but the demand functions $f^t_{\text{na}}, t > 0$ that are given to the individual (subject however to the restrictions upon the $f^0_{\text{na}}$ mentioned in 4. M. and required to clear the market). An equiproportional change in the prices of all consumption goods must be understood as a change in the prices for given quantities bought or sold by the individual, i.e., as the multiplication of all demand functions $f^t_{\text{na}}, (n < N)$ by a constant, say $k (> 0)$. Because of the definitions (4.15), (4.16) this is equivalent to multiplying by $k$ all functions $M^t_{\text{na}} (n < N)$.

As in 3. T., the $z^t_{\text{na}}$ of the individual $a$ (the successive changes in his money stock) cannot vanish for all marketing dates $t$: for he starts with a stock $y^0_{\text{na}} > 0$, and plans to end up with $y^T_{\text{na}} = 0$. Hence his budget restriction (4.18) is non-homogenous in the prices $f^t_{\text{na}} (z^t_{\text{na}})$ for at least one date, say $t = t'$. Thus the quantities $z^t_{\text{na}}$ will depend on the multiplier $k$. In particular, if $t' = 0$, i.e., if the individual changes his money stock at date 0, his demands $z^0_{\text{na}}$ will depend on the multiplier $k$, both when $k$ is applied only to all current demand functions $f^0_{\text{na}}, n < N$, or when it is applied to the whole sequence of demand functions, $f^0_{\text{na}}, ..., f^T_{\text{na}}, n < N$. Suppose, on the other hand that $t' > 0$. Consider the equations (4.11) which result from (4.20)-(4.22) if the case of unloading is neglected (*). If the multiplier $k$ is applied only to the functions $M^0_{\text{na}}$ but not to the later marginal revenue functions $M^t_{\text{na}} (t > 0)$, the system of equations that contains (4.11) will not be, in general, satisfied unless the $z^t_{\text{na}}$ for all $t$, including $t = 0$, are changed. If, on the other hand, the whole sequence of marginal revenue functions is multiplied by $k$, then the change in $z^t_{\text{na}}$ will in general involve, because of (4.11), also a change in the $z^0_{\text{na}}$. It is thus rational, in a dynamic market, to have a money illusion.

4. O. Since we conceive of each demand function $f^t_{\text{na}}$ as a univalued one, the given of the system that determine the demands $z^t_{\text{na}}$, determine also the absolute prices $p^t_{\text{na}} = f^t_{\text{na}} (z^t_{\text{na}})$. The question asked in 1. E. must be readapted to fit the imperfect market situation. We have to ask whether the set of demands ($z$), flows ($x$) and stocks ($y$) of all individuals that is consistent with certain utility functions ($u^a$), initial stocks ($y$) and demand functions ($f$), and that clears the market of date 0, will also clear that market when either all demand functions, or possibly only the demand functions of date 0 ($f^0_{\text{na}}$ for every $a, n < N$), are multiplied by a constant. We have seen in 4. N. that this is not the case.

(*) Unloading was a rule, rather than an exception, in the model of Section 3; but it is not essential for the present model, and we may leave its analysis aside.
§ 5. — Summary

5. **A.** The most general among the models that were treated explicitly in this paper is characterized by the same Plan (3;1) as in the perfect market model, but with the column of successive prices, $\delta^t_{na}$, replaced by a column of successive marginal revenue functions, $M^t_{na}(z^t_{na})$. These functions belong to the givens of the problem, together with the utility functions $u^a$ and the initial stocks $\bar{y}^a_{na}$. The unknowns are the stocks $x^t_{na}$ that the individual plans to sell, the stocks $\gamma^t_{na}$ that he plans to retain, and the consumption flows $s^t_{na}$. The marketing dates are $t = 0, \ldots, T$; the goods are $n = 1, \ldots, N$, with $N$ being the universal means of payment, i.e., $M^t_{na}(z^t_{Na}) = 1$, for all $a$ and $t$.

Special, mutually independent, assumptions taken singly or in combinations, modify this general model into more special ones. These assumptions are:

(I) Perfect market: $dM^t_{na}/dz^t_{na} = 0$ for all $n, a, t$.

(II) Paper Money: $u^t_{Na} = 0$ for all $a, t$.

(III) Static case: $T = 0$.

The implications of assumption (II), when it is introduced singly into the general model, were studied in Section 4. The implications of (I), alone or in combination with (II), were discussed in Section 3. In Section 2, assumptions (I) and (III) were introduced jointly, and treated both in the absence and in the presence of assumption (II). In all cases, we tried to find whether, and under what conditions, positive stocks of paper money are consistent with utility maximization and with positive and finite prices of consumption goods. We found this consistency present in the dynamic case with market imperfections (or transaction costs), i.e., when assumptions (I) and (III) are not made. When assumption (I) is introduced, only discontinuous and unstable stocks of paper money are seen to be justifiable. When, in addition, assumption (III) is introduced, positive and finite prices of consumption goods imply zero stocks of paper money; i.e., a regime of money-of-account.

5. **B.** Important omissions were registered as we went along. Not only did we disregard uncertainty, production and lending (1. **H.**-1. **I.**); we decreed the exclusion of barter (4. **C.**); we compressed the bargaining process into an instant of time (3. **B.**); and glossed over the nature of interdependence between marginal revenue functions of various individuals (4. **G.**); and, for reasons of mathematical simplicity, we approximated the discontinuous phenomenon of transaction costs (and the finite difference between selling and buying price) by piecing together in a continuous fashion the demand curve of a buyer and that to a seller (4. **E.**).
5. C. Using the expressions of the older economic literature the results obtained so far can be also summarized as follows: It was shown in Section 2 (confirming the finding of Patinkin [1]) that the existence of paper money is not explained by its function as a numéraire, a unit of price measurement. In Section 3, the store of values function of money was investigated but was not proved sufficient to explain paper money stocks that take values other than 0 or the money equivalent of all resources. In Section 4, money was viewed (in addition to being the numéraire, and to store value in expectation of favorable prices) as the universal means of payment (legal tender), and all other goods were assumed to have imperfect markets. Under these conditions continuous positive stocks of paper money were explained. Note that it was not necessary to introduce uncertainty.

5. D. Of all models studied, money illusion was absent only in a model involving money-of-account. Wherever positive stocks of money were present, we had also money illusion: the change of all prices in relation to money did affect the quantities of goods held or consumed by the individual.

This agrees with common sense. If people hold money stocks this holding must give them some advantages. Therefore, if the prices of all non-money goods change in relation to the price of money (though not necessarily in relation to each other), it may be advantageous to change the stock of money held— and, therefore, to change other quantities involved, i.e., stocks retained and sold, and the consumption flows, of some or all non-money goods. Thus money illusion is rational. We had better not call it illusion!