Cowles Foundation Paper 43b

Reprinted from
ECONOMETRICA, Journal of the Econometric Society, Vol. 18, No. 2, April, 1950
The University of Chicago, Chicago 37, Illinois, U. S. A.

Utility Analysis of Decisions Affecting Future Well-Being, TJALLING C. KOOPMANS, Cowles Commission for Research in Economics and The University of Chicago.

UTILITY analysis of consumer's choice is based on a complete ordering (admitting indifference as well as preference) of the objects of choice. In static analysis the objects of choice are bundles of commodity flows, that is, vectors x of which the nonnegative components $x_1, x_2, \cdots, x_n, \cdots, x_N$, are rates of consumption of specific commodities, supposed constant over an indefinite period. The intent of this analysis is to treat *preference* and *opportunity* as separate data of the choice problem. The opportunity is defined as a set Q of bundles x given as being accessible to the choosing individual. Best choice is confined to that subset Q_0 of Q (which often is a single bundle x_0 of Q) such that choice between any two bundles x, y in Q_0 is indifferent, while no bundle z in Q outside Q_0 is preferred to any bundle in Q_0. The opportunity set is a linear subspace if the individual buys at constant prices from a given income.

A frequently discussed dynamic generalization of this analysis is obtained by adopting as the objects of preference ranking sequences $X = \{x(1), x(2), \cdots, x(T)\}$ of bundles $x(t)$ of flows which remain constant only within each future time segment (for each value of t) considered. This formulation of the object disregards the desire for postponement of decisions not yet called for by circumstances. It burdens the analysis with an amount of detail regarding alternatives in a distant future such as never sways the decisions economic analysis is concerned with. Furthermore, since it does not allow consideration of assets, the end-of-period position (at time $t = T + 1$) cannot be taken into account, and the arbitrary length T of the period considered will affect the choice which is predicted by the analysis.

These objections can be overcome by adopting as the objects of preference ranking sets S of sequences X of bundles $x(t)$ of flows $x_n(t)$ with the following understanding regarding later choice within the set. A set S of such sequences is called t-uniform if the values $x(1), x(2), \cdots, x(t)$ of the first t vectors are the same for all sequences X in S. We apply the preference ranking only to 1-uniform sets S_1. Any such set S_1 can be exhaustively decomposed into nonoverlapping subsets S_2 which are 2-uniform, by combining into the same S_2 all those sequences for which $x(2)$ has some given value. The individual knows that his choice of S_1 at time $t = 1$ commits him to make a further choice of a 2-uniform subset S_2 of S_1 at time $t = 2$ (if there is more than one S_2 in S_1). S_2 again partitions into 3-uniform subsets S_3, between which further choice is required at time $t = 3$, etc. The preferability of postponement of choice is expressed by the following postulate: If $S_t = S_t' + S_t''$ is t-uniform for some $t \geqq 1$, then the individual will rank S_t at least as high as, and often higher than, either S'_t or S''_t.

While in principle all 1-uniform sets S_1 are assumed subject to preference ranking, the sets contained in the opportunity set Q of sets S_1 may be thought of as given by market anticipations, in the simplest case by single-valued anticipations of future commodity prices and of incomes derivable from specific occupational efforts. Assets can then be entered in the utility function as representative of the sets of consumption sequences they give access to, through direct enjoyment, resale and purchase of other assets or consumption flows, alternatively or in succession, subject to later decision. By introducing end-of-period assets as representing the anticipated level of well-being for $t \geqq T + 1$ for the individual or his heirs, the designation of the period of analysis T represents only a decision of the analyst as to how much of future consumption to introduce explicitly, how much to leave implicit in assets—and thus need not affect the choice predicted by the analysis.

Consumer financing adds to well-being because it gives access to the flows of services associated with possession of durable consumers' goods at an earlier time than would otherwise be possible at the same rate of saving. Compulsory pension plans diminish welfare to the extent that they prohibit the use of accumulated savings (while safeguarding principal by proper depreciation allowances) for gaining access to the services of durable consumers' goods.

Optimal Investment of a Firm, JACOB MARSCHAK, Cowles Commission for Research in Economics and The University of Chicago.

NOTATIONS, DEFINITIONS, ASSUMPTIONS. 1. Use Latin capitals for random variables. Denote the vector of *assets* held by a firm by $a \equiv [a_g] \equiv (a_1, \cdots, a_n)$, and its *profit* by $Y \equiv Y(a)$. Then, at $a = 0$, $Y = 0$; and write $\partial Y/\partial a_g \equiv Y_g$, $\partial^2 Y/\partial a_g \partial a_h \equiv Y_{gh}$, $(g, h = 1, \cdots, n)$. Expand $Y = \sum a_g Y_g + \frac{1}{2}\sum\sum a_g a_h Y_{gh} + \cdots$. Write $\mathcal{E}Y \equiv m_Y$; $\mathcal{E}(Y - \mu_Y)^2 \equiv m_{YY}$; $\mathcal{E}Y_g \equiv q_g \equiv$ the mean of the *marginal profitability* of the gth asset; its variance and covariances are $\mathcal{E}(Y_g - q_g)(Y_h - q_h) \equiv q_{gh}$, $(g, h = 1, \cdots, n)$.

2. Assume that *complementarity* $Y_{gh} = \text{const.} \equiv y_{gh}$, $(g, h = 1, \cdots, n)$. Then the profit function $Y(a)$ is approximately characterized by the vector $[q_g]$ and the two matrices $\| q_{gh} \|$, $\| y_{gh} \|$.

3. *Effective utility function* $u(Y) \equiv u_2[u_1(Y)]$ where $u_1(Y) \equiv$ profit after taxes, and u_2 is the subjective utility function.

4. Expand $v \equiv \mathcal{E}u(Y) = u(m_Y) + u''(m_Y) \cdot m_{YY}/2 + \cdots$. Hence approximately, for v given, $dm_Y/dm_{YY} = -\frac{1}{2}u''(m_Y)/u'(m_Y) \equiv \rho \equiv$ *risk aversion*. (If $u = u_1$, $\rho =$ *tax progressivity;* if $u = u_2$, $2\rho =$ *flexibility of marginal utility of profit, multiplied by profit.*)

5. *Rational behavior:* maximize v subject to constraints such as (in a simple case) $\sum a_g p_g = k$ (borrowing limit), where p_g are constant prices.

PROBLEM I. Find desirable properties of assets. $\rho > 0$, and, by (5), p_g is proportional to $\partial v/\partial a_g$ and therefore, by (4), also to the quantity $(q_g + \sum y_{gh}a_h - 2\rho\sum q_{gh}a_h)$. Hence high values are desired for q_g, y_{gh}, y_{gg}, $-q_{gh}$, $-q_{gg}$. Examples: comparisons between insurance companies and specialized and diversified investment trusts and producers.

PROBLEM II. Let $X \equiv X(a) \equiv$ the firm's physical output. In the national interest, maximize $\mathcal{E}X$ by choosing an appropriate profit-after-tax schedule, $u_1(Y)$, provided the expectation of tax revenue has a fixed level, c.

Simplifying assumptions: $u = u_1$ for any Y; $n = 2$; $0 \leqslant a_1 =$ risky plant; $0 \leqslant a_2 =$ riskless bonds. $X = X(a_1)$, $X'(a_1) > 0$; $Y = a_1Y_1 + a_1^2y_{11}/2 + a_2y_2$ approximately, where y_{11}, y_2, and the mean and variance of Y_1 are known. Put $p_1 = p_2 = 1$. Then by (5),

$$(\alpha) \qquad a_1 \leqslant k,$$

$$(\beta) \qquad \mathcal{E}u[Y(a_1)] \geqslant \mathcal{E}u[Y(a_1^*)] \text{ for any } a_1^*,\ 0 \leqslant a_1^* \leqslant k.$$

To find a function u that maximizes a_1 (and hence X), subject to (α), (β), and subject to

$$(\gamma) \qquad \mathcal{E}[Y - u(Y)] = c,$$

is a problem in the calculus of variations. A solution was given for a quadratic tax schedule admitting negative taxes, and with $u(0) = 0$. Instead, some or all of the following conventional constraints upon the tax schedule might be used: for $Y > 0$, $u \leqslant Y$, $0 < u' < 1$, $u'' < 0$; and for $Y \leqslant 0$, $u(Y) = Y$.

PROBLEM III. Let $Y^{(\tau)} \equiv$ *sequence of profits* in τ years, 0 through $\tau - 1$. Generalize Problem I by substituting $Y^{(t)}$ for Y ($t =$ "horizon"), and by redefining a as a matrix of *decision functions* ("strategy") $a \equiv \| a_g^{(\tau)}[Y^{(\tau)}] \|$, where $g = 1, \cdots, n$; $\tau = 1, \cdots, t$. The best value of a will depend on parameters $[q_g]$, $\| q_{gh} \|$, $\| y_{gh} \|$ [defined in (2) and properly generalized] and on additional asset properties, viz. the mutual conversion costs (illiquidities).

PROBLEM IV. Same as Problem III, but the parameters just mentioned are not known in advance. The materialized sequence $Y^{(\tau)}$ serves as a statistical sample of growing size.