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ROLE OF LIQUIDITY UNDER COMPLETE AND INCOMPLETE INFORMATION

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1. It is proposed to study how the demand of rationally acting men for a commitment (an asset or a contract) depends on its liquidity under various degrees of available information.

1.1 While the actual behavior of men is not rational, the implications of rational behavior, or the so-called "pure economic theory," deserve study for two reasons: (1) as a possible first approximation to the description of actual behavior; (2) as a set of practical norms, to be used by firms or governments.

2. *Liquidity*

Let x_0, x_1 = rates of input of a certain service (in man-hours, machine-hours, etc.) in "years" 0 and 1 respectively. If $x_1 > x_0$, and the asset or contract that yields the service is expanded, a price, P, is to be paid, per unit of input added. If $x_1 < x_0$, and the asset or contract that yields the service is reduced, a certain amount, Pl is released, per unit of input subtracted. The ratio $l (0 \leqq l \leqq 1)$ will be called liquidity. Of the cases drawn on Chart I, we shall consider line *bb* as sufficiently realistic, though *cc* is somewhat more general.

2.1 *Special forms*

2.1.1 Liquidity as *marketability* of an asset. Here Pl = second hand or scrap value of a machine (per machine-hour at maximum rate of use); the selling price of a (nonstandardized) real estate after advertising or agent costs, etc. In perfect market (single shares, bonds; grains) $l = 1$. Note (on Chart I) that continuous line *dd* ordinarily used for monopoly does not meet our case as well as do cases *bb* or *cc:* the transition from the buying to the selling role involves a break!

2.1.2 Liquidity as *physical convertibility* of an asset. Here $l = 1$ implies costless change of physical form or location. For raw materials, l is larger than for finished goods.

2.1.3 *Liquidity of contracts.* Here $l < 1$ if the dissolution of contract entails legal penalty or some other cost.

2.1.4 Liquidity and *nondurability*. It follows that if the contract's term is not longer than one "year," its liquidity $l = 1$; and $l < 1$ other-

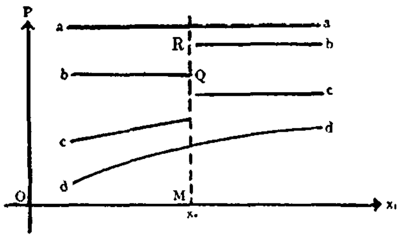
[1] With acknowledgments to Messrs. K. Arrow and H. Chernoff, of the Cowles Commission for Research in Economics and to Mr. James Tobin. The paper will be reprinted as Cowles Commission Paper, New Series, No. 37.

wise. Similarly, a plant with durability exceeding one "year" has $l<1$.

2.1.5 Thus liquidity is a physical or institutional property of an asset or contract, similar to the proportion of raw material in a final product. It is the ratio of two contemporary prices and therefore independent of any price changes in the time between the buying and the selling of an asset, or between entering and dissolving a contract.[2]

CHART I

P = price (per input unit) paid or retained by firm
x_1 = input rate in year 1
x_0 = initial input rate



aa: perfect liquidity: $l=1$
bb: imperfect liquidity (admitting discontinuity): $l=MQ/MR$ = constant <1
cc: ditto, l variable <1
dd: imperfect liquidity (continuous case).

2.2 We shall assume a single kind of input (or a bundle of inputs tied by constant proportions between quantities and between prices; but see also 4.5). We shall regard it, at its buying price, as the *numéraire;* i.e., put $P\equiv 1$.

2.2.1 If one drops this assumption, or at least admits preference for present versus future profits, then liquidity must be defined, not as a ratio between two money costs, but as a ratio between two marginal utilities (a "marginal rate of substitution"). See also 5.4.1, 5.4.1.2.

2.3 *Horizon* might be extended beyond two time units ("years") but this would not alter the essential results.

2.4 The *initial input* x_0 is identical with *investment*, up to a proportionality factor; viz., the maximum number of input-units (machine-hours, etc.) per year.

3. *Degrees of Information*

For brevity, the words "the firm knows" will be used instead of "the firm believes it knows." Suppose it considers a set $[u]$ of alternative events $u_1, u_2, \cdots$. Denote by $[p]$ the set of probabilities $p_1, p_2, \cdots$ of these events. We can then distinguish four degrees of information about $[u]$ (besides obvious mixed cases):

3.1 The firm does not know $[p]$.

[2] G. H. Evans has privately made the interesting suggestion of describing a commitment as more or less "reversible." The older term "liquid" (as opposed to "frozen") has the advantage of being universally used. It alludes to the ability of liquid bodies, to change their form freely: cash (and to a lesser extent a share, a bond, or a stock of raw materials) is freely transformed into other assets.

3.2 As above, but it knows data permitting it to estimate $[p]$.

3.3 The firm knows $[p]$.

3.4 As in 3.3, and every element of $[p]$ is either 0 or 1.

3.5 The degrees 3.4, 3.3 we call *complete information;* in particular, degree 3.4 constitutes *certainty*. The degrees 3.1, 3.2 we call *incomplete information* (at least one of them seems to be what F. H. Knight calls "uncertainty"); in particular, degree 3.1 will be called *ignorance*.

3.5.1 Note that the extreme highest and lowest degrees of information do not involve probabilities; while the two intermediate degrees involve probabilities. We can thus distinguish between stochastic and non-stochastic cases of complete information; and similarly for incomplete information. The non-stochastic cases can be considered as special cases of the corresponding stochastic ones (e.g., by letting all variances vanish).

3.5.2 In both stochastic cases (3.2, 3.3) information is *sequential:* more is known in year 1 than in year 0. As will be seen, it may pay to postpone investment (i.e., to have a smaller x_0) and wait for more information.

3.5.3 Note that all probabilities involved are *subjective.*

3.6 The model can be extended to the case when probabilities are *degrees of belief* not depending on observed frequencies; provided the firm can make choices between bets.

3.7 Extension to "*ordinal probabilities*" is possible but will not be discussed here.

3.8 The rational man (1.1) maximizes the expected (=mean) value of utility.[3]

4. *Effect of Liquidity on Demand in Case of Certainty*

4.1 The firm *knows* (as in 3.4) that x input units will produce in years 0 and 1, a revenue of, respectively, $\rho_0(x)$ and $\rho_1(x)$. Write $x_1 - x_0 \equiv y$. The firm chooses, at beginning of year 0, those values of x, y that maximize two years' profit $z = z_0 + z_1$ where $z_0 = \rho_0(x_0) - x_0$; $z_1 = \rho_1(x_0 + y) - x_0 - qy$, where $q = 1$ if $y \geqq 0$; $q = l$ if $y < 0$.

4.2 *Problem:* Investigate the effect of l upon x and y for a given change in the revenue function.

4.2.1 Assume a one-parametric "shift" in revenue function, thus:

$\rho_1(x) = \rho_0(x + u)$ (i.e., total cost needed to produce a given revenue is changed by a constant).

4.2.2 We might, instead, assume a multiplicative not additive "shift"

[3] See "Measurable Utility and the Theory of Assets" (mimeographed: Cowles Commission Discussion Paper, Economics 226, 226A), presented at Madison meeting of Econometric Society, 1948; abstract in *Econometrica*, 1949; von Neumann and Morgenstern, *Theory of Games and Economic Behavior*, Section 3.6, and Appendix to Second Edition.

and write for the revenue of the second year, $u \cdot \rho_0(x+y)$. Variations in u might be due to changes in production, as well as in the price-ratio between input and output. The latter case was suggested by A. G. Hart in "Risk, Uncertainty, and the Unprofitability of Compounding Probabilities" (in *Studies in Mathematical Economics and Econometrics*, in memory of Henry Schultz, 1942). The results are in essence the same as those obtained under assumptions 4.2.1. The most general case is $\rho_1(x)=\rho_0(x)+\delta\rho_0(x)$, where δ indicates functional variation. (See Kenneth May, "Technological Change and Aggregation," *Econometrica*, 1947.)

4.2.3 Assume, as on Chart II, that marginal revenue function is linear,

$$\rho'_0(x) = b - x/c; \qquad \rho'_1(x) = b - (x+u)/c; \qquad c > 0.$$

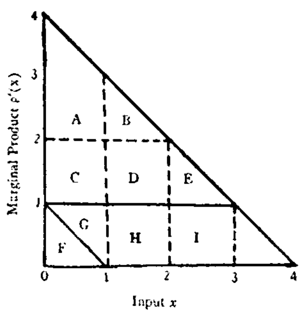
On Charts II–IV, $b=4$, $c=1$.

4.2.4 From now on we write x for x_0; $x+y$ for x_1; and ρ for ρ_0.

4.3 The problem becomes: "$\hat{x}$, $\hat{y}$ are values of x, y that maximize

$$z(x, y) = a + bx - x^2/2c + a + b(x+y+u) - (x+y+u)^2/2c - 2x - qy,$$

CHART II



Best inputs and profits under perfect liquidity and illiquidity.
It is known that marginal revenue function represented by line 44 in year 0 will shift by 3 input units (to line 11) in year 1. Input price = 1 throughout.

Case $l=1$ (perfect liquidity)
Best inputs: $x_0=3$, $x_1=0$;
Profits: $z_0=A+B+C+D+E$
$z_1=0$

Case $l=0$ (perfect illiquidity)
If again $x_0=3$, $x_1=0$, then
Profits: $z_0=A+B+C+D+E$
$z_1=-(F+G+H+I)$
A better (in fact, the best*) choice is:
$x_0=2$, $x_1=1$; then
Profits: $z_0=A+B+C+D$
$z_1=-(G+H)$. Then total profit (z_0+z_1) exceeds that of previous choice by $(F+I-E)$.

* See Chart III B, with $u=3$.

where $q=1$ if $y \geqq 0$; $q=l$ if $y<0$; $a=$constant.

Express $\hat{x}$, $\hat{y}$ as functions of both u and l."

4.3.1 Especially, show that, given the shift $u>0$ (decrease in total cost for given revenue), $\hat{x}$ is smaller when $l=0$ than when $l=1$: Chart II.

4.4 Solution of 4.3. Putting $\partial z/\partial x=\partial z/\partial y=0$, solve for x, y, when $y \neq 0$:

$$y = y(u, q) = 2c(1 - q) - u$$
$$x = x(u, q) = c(b - 2 + q).$$

When $y=0$, see (γ) below.

Case 1. Perfect liquidity: $l=1$, hence $q=1$.

$$\left.\begin{aligned} \hat{x} &= x(u, 1) = c(b - 1) \\ \hat{y} &= y(u, 1) = - u \end{aligned}\right\} \text{See Chart IIIA.}$$

Case 2. Imperfect liquidity (See Chart III B): $0 \leqq l<1$.

(α) for values of u such that $y(u, 1) \geqq 0$:

$$\hat{y} = y(u, 1) = - u, \text{ as in Case 1}$$
$$\hat{x} = x(u, 1) = c(b - 1), \text{ as in Case 1}$$
$$u \leqq 0;$$

(β) for values of u such that $y(u, 1)<0$ and that even $y(u, l)<0$:

$$\hat{y} = y(u, l) = 2c(1 - l) - u$$
$$\hat{x} = x(u, l) = c(b - 2 + l)$$
$$u > 2c(1 - l) > 0.$$

(γ) for values of u such that $y(u, 1)<0 \leqq y(u, l)$:

$$\hat{y} = 0,$$
$$\frac{dz(x, 0)}{dx} = 0, \qquad \hat{x} = c(b - 1) - u/2$$
$$0< u \leqq 2c(1 - l).$$

4.4.1 Meaning of the intervals (α), (β), (γ) in terms of *capacity*:

In (α) the shift of revenue function calls for an increase in input after year 0. This means also an increase in capacity as the asset acquired or contract made at the beginning of year 0 was such as to suit the initial input $\hat{x}$.

In (β) the shift calls for decrease of input. In this case, if the second-hand price of the asset$=0$, part of capacity created in year 0 may remain unused during year 1; that is, it was not absurd to plan in year 0 for unused capacity in year 1.

In (γ) the shift in revenue function would call for decrease of input if the second hand price equaled the price of new equipment; but with a lower, or even zero, second hand price it is preferable to start with smaller capacity in year 0 and to maintain it unchanged, and fully used, in year 1.

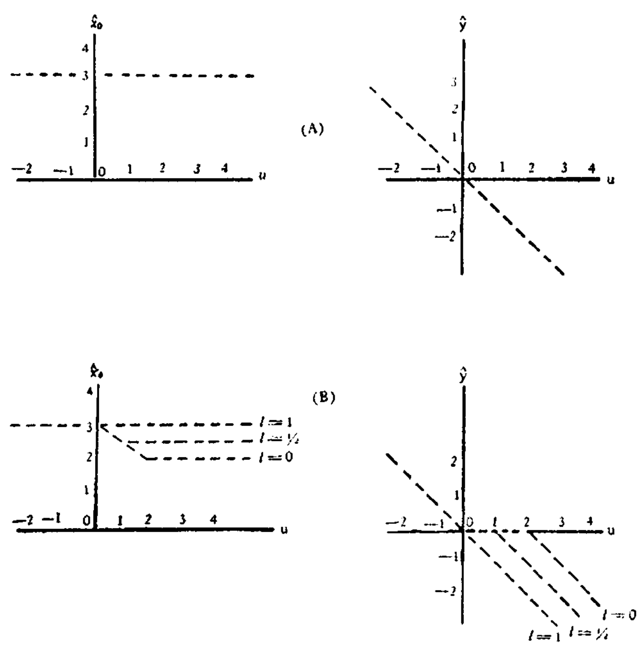
4.5 Problem of *several* (n) kinds of inputs (and, correspondingly, several kinds of assets or contracts), each having different liquidity $l^i(i=1, \cdots, n)$. Find the best initial inputs (proportional to investments) $\hat{x}^i$ and the best increments $\hat{y}_1{}^i$, $\hat{y}_2{}^i$ for successive years 1, 2, $\cdots$ as functions of l^i and of a known shift u in the revenue function. It is conjectured that (as on Chart III B), $\partial \hat{x}^i/\partial l^i \geqq 0$.

4.6 We conclude that differences in the liquidity of various assets affect the relative demand for them even under conditions of certainty. Examples: till money, and pipe-line stocks held to provide for predicted changes (seasonal or otherwise) in production and market conditions.

5. *Effect of Liquidity on Demand in the General Case of Complete Information* (3.3)

CHART III

Optimal initial input ($\hat{x}_0$) and optimal change in input ($\hat{y}$) as functions of a known shift (u) in revenue function.
(A): Perfect liquidity ($l=1$)
(B): Varying degrees of liquidity ($l=0, \frac{1}{2}, 1$).



5.1 Notation: Lower case letters: nonrandom variables (quantities and functions)

Capital letters: variables that are in general random

Italic letters: quantities (mostly)

Greek letters: functions (mostly)

5.2 At the beginning of year 0, the revenue function $\rho(x)$ for that year is known, but the revenue function for the next year is random. Specifically, replace equation in 4.2.1 by the following:

$$\rho_1(x) = \rho(x + U)$$

where the "random shift" U has probability density function $\phi(U)$.

5.3 We shall assume $\mathrm{E}U = 0$ and denote $\mathrm{E}U^2$ by σ^2. We shall discuss how the best investment (best initial input) $\hat{x}$ depends upon liquidity l and upon riskiness σ^2 of the contract or asset in question.

5.4 The two-years' profit z in 4.3 becomes

$$Z = Z(x, Y) = \rho(x) - x + \rho(x + Y + U) - x - QY,$$

where $Q = 1$ if $Y \geqq 0$; $Q = l$ if $Y < 0$. Here Y, and therefore Q, is random, because the choice of best value for Y, to be made at beginning of year 1, will depend upon the value which the shift U will have taken by then. At the beginning of year 0, the firm determines:

(a) the ("best") value $\hat{x}$ of x, and

(b) the ("best") function $\hat{\eta}(\ )$,—

such that if $x = \hat{x}$ and $Y = \hat{\eta}(U)$, then the expectation $\mathrm{E}Z$ has a maximum, i.e.

$$\mathrm{E}Z[\hat{x}, \hat{\eta}(U)] \geqq \mathrm{E}Z[x, \eta(U)]$$

for any x, η (cf. 4.3), or

$$\mathrm{E}Z[\hat{x}, \hat{\eta}(U)] = \mathrm{Max}_x \, \mathrm{Max}_\eta \, \mathrm{E}Z.$$

5.4.1 $Z = Z_1 + Z_2$ is used here as a simple case of utility function of money profits instead of a more general one, say $\omega(Z_1, Z_2)$: cf. 2.2.1. Expanding ω into Taylor series (as in Marshall, *Principles*, Math. Appendix IX) we approximate $\mathrm{E}\omega$ by a linear combination of $\mathrm{E}Z$, the variances of Z_1 and Z_2 (sometimes called "risks"), their correlation, etc. The firm is thus concerned with higher moments and not only with the means of profits. This does not contradict 3.8.

5.4.1.1 In the expansion just mentioned, the coefficient of the variance of $Z_i (i = 1, 2)$ is $\partial^2\omega/\partial Z_i^2$. Only if this is negative, i.e., if the marginal utility of profit decreases with profit, is there "risk aversion" (cf. Friedman and Savage, *Journal of Political Economy*, 1948).

5.4.1.2 Further generalization of the utility function might make it dependent on other commodities besides money—e.g. honor (for a firm's manager), consumers' goods (for the householder).

5.4.1.3 However, the simple utility function $\omega(Z_1, Z_2) = Z_1 + Z_2 = Z$ and hence the maximizing of EZ, will suffice to illustrate the main propositions of this paper.

5.4.1.4 The behavior in 5.4 is equivalent to the Neumann-Morgenstern optimal strategy.

5.4.1.5 We have here "sequential information"—the particular value taken by U becomes known after the end of year 0. The result of 5.3 will mean that if liquidity of the asset is low, it may pay to wait with investment (i.e., to invest in another, more liquid asset), pending further information.

5.5 We shall show that

$$\text{Max}_x \text{Max}_\eta \text{E}Z[x, \eta(U)] = \text{Max}_x \text{EMax}_\eta Z[x, \eta(U)]; \tag{5.5.1}$$

for this, it suffices to show that

$$\text{Max}_\eta \text{E}Z[x, \eta(U)] = \text{EMax}_\eta Z[x, \eta(U)]. \tag{5.5.2}$$

Suppose that

$$\text{Max}_\eta Z[x, \eta(U)] = Z[x, \hat{\eta}(U)]; \tag{5.5.3}$$

that is, for any U, x, η

$$Z[x, \hat{\eta}(U)] \geqq Z[x, \eta(U)];$$

multiplying by $\phi(U)$ (nonnegative) and summing over U,

$$\text{E}Z[x, \hat{\eta}(U)] \geqq \text{E}Z[x, \eta(U)],$$

$$\text{E}Z[x, \hat{\eta}(U)] = \text{Max}_\eta \text{E}Z[x, \eta(U)];$$

This, by (5.5.3) is equivalent to (5.5.2); hence (5.5.1) is proved.

5.6 Therefore, to find $\hat{x}$, $\hat{\eta}$, we proceed as follows: solve

$$\partial Z(x, Y)/\partial Y = 0 \tag{5.6.1}$$

for Y, obtaining $Y = \hat{\eta}(U)$, which involves x. Substituting,

$$\text{E}Z(x, Y) = \text{E}Z[x, \hat{\eta}(U)] = \text{E}Z, \tag{5.6.2}$$

now a function of x only. Then solve for x the equation

$$d\text{E}Z/dx = 0; \tag{5.6.3}$$

it will be satisfied by $x = \hat{x}$.

5.7 Apply this to the case of linear marginal revenue function (4.2.3.)

The problem in 4.3 becomes:

"Write $Y \equiv \eta(U; l)$; $V = x + Y + U$; then $\hat{x}$ and $\hat{\eta}(U; l)$ are a constant and a function that maximize $EZ(x, Y)$, where

$$Z(x, Y) = a + bx - x^2/2c + a + bV - V^2/2c - 2x - QY,$$

where $Q = 1$ if $\eta(U, l) \geqq 0$; $Q = l$ if $\eta(U, l) < 0$; and $EU = 0$.

Express $\hat{x}$ as a function of liquidity l and of the parameters of the probability function $\phi(U)$, such as the variance, $\sigma^2 = EU^2$, of random shifts of the revenue function. In particular, find the signs of $\partial\hat{x}/\partial l$ and $\partial\hat{x}/\partial\sigma$."

5.8 Proceeding as in (5.6.1), we obtain, for the three intervals analogous to those in 4.4:

(α) for $U \leqq c(b-1) - x$:

$$\hat{\eta}(U) = c(b-1) - x - U; \qquad V = c(b-1); \qquad Q = 1.$$

(β) for $U \geqq c(b-l) - x$:

$$\hat{\eta}(U) = c(b-l) - x - U; \qquad V = c(b-l); \qquad Q = l.$$

(γ) for $c(b-1) - x < U < c(b-l) - x$:

$$\hat{\eta}(U) = 0; \qquad V = x + U;$$

5.9 Proceeding further, as in (5.6.2), (5.6.3):

$$dEZ/dx = b - 2 - \hat{x}/c + \int_{m_1}^{m_l} [b - (\hat{x} + U)/c]\phi(U)dU$$
$$+ \int_{-\infty}^{m_1} \phi(U)dU + \int_{m_l}^{\infty} l\phi(U)dU = 0,$$

where $m_l = (b-l)c - \hat{x}$, $m_1 = (b-1)c - \hat{x}$.

5.9.1 *Note:* At perfect liquidity $l=1$, the equation in 5.9 yields

$$\hat{x} = (b-1)c,$$

independent of the random shifts, and identical with the solution in 4.4, Case 1.

5.9.2 On Chart IV, the equation in 5.9 is plotted for $\phi(U)$ normal and for $l=0$ and $l=1$: $\hat{x}$ is expressed as a function of σ.

5.10 To find the sign of $\partial\hat{x}/\partial l$, differentiate the equation in 5.9 with respect to l, and write for the cumulative probability function

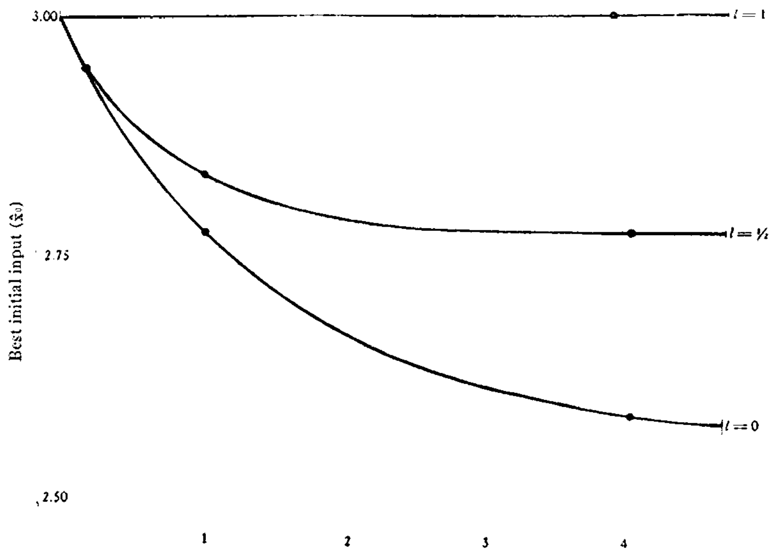
$$\int_{-\infty}^{m} \phi(U)dU = \psi(m) \geqq 0.$$

We obtain

$$(\partial\hat{x}/\partial l) \cdot [1 + \psi(m_l) - \psi(m_1) + c\phi(m_l)] = c\psi(m_l);$$

and since $c>0$, and $m_1 < m_l$, $\psi(m_1) < \psi(m_l)$, $\partial\hat{x}/\partial l \geqq 0$. That is, initial investment increases with liquidity.

CHART IV



Dependence of best initial input ($\hat{x}_0$) upon liquidity (l) of the asset or contract, and upon the known standard deviation (σ) of the probability distribution (known to be normal, with zero mean) of a future shift in the production function.

5.11 It can be further shown that investment $\hat{x}$ decreases with increasing variance of the random shift. By introducing $U^* = U/\sigma$, a random variable with zero mean, unit variance and probability density function $\phi^*(U^*)$, Herman Chernoff has shown that for continuous distribution functions,

$$\frac{\partial\hat{x}}{\partial\sigma} = -\int_{n_1}^{n_l} U^*\phi^*(U^*)dU^* \Big/ \left[1 + \int_{n_1}^{n_l} \phi^*(U^*)dU^*\right] \leqq 0,$$

where $n_i = m_i/\sigma (i=1,\ l)$. (The equality sign applies when $l=1$: see 5.9.1).

5.12 Chernoff has shown, moreover, that investment $\hat{x}$ is bounded as follows:

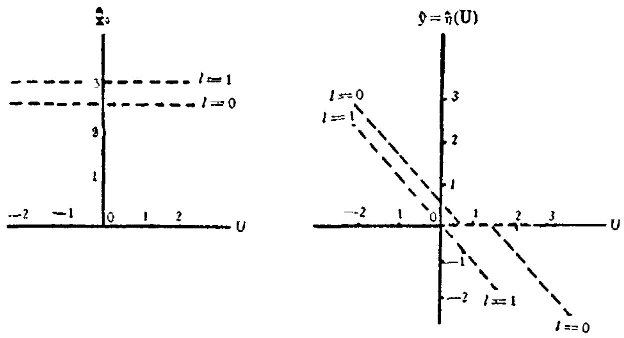
(5.12.1) as $\sigma \rightarrow 0$, $\hat{x} \rightarrow c(b-1)$;

(5.12.2) as $\sigma \rightarrow \infty$, $x \rightarrow c[b - 2 + l + (1-l)\psi^*(0)]$, where $\psi^*(m) = \int_{-\infty}^{m} \psi^*(U^*)dU^*$, so that, for symmetrical functions, $\psi^*(0) = 1/2$.

While (5.12.1) simply repeats the result obtained in the case of certainty (4.4: Case 2, for $u=0$), the result (5.12.2) is interesting: as the variance of the random shift increases indefinitely, investment does not fall to zero; for example, if $l=0$ and the distribution is symmetrical, $\hat{x}$ approaches $(b-3/2)c$. (See Chart IV for the more special case of normal distribution and for $b=4$, $c=1$.)

CHART V

Limiting values of best initial input $\hat{x}_0$ and best change in input $\hat{y} = \hat{\eta}(U)$ as functions of a random shift U, symmetrically distributed with zero mean,* and with standard deviation increasing indefinitely ($\sigma \to \infty$); two alternative degrees of liquidity ($l=0$; 1) of the asset or contract in question are assumed.



* *Note:* Since this chart is constructed for mean $U=0$, it can be compared with the case of certainty ($\sigma=0$) plotted on Chart III, for $u=0$ only. The comparison is as follows:

$\sigma=0;\ 0 \leqq l \leqq 1 \quad : \quad \hat{x}_0 = 3 \quad , \quad \hat{y}=0$

$\sigma \to \infty;\ l=1 \qquad \hat{x}_0 = 3 \quad , \quad \hat{y} = \hat{\eta}(0) = 0$

$\sigma \to \infty;\ l=0 \qquad \hat{x}_0 = 2\frac{1}{2} \quad , \quad \hat{y} = \hat{\eta}(0) = \frac{1}{2}$

6. *The Case of Ignorance*

6.1 We proceed now to the cases of incomplete information, and begin with the simple, non-stochastic one: the case of ignorance (3.4, 3.5.1). Consider two examples:

6.1.1 The firm knows the revenue function $\rho(\ )$, the liquidity l of the asset or contract in question, and a set $[u]$ of values, that the shift u of the revenue function can take; but it has no information from which to estimate probabilities attached to each of these values. It has to choose x, y.

6.1.2 Same as before but, instead of a set $[u]$ of constants, the firm knows that $\phi(U)$, the probability distribution of the random shift U, has zero mean and is normal, and that standard deviation σ is an element of a set $[\sigma]$ of constants.

6.1.3 The two cases 6.1.1 and 6.1.2 can be considered as special cases of the following: The form and all but one parameters of $\phi(U)$ are known. In case 6.1.1, $\sigma=0$, and EU is unknown; in 6.1.2, $EU=0$, and σ is unknown.

6.2 If the unknown conditions (u, σ) were determined by a rational opponent in a game, a rational choice by our firm, in case 6.1.1, would be as was shown by von Neumann and Morgenstern: it would "minimax the loss," i.e., make a choice of x, y such as to obtain:

$$\text{Min}_{x,y}\ \text{Max}_u\ [-z(x, y; u)];$$

and, in case 6.1.2, it would "minimax the expected value of the loss,"

$$\text{Min}_{x,\eta}\ \text{Max}_\sigma\ [-EZ(x, \eta(U); \sigma].$$

6.2.1 In words: In a game, choose a policy (x and y, or x and η) such that, even if the unknown conditions (u or σ) should turn out to be the ones most adverse to this policy, the expected gain (z or EZ) would be at least as large as the expected gain that would result from any other policy, and from conditions most adverse to that other policy.

6.3 If conditions are not determined by an inimical rational opponent, there is no reason for this maxim of behavior. L. J. Savage has suggested (orally) that not the loss but the "regret" (or "miss") be minimaxed. In the case 6.1.1, the "regret," r is the difference between the gain actually obtained and the gain that would be obtained if choice had been made in full knowledge of conditions. We shall work out this case.

6.4 We define regret:

$$\begin{aligned} r &= \text{Max}_{x,y}\ z(x, y; u) - z(x, y; u) = r(x, y, u) \\ &= z(\hat{x}, \hat{y}; u) - z(x, y; u). \end{aligned}$$

The firm chooses x, y so as to achieve

$$\text{Min}_{x,y}\ \text{Max}_{u \text{ in } [u]}\ r(x, y, u) = r(\bar{x}, \bar{y}, \bar{u}), \text{ say.}$$

We are interested in the effect of liquidity l upon best investment, $\bar{x}$. Obviously if $l=1$, or if u cannot take positive values the case is identical with 4.4, Case 1: the best investment in this case will be called $\hat{x}_\alpha$. Consider now the case $l=0$ and show that then the regret will be minimax at a certain value of x, $\bar{x} \leqq x_\alpha$: investment in an illiquid asset is equal to or smaller than that in a liquid one, all other things being equal.

6.4.1 Suppose first that $[u]$ consists of two values only: u_α in the interval (α), and u_β in the interval (β), as defined in 4.4; $u_\alpha \leqq 0$, $u_\beta > 2c$. Then, using the respective optimal inputs (call them $\hat{x}_\alpha$, $\hat{y}_\alpha$; $\hat{x}_\beta$, $\hat{y}_\beta$), given in 4.4, we can compute the corresponding maximum profits, $\hat{z}_\alpha$, $\hat{z}_\beta$, and hence obtain the regrets, $r_\alpha = \hat{z}_\alpha - z_\alpha$, $r_\beta = \hat{z}_\beta - z_\beta$. The difference

$$r_\alpha - r_\beta = u_\alpha - c + \rho(x + y + u_\beta) - \rho(x + y + u_\alpha),$$

is an increasing function of $x+y$. It vanishes when $x+y$ has a certain value, w (say), depending on u_α, u_β. Therefore $r_\alpha >, <, = r_\beta$ according as $x+y >, <, = w$. Minimax regret is therefore found as follows: minimize r_α for the cases $x+y>w$; minimize r_β for the cases $x+y<w$; then choose the smaller of the two minima. Note that r_α can itself have two minima, one at some $y \geqq 0$, and one at some $y<0$; and similarly for r_β. It turns out that all minima considered are either at $x=\hat{x}_\alpha=c(b-1)$ or at $x=\hat{x}_\beta=c(b-2)$. And, depending on the size of the possible shifts u_α, u_β, either $\hat{x}_\alpha$ or $\hat{x}_\beta$ yields the smallest of the minima. Therefore the minimax regret is obtained at an investment value which is $\leqq \hat{x}_\alpha$, and thus best investment under illiquidity is smaller than or equal to the best investment under full liquidity.

6.4.2 This result is easily shown to be true also in the more general case: when $[u]$ contains any number of elements, belonging to all three intervals defined in 4.4: (α), (β), and the intermediate interval (γ). Again, minimax regret will be obtained when investment is either $\hat{x}_\alpha$ or $\hat{x}_\beta$: the possibility of events of intermediate kind (γ) has no effect!

6.4.2.1 It follows that, for example, a rational man ignorant of the probabilities of three alternative events should *not*, in general, apply the rule that each of these events has probability 1/3. Thus the classical doctrine (Laplace) which declared events with unknown probabilities equiprobable cannot be a guide to action.

7. *The General Case of Incomplete Information*

7.1 We now modify the example 6.1.2 as follows:

The firm knows the revenue function ρ_0, the liquidity l of the asset or contract in question, and knows that $\phi\ (U)$, the distribution of random shifts, is normal with zero-mean. In addition, it can obtain, in the year 0, certain data, to be denoted by D_0, that are related to the future random shifts U. For example, the numbers D_0 may be sample previews of crops. Denote the (unknown) joint distribution of D_0, U by $\delta(D_0, U)$. In particular, δ may be characterized by variances and regression coefficients. To choose a policy means to decide how to react to information D_0, i.e., what functions ξ, ϵ will yield optimal initial investment $\hat{X}=\xi(D_0)$ and optimal increment in year 1, $Y=\epsilon(D_0)$, determined on the basis of information in year 0. Applying the principles of 6.4, we have to choose ξ, ϵ so as to obtain

$$\text{Min}_{\xi,\epsilon}\ \text{Max}_\delta\ (\text{Max}_{\xi,\epsilon}\ \text{E}Z - \text{E}Z),$$

where $Z=Z[\xi(D_0), \epsilon(D_0)]$ and $\delta(U, D_0)$ is an unknown joint distribution function.

7.2 Again, it is conjectured that if the optimal decision functions—say, $\hat{\xi}$, $\hat{\epsilon}$—are actually evaluated for our economic examples, they will turn out to depend on liquidity.

7.3 In 7.1, the only data considered were those available at beginning of year 0, D_0. Actually, further data, say D_1, will become available in the course of that year. Then D_1 must be included as argument in the joint distribution function δ and in the decision function ϵ. This will affect the optimal values of ξ, ϵ.

7.3.1 In particular, it will probably turn out that if the liquidity of the asset is low, it is advantageous (i.e., it will "minimax the regret") to have less of it during the year 0, in order to *wait for additional information*. The fact of "sequential information" (3.5.2) is likely to be of importance for the theory of liquidity as was already seen in the (stochastic) case of complete information (5.4.1.5).

7.4 This section (7) merely outlines a program. Its problem is, in fact, identical with the general theory of statistical inference, interpreted as the theory of a choice of action rather than choice of hypotheses. (The statistician is merged with the entrepreneur.) This is the approach of Abraham Wald (continuing the work of J. Neyman and E. Pearson); for most recent writings, see Wald, "Foundations of a General Theory of Sequential Decision Functions," *Econometrica*, 1948; and Arrow, Blackwell, and Girshick: "Bayes and Minimax Solutions of Sequential Decision Problems" (to be published in *Econometrica*).