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THE INDETERMINACY OF ABSOLUTE PRICES IN CLASSICAL ECONOMIC THEORY*

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SUMMARY

CLASSICAL economic theory postulates two parallel dichotomies: the real and monetary sectors of the economy on the one hand, and relative and absolute prices on the other. In the real sector all economic behavior depends solely on relative prices; conversely, once the behavior of the real sector is specified (in the form of demand and supply functions), relative prices are uniquely determined. Similarly, absolute prices play a role in and are determined by the monetary sector alone.

Something is wrong with this neat picture. It lies in the fact that in a monetary economy a bridge is inevitably created between the real and monetary sectors: individuals cannot make decisions in the real sector independently of their decisions in the monetary sector. In particular, the only way people can obtain money is by selling goods; hence the demand for money is identical with the supply of all goods. That is, when people determine how much to supply of every good, they simultaneously determine how much money to demand. Classical economists recognized this dependence, and in fact made use of it. But they overlooked one of its simple implications: If the supply of all goods depends only on relative prices, then, of necessity, the demand for money can depend only on relative prices. Thus absolute prices appear nowhere in the system, and hence obviously cannot be "determined" by it.

In brief, the only way to have behavior in the monetary sector depend on absolute prices is to have these prices appear in the real sector. Conversely, if the real sector depends only on relative prices, then so must the monetary sector. The classical dichotomy is self-contradictory (section 10).

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Once this point is recognized it is immediately seen that the only way to have the system determine absolute prices is to have them appear in the real sector of the economy too. Nor will any patchwork attempt to retain the "main sense" of the dichotomy by introducing just "a few" absolute prices work. For it will be shown that the only way in which all absolute prices can be determined is to have each and every one play a role in the real sector of the economy. The purge of relative prices from the real sector must be complete. *A money economy without a "money illusion" is an impossibility* (sections 11 and 14).

The classical school frequently introduced the assumption of Say's law. This law served to remove other contradictions from the classical system; but it did so at the expense of making it impossible to determine all prices. The reason for this can be seen intuitively in the following way: The meaning of Say's law is that people spend all they receive, regardless of prices. Another way of saying the same thing is that people maintain their money stocks constant regardless of prices. Thus prices play no role in the monetary sector; consequently the monetary sector can have no influence on the determination of prices. This throws the whole burden of determining prices on the real sector alone. But the real sector does not provide enough information to complete this task; at most it can determine all but one of the prices as functions of the remaining one. Hence the assumption of Say's law renders the system incomplete. This difficulty remains even if the real sector depends also on absolute prices (section 13).

From the above it is clear that if absolute prices are to be determinate, the classical system must be modified. However, modifications that do not completely break away from the classical dichotomy are doomed to failure. In particular, it will be shown that Lange's system runs into exactly the same type of difficulties as the classical one and hence is inconsistent (section 12). The only way out of this difficulty is to discard completely the classical dichotomy between the real and monetary sectors, and to recognize that prices are determined in a truly general-equilibrium fashion, by both sectors simultaneously (section 14). Nevertheless, it is possible to reconstitute the classical theory in such a way that the following familiar proposition still holds: An increase in the amount of money will merely cause a proportionate increase in the prices of all commodities, without in any way affecting the demand and supply for these commodities or the rate of interest. (Section 14.)

* * *

I. INTRODUCTION

1. When Walras and Pareto began the task of formalizing economic theory in a rigorous, mathematical structure, they were concerned with assuring an equality between the numbers of variables and equations.

They sought this equality in order to be sure that their systems should be both *complete* (in the sense of determining a specific value for each of the variables) and *consistent* (in the sense that there should exist a set of values for the variables which would satisfy simultaneously all the stipulated equations). To insure the completeness, they specified at least as many equations as unknowns (i.e., the system should not be underdetermined); to insure the consistency, they were careful not to impose more restrictions (in the form of equations) than could be satisfied by the variables (i.e., the system should not be overdetermined). A simple example will help clarify these concepts.

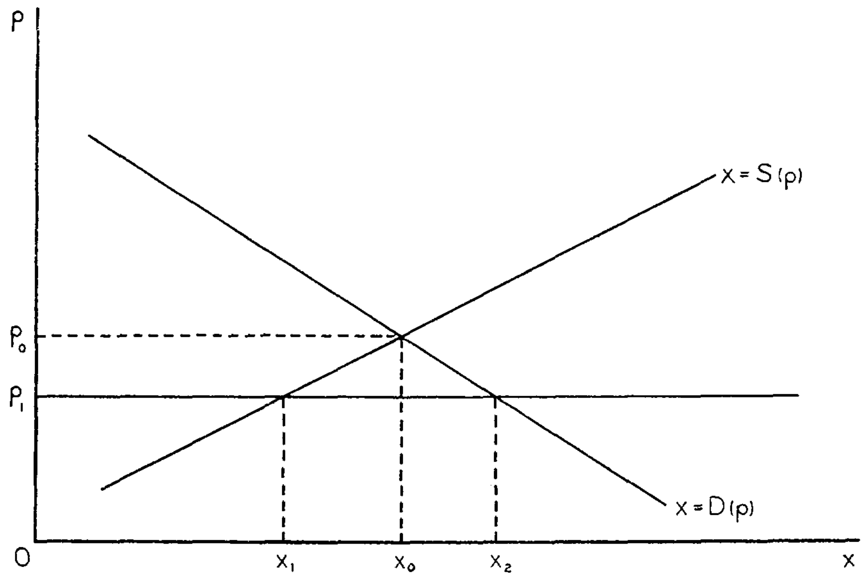

FIGURE 1

Consider the market for sugar, and assume that our theory states that the amount of sugar demanded (x) is a function of its price (p), i.e.,

$$(1.1) \qquad x = D(p).$$

If this were the extent of our theory, we should have an underdetermined system. That is, the price and quantity could correspond to any one of the infinite number of points lying on the curve $D(p)$ in Figure 1. If we went on to postulate a supply function

$$(1.2) \qquad x = S(p),$$

then the price and quantity would definitely be established at (p_0, x_0), since this is the only point satisfying both (1.1) and (1.2). If in addition we should state that the government fixes the price at

$$(1.3) \qquad p = p_1,$$

we should have an inconsistent (overdetermined) system. For by postulating (1.1)–(1.3), we are in effect saying that there exists at least one set of values for p and x that will simultaneously satisfy (1.1)–(1.3); but from Figure 1, it is clear that such a set cannot exist; hence the inconsistency.

2. Unfortunately, Walras and Pareto failed to realize that, within the domain of real numbers, equality in number of variables and equations is neither a necessary nor a sufficient condition for a complete and consistent system. The nonnecessity is proved by the following example of a single equation in two variables:

$$x^2 + y^2 = 0. \tag{2.1}$$

For the domain of real numbers, equation (2.1) uniquely determines $x = 0$ and $y = 0$. Similarly, even if there are more equations than unknowns, the system may still possess a consistent solution. Consider, for example, the following system:

$$\begin{aligned} x^2 - 6x + 9 &= 0, \\ x^3 - 3x - 18 &= 0. \end{aligned} \tag{2.2}$$

Even though the system consists of two equations in only one variable, it nevertheless possesses the consistent solution $x = 3$. Similarly, the insufficiency is proved by the following system

$$\begin{aligned} x^2 + y^2 &= 1, \\ x + y &= 20, \end{aligned} \tag{2.3}$$

which cannot be satisfied for any pair of real values (x, y).

3. There are no simple criteria for determining when a system of equations is consistent. (Suffice it to say that except in the linear case such criteria are not provided by the Jacobians of the system.) In this paper we shall not deal with this general problem. Instead we shall assume throughout that for the systems under study equality in numbers of equations and variables is a necessary, but not sufficient, condition for the existence of a unique solution. In particular, we shall assume that any system considered here with more independent equations than unknowns is inconsistent; and that any system with fewer equations than unknowns cannot yield a unique solution for all its variables.

4. The primary purpose of this essay is to examine critically the "classical system." As in the examination of any school of thought, the problem of textual interpretation immediately arises. To minimize this problem, I shall confine myself to the mathematical economists of this school. These can be classified as follows: (a) those who start from the theory of individual behavior and build up to market relations on this

basis (e.g., Walras, Pareto); (b) those who start from the market relations themselves (e.g., Cassel).

This essay will deal primarily with the classical system as formulated by Cassel;[1] this is the form in which it is most familiar to us now. In addition some criticisms will also be made of modifications that have been proposed for this system.

II. SOME PRELIMINARY THEOREMS

5. In this chapter I shall construct a general model with respect to which models considered in the following chapter can be considered as special cases. The purpose of constructing this model here is to develop several preliminary theorems that are fundamental to the subsequent argument. Specifically, I shall first discuss the relationship between the static models discussed in this essay, and dynamic economic models. This will be followed by an analysis of the relationship between two alternative ways of looking at the demand for money: the demand for money considered as a flow, and the demand for money considered as a stock. It is shown that these two alternatives are really equivalent and consequently can be substituted for one another without affecting the theory. Finally the class of functions known as homogeneous functions is studied, and several fundamental theorems deduced.

6. Consider an isolated economy with n commodities, the nth commodity being paper money. Let p_i = the number of units of money necessary to purchase one unit of the ith commodity, i.e., the price of the ith commodity. Then $p_n = 1$. Assets are assumed to be nonmarketable, with only their services saleable. (I follow Keynes in ignoring the effect of the production of new assets—net investment—on current supply functions.) Let D_i (S_i) represent the demand for (supply of) the ith commodity or service per unit of time. Express D_i and S_i as functions of the prices p_i.

$$D_i = f_i(p_1, p_2, \cdots, p_{n-1}) \qquad (i = 1, \cdots, n), \tag{6.1}$$

$$S_i = g_i(p_1, p_2, \cdots, p_{n-1}) \qquad (i = 1, \cdots, n), \tag{6.2}$$

$$D_i = S_i \qquad (i = 1, \cdots, n). \tag{6.3}$$

These are the demand functions, supply functions, and equilibrium conditions, respectively. (6.1), (6.2), (6.3) each consist of n equations.

[1] The examination of the Walras-Pareto system is presented in an earlier article. "Relative Prices, Say's Law, and the Demand for Money," ECONOMETRICA, Vol., 16, April, 1948, pp. 135–154. This will be referred to henceforth as "The Demand for Money." The present paper is referred to in the earlier one by the title "The Indeterminacy of Absolute Prices in the Casselian System." This original title has since been changed to the one now appearing at the head of this article.

Thus there is a total of $3n$ equations in the $3n - 1$ variables: $p_i(i = 1, \cdots, n - 1)$, $D_i(i = 1, \cdots, n)$, $S_i(i = 1, \cdots, n)$. However, not all the equations are independent. Following Lange[2] we note that within this system the only way people can acquire money is by supplying commodities; and the only way to dispose of money is by demanding commodities. Thus the demand for money per unit of time is identically equal to the aggregate money value of all goods supplied during the period: when people determine how much to supply of every good at different prices and incomes, they simultaneously determine how much money to acquire at different prices. A corresponding statement holds for the supply of money. Thus we have for the demand and supply of money, respectively,

$$(6.4) \qquad D_n \equiv f_n(p_1, p_2, \cdots, p_{n-1}) \equiv \sum_{i=1}^{n-1} p_i g_i(p_1, p_2, \cdots, p_{n-1})$$

identically in the p_i, and

$$(6.5) \qquad S_n \equiv g_n(p_1, p_2, \cdots, p_{n-1}) \equiv \sum_{i=1}^{n-1} p_i f_i(p_1, p_2, \cdots, p_{n-1})$$

identically in the p_i. Subtracting (6.5) from (6.4) we have what Lange has called "Walras' law":

$$(6.6) \qquad D_n - S_n \equiv \sum_{i=1}^{n-1} p_i(S_i - D_i).$$

Thus in (6.3), if $D_i = S_i$ is satisfied for $i = 1, 2, \cdots, n - 1$, then it follows from (6.6) that the equation $D_n = S_n$ is simultaneously satisfied and is not an additional restriction. Assume that the remaining equations are independent, so that there are $(n - 1)$ independent equations in (6.3). For the moment I shall assume that as a result of this dependence any one equation of (6.3) can be dropped. (I shall return to this assumption below in section 12.) We have then $3n - 1$ independent equations in $3n - 1$ variables. The system (6.1)–(6.3) thus enables us to solve for the quantities D_i and S_i, and the absolute prices p_i. Insofar as the "counting" criterion applies, our general model is thus exactly determinate.

7. Before continuing with the analysis of (6.1)–(6.4), it will be necessary to deal (in this and the two following sections) with several preliminary points.

First I must make clear in what sense the system (6.1)–(6.3) [and

[2] Oscar Lange, "Say's Law: A Restatement and Criticism," *Studies in Mathematical Economics and Econometrics*, ed. by O. Lange, F. McIntyre, and T. O. Yntema, Chicago, University of Chicago Press, 1942, pp. 49–69.

especially (6.3)] is supposed to hold. The system (6.1)–(6.3) is a static model; but it is directly related to a dynamic one. Assume that individuals' plans (decisions) are made not continuously, but at discrete points of time, with the intention of carrying out the plan during an ensuing finite time interval. Call this time interval a "period." Then at the beginning of the period, individuals plan their demand and supply flows (D_i and S_i) for the entire period. The system (6.1)–(6.3) can then be considered as the limiting position of that dynamic model which has become familiar through the work of Samuelson and Lange.[3] This dynamic model is identical with (6.1)–(6.3) except that it replaces the $n - 1$ independent equations of (6.3) by the dynamic market-adjusting equations

$$\frac{dp_i}{dt} = H_i(D_i - S_i), \tag{7.1}$$

where

$$\text{sign } \frac{dp_i}{dt} = \text{sign } (D_i - S_i), \tag{7.2}$$

i.e., price rises with excess demand and falls with excess supply. As long as (6.3) is not satisfied for all commodities, we see from (7.2) that the system will not be in stationary equilibrium but will continue to fluctuate. Thus the existence of a solution to the static system (6.1)–(6.3) is a necessary condition for the existence of a stationary solution for the dynamic system (6.1)–(6.2), (7.1). Throughout this paper, all models in which the equality of demand and supply for different commodities is postulated must be understood in this same sense: viz, that the existence of a solution to the postulated static model is a necessary condition for the existence of a stationary solution for the dynamic system underlying the static model.

8. As can be seen from the beginning of the previous section, the demand and supply for money, D_n and S_n, are considered in the sense of *flows* per planning period. D_n is the planned flow of money in exchange for commodities supplied (i.e., planned receipts), and S_n the planned flow in exchange for commodities demanded (i.e., planned expenditures). It is more usual in economic theory to discuss the demand for and supply of money as a *stock*. From this viewpoint, instead of assuming the individual to plan, at the beginning of the period, the *flows* of money

[3] P. A. Samuelson, "The Stability of Equilibrium: Comparative Statics and Dynamics," ECONOMETRICA, Vol. 9, April, 1941, pp. 97–120. Oscar Lange, *Price Flexibility and Employment*, Bloomington, Indiana, Principia Press, 1944, pp. 91 ff.

during the ensuing period, assume that he plans at the beginning of the period the stock of money (cash balances) he will hold at the end of the period. Denote this by M_D. Similarly, those economic units (e.g., government and banks) which control the stock of money in existence are also assumed to plan at the beginning of the period the stock they will supply (M_S) at the end—i.e., the amount of money they will permit in circulation. Assume M_D and M_S to be functions of prices. We have then

$$M_D = F_n(p_1, p_2, \cdots, p_{n-1}), \tag{8.1}$$

$$M_S = G_n(p_1, p_2, \cdots, p_{n-1}). \tag{8.2}$$

Divide the economy into two sectors: the private P sector; and the bank-government B sector. Denote the planned demand of the P sector for cash balances by M_D^P, the demand flow planned by the B sector by D_n^B, etc. Assume that the B sector has no demand for cash balances; therefore

$$M_D = M_D^P + M_D^B = M_D^P. \tag{8.3}$$

Let M_S^0 be the supply of cash balances at the beginning of the period and M_S the supply planned for the end of the period. Then we have the following relationships:

$$M_D \equiv M_S^0 + D_n^P - S_n^P \quad \text{identically in the } p_i, \tag{8.4}$$

$$M_S - M_S^0 \equiv S_n^B - D_n^B \quad \text{identically in the } p_i. \tag{8.5}$$

The first equation states that the planned excess of inflow over outflow of the private sector ($D_n^P - S_n^P$) is identically equal to the planned in crease in cash balances of the P sector ($M_D - M_S^0$). The second states that the B sector plans to increase (decrease) the amount of money in circulation (cash balances) by planning a greater injection (withdrawal) of money into the economy than a withdrawal (injection). Subtracting (8.5) from (8.4) we have

$$M_D - M_S \equiv D_n - S_n \quad \text{identically in the } p_i, \tag{8.6}$$

where

$$D_n = D_n^P + D_n^B \tag{8.7}$$

and

$$S_n = S_n^P + S_n^B. \tag{8.8}$$

Denote the left-hand member of (8.6) by M_x and the right-hand by X_n. Then we can write

$$M_x \equiv X_n \quad \text{identically in the } p_i. \tag{8.9}$$

The equation (8.9) relates the demand for and supply of money considered as a stock and the demand for and supply of money considered as a flow. By virtue of (8.9) we can replace, in (6.3), the equation

$$X_n = 0 \tag{8.10}$$

by the equivalent restriction

$$M_x = 0. \tag{8.11}$$

In the subsequent exposition we shall make frequent use of this substitution.

9. In economic theory frequent use is made of a class of functions known as "homogeneous functions." In this section I shall develop certain properties of these functions which will subsequently prove useful.[4]

A function

$$w = f(x_1, x_2, \cdots, x_m, y_1, \cdots, y_n) \tag{9.1}$$

is homogeneous of degree t in $x_1, x_2, \cdots, x_m$ if

$$\begin{aligned} &f(\lambda x_1, \lambda x_2, \cdots, \lambda x_m, y_1, \cdots, y_n) \\ &\qquad \equiv \lambda^t f(x_1, \cdots, x_m, y_1, \cdots, y_n) \end{aligned} \tag{9.2}$$

identically in the x_i, y_j, and λ, where λ may be any number. Putting in particular $\lambda = 1/x_m$ and substituting in (9.2) we get

$$\begin{aligned} &f\left(\frac{x_1}{x_m}, \cdots, \frac{x_{m-1}}{x_m}, 1, y_1, \cdots, y_n\right) \\ &\qquad = \frac{1}{x_m^t} f(x_1, \cdots, x_m, y_1, \cdots, y_n). \end{aligned} \tag{9.3}$$

Consider now the system of $n + m$ independent equations in $n + m$ variables

$$\begin{aligned} &f_i(x_1, \cdots, x_m, y_1, \cdots, y_n) = 0 \\ &\qquad (i = 1, \cdots, m, m + 1, \cdots, m + n), \end{aligned} \tag{9.4}$$

[4] The development in this section follows more or less that of an unpublished note on homogeneous functions, prepared by Leonid Hurwicz and circulated among members of the Cowles Commission in June, 1945. [The specialization consists in assuming *t constant*. If t is considered as any function of the variables involved, no special "class" of functions is segregated. EDITOR.]

where f_i is homogeneous of degree t_i in the variables $x_1, \cdots, x_m$. By (9.3) we can then write

$$f_i\left(\frac{x_1}{x_m}, \cdots, \frac{x_{m-1}}{x_m}, 1, y_1, \cdots, y_n\right) \equiv g_i(z_1, \cdots, z_{m-1}, y_1, \cdots, y_n) = 0 \qquad (i = 1, \cdots, m+n), \tag{9.5}$$

where

$$z_j = \frac{x_j}{x_m} \qquad (j = 1, \cdots, m-1). \tag{9.6}$$

Thus the system of equations $g_i = 0$ consists of $m + n$ equations in $m + n - 1$ variables and is therefore inconsistent (cf. above, section 2). We have proved the following

THEOREM: *If every equation of a system of K independent equations in K variables is homogeneous of some degree t in the same set of variables, then the system possesses no solution (i.e., it is inconsistent), with the possible exception of the one which sets each of the variables equal to zero.*[4a]

The theorem obviously holds when every equation is homogeneous in all the variables. However, if every equation is homogeneous, but not in the same set of variables, the theorem does not hold. Thus, for example, the system of independent equations

$$h_i(x, y, z) = 0 \qquad (i = 1, 2, 3) \tag{9.7}$$

(where h_1 and h_2 are homogeneous of degree t_1 and t_2 (respectively) in x and y, and h_3 is homogeneous of degree t_3 in x, y, and z) is not overdetermined. For by (9.3) the system (9.7) can be rewritten as

$$\begin{aligned} q_1\left(\frac{x}{y}, z\right) &= 0, \\ q_2\left(\frac{x}{y}, z\right) &= 0, \\ q_3\left(\frac{x}{y}, \frac{z}{y}\right) &= 0. \end{aligned} \tag{9.8}$$

The first two equations determine x/y and z; the last determines z/y. From these we can then derive the values for x and y separately.

[4a] The zero solution will definitely hold in the case $t > 0$, and might possibly hold in the case $t = 0$. Since the zero solution is economically unimportant, we shall disregard it in the future discussion.

I am indebted to Professor Ragnar Frisch for pointing out the necessity of adding the qualifying phrase at the end of this theorem.

As a corollary of this theorem note that if we have $n + m - 1$ independent equations

$$\phi_j(x_1, \cdots, x_m, y_1, \cdots, y_n) = 0 \quad (j = 1, \cdots, m, m + 1, \cdots, m + n - 1), \tag{9.9}$$

where ϕ_j is homogeneous in $x_1, \cdots, x_m$ of degree t_j, then (9.9) can be solved for $z_r = x_r/x_m$ $(r = 1, \cdots, m - 1)$ and the y_i $(i = 1, \cdots, n)$.

Finally, we should note that a linear combination of any finite number of functions, each homogeneous of the tth degree in the same variables, is homogeneous of the tth degree in the same variables. That is, if

$$w_j = w_j(x_1, \cdots, x_m, y_1, \cdots, y_n) \qquad (j = 1, \cdots, k) \tag{9.10}$$

is homogeneous of degree t in $x_1, \cdots, x_m$, then $W = \sum_{j=1}^{k} \alpha_j w_j$ (where the α_j are some designated constants) is also homogeneous of degree t in the same variables; for

$$\begin{aligned} W(\lambda x_1, \cdots, \lambda x_m, y_1, \cdots, y_n) &\equiv \sum_{j=1}^{k} \alpha_j w_j(\lambda x_1, \cdots, \lambda x_m, y_1, \cdots, y_n). \\ &\equiv \lambda^t \sum_{j=1}^{k} \alpha_j w_j(x_1, \cdots, x_m, y_1, \cdots, y_n) \equiv \lambda^t W. \end{aligned} \tag{9.11}$$

III. ANALYSIS OF THE CLASSICAL SYSTEM

10. This chapter begins with an examination of the classical system as presented by Cassel and demonstrates its inconsistency. This inconsistency is shown to be due to the traditional assumption that the demand for goods depends only on relative prices. The demonstration is then generalized to prove that even when some absolute prices enter the demand functions, the system may still be inconsistent. A modified classical system proposed by Lange is then considered and its inconsistency shown. Finally, the role of Say's law is examined, and its effect on the consistency of the system described.

In contrast with Walras and Pareto, Cassel does not concern himself with microeconomic analysis. Instead his system consists of equations of the form (6.1)–(6.3). (For simplicity, I consider only his analysis of an exchange economy.) The "classical" element is introduced into this system by his particular assumptions about the properties of the functions (6.1)–(6.2). These stipulated properties (supposedly holding for a paper-money economy) are that the actual values of D_i and $S_i(i =$

$1, \cdots, n-1$) depend only on relative prices, and are independent of the absolute price level.[5] Whether the latter is 100 or 200 should not affect the working of the economy. In other words, the functions (6.1)–(6.2) (for $i = 1, \cdots, n-1$) are homogeneous of degree 0: instead of depending on the absolute values of the $(n-1)$ variables $p_1, p_2, \cdots, p_{n-1}$, they depend only on the $(n-2)$ ratios $p_1/p_{n-1}, \cdots, p_{n-2}/p_{n-1}$. Thus when each of the $n-1$ variables is changed in the same proportion μ, so that the general price level is also changed in the proportion μ, the relative prices (and therefore D_i and S_i for $i = 1, \cdots, n-1$) remain the same. This follows directly from (9.2) by setting $t = 0$.

The classical analysis assumed that the equation dropped in (6.3) [by virtue of the interdependence shown by (6.6)] was one of those referring to commodities (as distinct from money), say the equation for $i = 1$. This left $3n - 4$ commodity equations determining the $3n - 4$ variables D_i, S_i ($i = 1, \cdots, n-1$), and p_j/p_{n-1} ($j = 1, \cdots, n-2$).[6]

[5] Gustav Cassel, *The Theory of Social Economy*, translated by S. L. Barron, new revised edition, New York, Harcourt, Brace and Co., 1932, pp. 154–55: "It is clear that the functions [D_i and S_i] . . . will remain unchanged if all the [p_i] expressed in the money unit are multiplied by any multiplier whatever. . . . The demand can only be determined by the relative prices."

Other mathematical economists make similar statements. Thus F. Divisia, *Economique rationelle*, Paris, Gaston Doin et Cie, 1928, pp. 413–416, makes this assumption most explicitly. Walras is less clear. He concludes that the D_i and S_i are functions of the absolute price level. But he maintains that this dependence is very weak ("Ils n'en dépendent que très indirectement et très faiblement") and may be disregarded, so that the general price level is determined apart from the demand and supply equations for commodities. ("En ce sens il s'en faut de peu que l'équation de la circulation monétaire [which determines the general price level], dans le cas d'une monnaie non marchandise, ne soit en réalité extérieure au système des équations de l'équilibre économique.") [L. Walras, *Elements d'économie politique pure*, definitive edition, Paris, Pichon et Durand-Auzias, 1926, p. 311.] (Walras's logic is really incorrect, as was pointed out in "The Demand for Money," (Section 3).)

Cf. also K. Wicksell, *Lectures on Political Economy*, translated by E. Classen and edited with an introduction by L. Robbins, London, George Rutledge and Sons, Ltd., 1934, I, pp. 65–68: ". . . all the quantities involved can . . . be expressed in terms of the n-1 relative prices of the commodities." By these prices are meant "the $n-1$ ratios between the money prices of the n commodities." [Wicksell is dealing with a case where there are a total of $n+1$ commodities. The $(n+1)$th being money.]

I must emphasize that what these classical economists mean by relative price are the $n-2$ ratios p_j/p_{n-1} ($j = 1, \ldots, n-2$), and *not* the $n-1$ "ratios" p_i/p_n ($i = 1, \ldots, n-1$). Since p_n is by definition equal to unity, these latter are *absolute* prices. The last quotation from Wicksell (recalling that he is concerned with a system of $n+1$ commodities) should make this clear.

[6] Cf., for example, F. Modigliani, "Liquidity Preference and the Theory of Interest and Money," Econometrica, Vol. 12, January, 1944, pp. 45–88, esp. p. 69.

Thus the quantities of goods bought and sold were determined in the "real" part of the model, independently of what happened in the money market. The equations referring to the money market were used only to determine the absolute level of prices. In reality, the theory discussed the demand and supply for money in terms of stocks, and assumed (using the notation of section 8)

$$M_S = M = \text{const}, \tag{10.1}$$

$$M_D = K \sum_{i=1}^{n-1} p_i S_i = p_{n-1} K \sum_{i=1}^{n-1} \frac{p_i}{p_{n-1}} S_i, \tag{10.2}$$

$$M_S = M_D. \tag{10.3}$$

Therefore

$$M - p_{n-1} K \sum_{i=1}^{n-1} \frac{p_i}{p_{n-1}} S_i = 0. \tag{10.4}$$

This is essentially the familiar "Cambridge equation" for the demand for cash balances, where K is an institutionally determined constant. The p_i/p_{n-1} and S_i being given by the "real" part of the system, (10.4) determined p_{n-1} and thereby all the absolute prices.[7]

Despite its apparent elegance, the preceding theory involves logical contradictions on several scores. We will discuss two of these here. In the following section we will discuss a third.

(a) In (6.4) and (6.5) make the classical assumption that the f_i and g_i ($i = 1, \cdots, n-1$) are homogeneous of degree 0 in all the variables. Then

$$\begin{aligned} f_n(\lambda p_1, \cdots, \lambda p_{n-1}) &\equiv \sum_{i=1}^{n-1} \lambda p_i g_i(\lambda p_1, \cdots, \lambda p_{n-1}) \\ &\equiv \lambda \sum_{i=1}^{n-1} p_i g_i(p_1, \cdots, p_{n-1}) \\ &\equiv \lambda f_n(p_1, \cdots, p_{n-1}) \end{aligned} \tag{10.5}$$

identically in λ and the p_i. A similar statement holds for g_n. We have thus proved the following

[7] Cassel, *op. cit.*, pp. 454–459. Cf. also F. Divisia, *op. cit.*, ch. 19; Walras, *op. cit.*, pp. 302–312. The interpretation of Walras as supporting a cash-balance equation follows A. W. Marget, "Leon Walras and the 'Cash Balance Approach' to the Problem of the Value of Money," *Journal of Political Economy*, Vol. 39, 1931, 569–600. Note again that in order to fit Walras' analysis into the mold of the classical analysis as presented here, we must apply rigorously his assumption that the dependence of the S_i on absolute prices can be ignored. Cf. above footnote 5.

LEMMA: *If the g_i $(f_i)(i = 1, \cdots, n - 1)$ are homogeneous of degree 0 in all the variables, then f_n (g_n) is homogeneous of degree 1 in the same variables. More generally, if the g_i $(f_i)(i = 1, \cdots, n - 1)$ are homogeneous of degree t in all the variables, then f_n (g_n) is homogeneous of degree $t + 1$ in the same variables.*

Note, however, that if each of the functions is homogeneous of the tth degree in the same *subset* of variables, then f_n (g_n) is not necessarily homogeneous of any degree in any of the variables. For example assume the $g_i(i = 1, \cdots, n - 1)$ to be homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$. Then, if $\lambda \neq 1$,

$$f_n(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1}) \equiv \sum_{j=1}^{n-2} \lambda p_j g_j(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1})$$

$$+ p_{n-1} g_{n-1}(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1}) \equiv \sum_{j=1}^{n-2} \lambda p_j g_j(p_1, \cdots, p_{n-1}) \tag{10.6}$$

$$+ p_{n-1} g_{n-1}(p_1, \cdots, p_{n-1}) \not\equiv \lambda f_n(p_1, \cdots, p_{n-1}).$$

A similar statement holds for $g_n(p_1, \cdots, p_{n-1})$.

For convenience form the excess-demand functions

$$X_i(p_1, \cdots, p_{n-1}) = f_i(p_1, \cdots, p_{n-1}) - g_i(p_1, \cdots, p_{n-1}) \tag{10.7}$$

and rewrite (6.1)–(6.3) as

$$X_i(p_1, \cdots, p_{n-1}) = 0 \qquad (i = 1, \cdots, n). \tag{10.8}$$

Employing the classical homogeneity assumption, the preceding lemma, and the last paragraph of section 9, the first $n - 1$ of these equations are homogeneous of degree 0 in all the variables, and the last equation homogeneous of degree 1 in all the variables. Thus no matter what equation of (10.8) we drop (by virtue of their interdependence) we are left with $(n - 1)$ independent equations in $(n - 1)$ variables, where each of the equations is homogeneous in all the variables. By virtue of the theorem proved in section 9 and the preceding lemma we can then state the following

THEOREM: *If the f_i and g_i $(i = 1, \cdots, n - 1)$ in (6.1) and (6.2) are independent and homogeneous of degree t' in all the variables, then the system (6.1)–(6.3) is overdetermined. In particular, the Casselian system (6.1)–(6.3) is inconsistent.*

An additional word of explanation is in place with reference to this last theorem. There is a belief that Wald has proved the consistency of the Casselian system under certain specified assumptions as to the properties of the demand functions.[8] Undoubtedly Wald does prove the consistency of the system he considers; *but this system is not the Casselian system.* Specifically, in the system considered by Wald, *the assumption is tacitly made that the demand functions are not homogeneous of degree* 0 *in* $p_1, \cdots, p_{n-1}$. Wald's system is stated in terms of the inverse functions of f_i (i.e., price as a function of quantities); and a necessary condition for the existence of these inverse functions is that the functions f_i $(i = 1, \cdots, n - 1)$ not be homogeneous of degree 0 in $p_1, \cdots, p_{n-1}$. This can easily be proved as follows:

A necessary condition for the existence of the inverses of $f_i(i = 1, \cdots, n - 1)$ is that the Jacobian

$$\begin{vmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n-1} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n-1} \\ \cdot\cdot & \cdots & & \cdot\cdot \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n-1} \end{vmatrix} \tag{10.9}$$

(where $f_{i,j}$ is the partial derivative of f_i with respect to its jth argument) should not vanish identically. Now assume the f_i to be homogeneous of degree 0 in $p_1, \cdots, p_{n-1}$. Multiply the ith column of (10.9) by p_i $(i = 1, \cdots, n - 1)$ and add it to the last column. The determinant (10.9) then becomes

$$\begin{vmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n-2} & \sum_{r=1}^{n-1} p_r f_{1,r} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n-2} & \sum_{r=1}^{n-1} p_r f_{2,r} \\ \cdot\cdot & & \cdots\cdots & & \cdot\cdot \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n-2} & \sum_{r=1}^{n-1} p_r f_{n-1,r} \end{vmatrix}. \tag{10.10}$$

But by Euler's theorem on homogeneous functions we have

$$\sum_{r=1}^{n-1} p_r f_{i,r} \equiv 0 \qquad (i = 1, \cdots, n - 1) \tag{10.11}$$

identically in the p_r. Consequently the last column of (10.10) becomes zero, and the Jacobian (10.9) vanishes identically. This proves that a necessary condition for the existence of the inverse function of f_i $(i =$

[8] A. Wald, "Ueber die eindeutige positive Lösbarkeit der neuen Produktionsgleichungen," *Ergebnisse eines mathematischen Kolloquiums* (edited by Karl Menger), Vol. 6, 1933–34, pp. 12–20; "Ueber die Produktionsgleichungen der ökonomischen Wertlehre," *ibid.*, Vol. 7, 1934–35, pp. 1–6; "Ueber eines Gleichungssysteme der mathematischen Oekonomie," *Zeitschrift fur Nationalökonomie*, Vol. 7, 1936, pp. 637–670.

$1, \cdots, n-1$), is that the f_i not be homogeneous of degree 0 in $p_1, \cdots, p_{n-1}$.

(b) From (a) we have seen that under the classical homogeneity assumptions, the equation

$$X_n(p_1, \cdots, p_{n-1}) = 0 \tag{10.12}$$

is homogeneous of degree 1. But by (8.9)

$$M_X \equiv X_n . \tag{10.13}$$

Thus the classical homogeneity assumptions imply that M_X is homogeneous of degree 1. That is, (10.4) must be homogeneous of degree 1. But this is impossible if K and M are constants. Thus we have shown that *the classical homogeneity assumption is logically inconsistent with the classical monetary equation.*

11. The discussion in the last section has shown that the overdeterminacy of (10.8) [and consequently of (6.1)–(6.3)] is due to the fact that the homogeneity of the first $n-1$ equations implies the homogeneity of the nth. Let us now consider a situation in which this will not be true. Specifically, assume that the X_i ($i = 1, \cdots, n-1$) are homogeneous of degree 0 in all the prices but one, say p_1. Then by (10.6) we see that X_n is not homogeneous. Therefore the theorem of section 9 does not apply. Nevertheless I shall show that even under these assumptions the system (10.8) will not in general possess a solution.

For convenience rewrite (10.8) as

$$X_j(p_1, \cdots, p_{n-1}) = 0 \quad (j = 1, \cdots, n-2), \tag{11.1}$$

$$X_{n-1}(p_1, \cdots, p_{n-1}) = 0, \tag{11.2}$$

$$X_n(p_1, \cdots, p_{n-1}) = 0, \tag{11.3}$$

where the X_j and X_{n-1} are homogeneous of degree 0 in $p_2, \cdots, p_{n-1}$. The approach of, for example, Modigliani[9] is to deal with the system (11.1)–(11.3) as follows: "The excess demand function to be eliminated, by virtue of the interdependence, is arbitrary; we may, if we choose, eliminate one of the $n-1$ referring to commodities, say $X_{n-1} = 0$; we are then left with $n-2$ commodity equations (11.1) to determine the $n-3$ price ratios p_i/p_{n-1}, $i = 2, \cdots, n-2$, and p_1. To determine the absolute prices we use (11.3) as was done with (10.4)."

[9] Modigliani, *op. cit.*, pp. 68–70. In the next sentence of the text I have paraphrased Modigliani's words to fit in with the models set forth here. Otherwise there is no change from his original statement. In reality, Modigliani's analysis is incorrect even without the analysis of this section; for he assumes that the first $n-1$ equations are homogeneous of degree 0 in *all* the variables (*op. cit.*, p. 68, especially footnote 24).

The procedure seems straightforward. But an obvious hint of an inconsistency somewhere in the argument follows from the fact that if the dropping of the extra equation is as symmetrical a process as Modigliani implies, we should get the same results no matter which equation we choose. But if we drop (11.3) instead of (11.2) we are left with $n - 1$ homogeneous equations in $n - 1$ variables, which (by section 9) are overdetermined and thus in general do not possess a consistent solution!

Let us analyze in detail what is involved in the process of dropping (11.2). Consider first (6.6) which we rewrite [using (10.7)] as

$$(11.4) \qquad - X_n(p_1, \cdots, p_{n-1}) \equiv \sum_{i=1}^{n-1} p_i X_i(p_1, \cdots, p_{n-1})$$

identically in the p_i. From the system (11.1), (11.3) (which is the system Modigliani considers) we obtain solutions for the p_i $(i = 1, \cdots, n - 1)$ which when substituted in (11.4) yield

$$(11.5) \qquad p_{n-1} X_{n-1}(p_1, \cdots, p_{n-1}) = 0.$$

From this Modigliani concludes that (11.2) is satisfied with the same set of values p_i $(i = 1, \cdots, n - 1)$ obtained by solving (11.1), (11.3). But (11.5) implies (11.2) *only if* $p_{n-1} \neq 0$. It will, of course, be answered that since this is an economics problem, we can assume that all the $p_i \neq 0$. Thus implicit in Modigliani's procedure is the asumption that *in general the system (11.1), (11.3) has a solution with* $p_{n-1} \neq 0$. I shall show that *in general this assumption is not true.*

Consider the systems (11.1), (11.2) and (11.1), (11.3). If the system (11.1), (11.3) has a solution with $p_{n-1} \neq 0$, then from (11.5) we see that (11.2) is also satisfied so that the system (11.1), (11.2) has a solution. Thus we have proved

THEOREM: *A necessary condition for (11.1), (11.3) to have a solution with* $p_{n-1} \neq 0$, *is that (11.1), (11.2) have a solution.*

Now since (11.1), (11.2) is a system of $(n - 1)$ homogeneous equations in $(n - 1)$ variables it will in general (by section 9) beinconsistent and not possess a solution. Therefore *in general* (11.1), (11.3) does not possess a solution with $p_{n-1} \neq 0$. That is, in general, the solution (if any) of (11.1), (11.3) will yield $p_{n-1} = 0$, and therefore [cf. (11.5)] the solution of (11.1), (11.3) will *not* in general satisfy (11.2).[10]

The reader can immediately generalize these results to the following

THEOREM: *If in (11.1)–(11.3), each of the* X_i $(i = 1, \cdots, n - 1)$ *is independent and homogeneous of degree t in the same subset of* $2 \leqslant m \leqslant n - 1$ *price variables, then the system (11.1)–(11.3) will in general be overdetermined.*

[10] I am indebted to D. Zelinsky (Institute for Advanced Studies, Princeton) for his advice on formulating the preceding theorem rigorously.

These results are also useful as a guide to general procedure. For note that any solution of (11.1)–(11.2) will always (regardless of the values of the p_i) satisfy (11.3). This is true because X_n enters (11.4) without a price coefficient. Thus, as a general rule, in dropping the equation due to the interdependence shown by (11.4), we should always first drop the excess-demand equation for money. Then we should examine the remaining equations. If, in general, they have a solution, then (and only then) we can reinstate the money equation and drop any other equation we might desire without affecting the results.

12. I shall now consider another set of assumptions about the system (10.8) and examine it for consistency. These assumptions (and the resulting system) are the ones set forth by Oscar Lange in his recent book.[11]

In the beginning of section 6 I assumed that no assets could be bought or sold. Assume now that there does exist one (and only one) asset that is marketable. Let this asset be bonds that are perpetuities paying one dollar per period. Represent these bonds by the $(n - 1)$th commodity. Thus bonds and money are assumed to be the only nonphysical assets. Abstracting from uncertainty, and assuming all bonds to be identical we have

$$p_{n-1} = \sum_{t=1}^{\infty} \frac{1}{(1 + r)^t} = \frac{1}{r} , \tag{12.1}$$

where r is the rate of interest. Thus the price of bonds, p_{n-1}, is the reciprocal of the rate of interest. Our commodities are then divided into three types: goods, bonds, and money. Assume with Lange[12] that the excess demand functions for goods are homogeneous of degree 0 only in the first $n - 2$ prices—i.e., in all prices except the rate of interest. Consider now the demand for bonds. Assume that a given individual possesses q bonds, each paying \$1 per period. The real value of this payment measured in terms of the ith good is $\$q/p_i$. If all prices increase by the proportion ϵ, then the real value of $\$q$ in terms of the ith good decreases to $1/(1 + \epsilon)$ of its former value. Since prices of all goods are assumed to increase in the same proportion ϵ, the real value of $\$q$ in terms of any good decreases to the same extent. Assume again with Lange[13] that the individual will then increase the number of bonds he holds in the proportion ϵ so that the real value of the bond yield after the price rise will be the same as it was originally, i.e., $\$q(1 + \epsilon)/p_i(1 + \epsilon) = \$q/p_i$.

[11] Lange, *Price Flexibility and Employment*.

[12] *Ibid.*, ch. 3.

[13] *Ibid.*, pp. 15–16, especially footnote 6, p. 16. Lange's mathematical argument in this footnote is really inconsistent with the assumptions of the rest of his book. Cf. "The Demand for Money," footnote 22.

This implies that the excess demand for bonds is homogeneous of degree 1 in the prices $p_1, \cdots, p_{n-2}$.

For convenience, we can again consider the system (11.1)-(11.3) where now the X_j are homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$, and X_{n-1} is homogeneous of degree 1 in the same variables. How do these assumptions affect X_n? From (11.4) we have

$$\begin{aligned} -\ X_n(p_1, \cdots, p_{n-1}) & \\ (12.2) \qquad &\equiv \sum_{i=1}^{n-1} p_i X_i(p_1, \cdots, p_{n-1}) \\ &\equiv \sum_{j=1}^{n-2} p_j X_j(p_1, \cdots, p_{n-1}) + p_{n-1} X_{n-1}(p_1, \cdots, p_{n-1}) \end{aligned}$$

identically in the p_i. Consequently

$$\begin{aligned} -\ X_n(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1}) & \\ (12.3) \qquad &\equiv \sum_{j=1}^{n-2} \lambda p_j X_j(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1}) \\ &\quad + p_{n-1} X_{n-1}(\lambda p_1, \cdots, \lambda p_{n-2}, p_{n-1}). \end{aligned}$$

Since the X_j are homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$, and X_{n-1} is homogeneous of degree 1 in these same variables, this reduces to

$$\begin{aligned} (12.4) \quad \lambda\left[\sum_{j=1}^{n-2} p_j X_j(p_1, \cdots, p_{n-2}, p_{n-1}) + p_{n-1} X_{n-1}(p_1, \cdots, p_{n-2}, p_{n-1})\right] & \\ \equiv \lambda[-\ X_n(p_1, \cdots, p_{n-2}, p_{n-1}) & \end{aligned}$$

identically in λ and the p_i. Therefore X_n is homogeneous of degree 1 in the variables $p_1, \cdots, p_{n-2}$. This is a generalization of the lemma proved in part (a) of section 10.[14]

Thus Lange's assumptions imply that the X_j are homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$, and X_{n-1} and X_n are homogeneous of degree 1 in the same variables. Then we can immediately apply the overdeterminacy theorem of section 9; for no matter which equation in (11.1)-(11.3) is eliminated by virtue of the interdependence between them, we are still left with $(n - 1)$ equations homogeneous of some degree in the same subset of variables. Consequently Lange's system is overdetermined.[15]

[14] Cf. Lange, pp. 99-100. The difference between this result and that of the preceding section is due to the fact that there X_{n-1} was assumed to be homogeneous of degree 0.

[15] Lange's system is presented mathematically in his *Price Flexibility and Employment*, Appendix, section 4. L. Hurwicz was the first one to point out its overdeterminacy (cf. above, footnote 4).

Lange, in fact, generalizes the results of (12.3)–(12.4) to the following:[16] If the X_r $(r = 1, \cdots, m)$ are homogeneous of degree 0 in $p_1, \cdots, p_r$; and the X_s $(s = m + 1, \cdots, n - 1)$ are homogeneous of degree 1 in the same variables, then X_n is homogeneous of degree 1 in $p_1, \cdots, p_r$. Using this and the theorem of section 9, we can generalize the theorem of section 10(a) as follows:

THEOREM: *If in (10.8) the $X_r(r = 1, \cdots, m)$ are homogeneous of degree 0 in $p_1, \cdots, p_r$ and the $X_s(s = m + 1, \cdots, n - 1)$ are homogeneous of degree 1 in the same variables and if all these equations are independent, then the system (10.8) is overdetermined.*

Similarly, it is readily seen that the objections of section 10(b) are still valid: for with a constant K (10.4) cannot be homogeneous of degree 1 in $p_1, \cdots, p_{n-2}$. Lange's system thus involves exactly the same contradictions as were discussed in section 10.

13. In the classical theory another special assumption (in addition to the homogeneity assumptions) was sometimes made with reference to the system (6.1)–(6.3). This was the assumption of Say's law.[17] According to this law the only reason people supply commodities is in order to use the receipts to purchase other commodities. The decision to supply simultaneously involves a decision to spend the receipts. People do not sell to obtain and hold money; money is only a "veil" concealing the true barter nature of the economy. Thus aggregate de-demand for all commodities must always equal aggregate supply—re-regardless of prices. That is,

$$\sum_{i=1}^{n-1} p_i f_i(p_1, \cdots, p_{n-1}) \equiv \sum_{i=1}^{n-1} p_i g_i\, p_1, \cdots, p_{n-1}) \tag{13.1}$$

identically in the p_i, or, using the notation of (10.7),

$$\sum_{i=1}^{n-1} p_i X_i\, (p_1, \cdots, p_{n-1}) \equiv 0 \tag{13.2}$$

identically in the p_i. This is the mathematical formulation of Say's law.[18]

What is the effect of Say's law on our system? It can be shown that it reduces the number of independent equations in (10.8) to $(n - 2)$. For from (11.4) we see that when (13.2) is satisfied, $X_n = 0$ is simultaneously satisfied. Therefore this equation is not independent. Consider now the first $(n - 2)$ equations of (10.8), homogeneous of degree 0 in the $n - 1$ variables $p_1, \cdots, p_{n-1}$. By the corollary to the theorem of

[16] Lange, *ibid.*, pp. 99–100

[17] Cf. Divisia, *op. cit.*, pp. 411–412.

[18] Cf. Lange, "Say's Law," *op. cit.*, pp. 49–53.

section 9, these equations can be solved for the price ratios p_j/p_{n-1} ($j = 1, \cdots, n - 2$). [Or if the functions are homogeneous in only $p_1, \cdots, p_{n-2}$, then they can be solved for p_r/p_{n-2} ($r = 1, \cdots, n - 3$) and p_{n-1}.] In either case, by substituting this solution in (13.2) we obtain

$$p_{n-1}X_{n-1}(p_1, \cdots, p_{n-1}) = 0. \tag{13.3}$$

Since in general we can assume that none of the prices (or price ratios) determined by the first ($n - 2$) equations are 0 or infinity, we can divide both sides of (13.3) by p_{n-1} to yield

$$X_{n-1}(p_1, \cdots, p_{n-1}) = 0. \tag{13.4}$$

Consequently the solution derived from the first $n - 2$ equations will also satisfy the ($n - 1$)th. So this equation is not independent either.

Thus the assumption of Say's law removes the overdeterminacy from the system; for it reduces (10.8) to $n - 2$ independent homogeneous equations in the $n - 1$ variables $p_1, \cdots, p_{n-1}$. By the corollary to the theorem of section 9, (10.8) can then be solved for the p_i/p_{n-1} ($i = 1, \cdots, n - 2$). *But Say's law removes the overdeterminacy only at the expense of leaving the system definitely underdetermined.* For as a result of Say's law we are left with $n - 2$ independent equations in $n - 1$ variables $p_1, \cdots, p_{n-1}$; therefore at the most we can solve only for the price ratios. The absolute prices must remain undetermined.

IV. CONCLUSIONS

14. From the preceding analysis it is clear that it has not been possible to construct a system satisfying the classical dichotomy of determining relative prices in the real part of the model, and absolute prices through the money equation. It must be emphasized that *there is no monetary equation that we can use to remove this indeterminacy of the absolute prices.* For any monetary equation will either be (a) homogeneous or (b) not homogeneous. It is impossible for (b) to be true, since as has repeatedly been shown, this contradicts the homogeneity assumptions made with reference to the nonmonetary equations. Therefore (a) must hold. But if (a) holds, we see from the second paragraph of section 10 [or (9.4)] that the monetary equation itself is a function of relative prices only, and therefore cannot determine absolute prices. We are caught on the horns of a dilemma: if the monetary equation is useful (in determining absolute prices) then it is inconsistent with the rest of the system; on the other hand, if it is consistent with the rest of the system, it is useless. Furthermore, although the assumption of Say's law removes the over-

determinacy of the system, it simultaneously renders absolute prices indeterminate.

What are the implications of these conclusions? They are, in brief, that the classical theory never really dealt with the monetary aspects of our economy. Classical analysis was restricted to examining those aspects of an economy which are similar to a barter economy; or, at most, to an economy in which transactions take place with goods against goods, with money acting only as a counting unit. But it did not explain why people held actual cash balances.[19]

It is equally clear that for a real monetary theory, at least one of the $n - 1$ equations of (11.1)–(11.2) cannot be homogeneous.[20] From the viewpoint of the determination of absolute prices, it makes no difference which one this is. However, in accordance with the results of section 11, we must be sure that the introduction of the nonhomogeneous equation effectively eliminates all relative prices from the real sector.

By proper selection of the nonhomogeneous equation it is, in fact, possible to achieve results very close to those desired by the classical school. For example, assume again (cf. section 12) that the $(n - 1)$th commodity is bonds. Let X_j $(j = 1, \cdots, n - 2)$ each be homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$, with p_{n-1} again representing the reciprocal of the interest rate. However, assume that, for some unspecified reason, the excess-demand equation for the bond market, X_{n-1}, is not homogeneous of degree 1 in $p_1, \cdots, p_{n-2}$. Then the development (12.3)–(12.4) is no longer valid, so that X_n is not homogeneous of degree 1 in $p_1, \cdots, p_{n-2}$, and the overdeterminacy theorem of section 9 cannot be applied.

Owing to the interdependence following from Walras' law (6.6) drop (11.3). Then (11.1) consists of $n - 2$ homogeneous equations in $n - 1$ variables. By (9.9) they can in general be solved for the relative prices. Then the absolute prices can be determined from the nonhomogeneous equation (11.2). Since (11.1)–(11.2) is thus in general consistent, we can, according to the procedure justified in section 11, reinstate the money equation (11.3) and eliminate the bond equation (11.2). Then the classical dichotomy would hold—with relative prices determined in the "real" part (11.1), and absolute prices through the monetary equation (11.3).

Keynesian theory usually assumes nonhomogeneity for another equa-

[19] The rigorous proof of this last statement requires microeconomic analysis. Cf. "The Demand for Money," section 2.

[20] Once again, the full implications of this nonhomogeneity can be developed only through microeconomic analysis. Cf. "The Demand for Money," Section 5, especially Theorem XVI.

tion: the supply of labor, which is assumed to be a function of money, and not real, wages.[21] The results are completely analogous to those just discussed and need not be repeated. But neither of these approaches is really acceptable: they both resort to tricks. What justification is there for singling out one particular equation and assuming it to be nonhomogeneous? A satisfying solution to our problem cannot be achieved by such *ad hoc* and arbitrary assumptions.

There is, however, a straightforward solution that can be readily formulated. The key to the problem lies in distinguishing between two assumptions of classical monetary theory which have hitherto been indiscriminately lumped together. The first postulates a twofold dichotomy between relative and absolute prices on the one hand, and the real and monetary sectors on the other. The second asserts that the quantity of money makes no difference for the determination of the equilibrium flows of goods and services. This last assumption is as basic and intu tively obvious today as it was in "classical" times. It is equivalent to the proposition that no difference will be made for the functioning of the economy if the dollar is replaced throughout by the peso. But the first assumption is neither obvious nor helpful. In fact, the inconsistencies and inadequacies of the classical system that have been repeatedly demonstrated in this paper are due entirely to this assumption.

The usual confusion of these two assumptions arises from the fact that in the classical system the first is made as a means of implying the second. Hence, the classical theory did not distinguish between them. But if we are to solve our difficulties we must find a way in which we can have the logically necessary results of the second assumption, without involving the treacherous implications of the first.

Fortunately, this can be done. First, the absolute price level is introduced into every equation of the system; this immediately eliminates the dichotomies. In particular, we argue that the excess demand for each commodity is affected by the value of all assets (monetary as well as nonmonetary) in the economy.[22] In other words, every excess-demand function depends on the money value of these assets divided by the absolute price level. The modified classical system can now be written down with the aid of the following symbols: r = rate of interest, p =

[21] J. M. Keynes, *The General Theory of Employment, Interest and Money*, New York, Harcourt, Brace, and Co., 1935, pp. 7–14. Cf. also W. Leontief, "The Fundamental Assumption of Mr. Keynes' Monetary Theory of Unemployment," *Quarterly Journal of Economics*, Vol. 5, (1936–37), pp. 192–197.

[22] The rationale of this hypothesis is provided by A. C. Pigou, "The Classical Stationary State," *Economic Journal*, Vol. 53, 1943, pp. 343–352. Cf. also D. Patinkin, "Price Flexibility and Full Employment," *American Economic Review*, Vol. 38, 1948, pp. 543–564.

absolute price level, Y = real national income,[23] M = amount of money, and A = *money* value of all other assets. We now write our system

(14.1) $X_i(p_1/p, \cdots, p_{n-2}/p, r, M/p, A/p) = 0 \quad (i = 1, \cdots, n-2)$,
(14.2) $X_{n-1}(p_1, \cdots, p_{n-2}, r, p, A, M) = 0$,
(14.3) $X_n(p_1, \cdots, p_{n-2}, r, p, A, M) = 0$,
(14.4) $p = \sum_{i=1}^{n-1} w_i p_i$.

Equation (14.4) defines the absolute price level p as a weighted average of all the prices, where the w_i represent the given weights. Following section 12, we assume the first $n - 2$ equation to be homogeneous of degree 0 in $p_1, \cdots, p_{n-2}$, *and* p, A, and M. This assumption has already been effected by writing the arguments of equations (14.1) in ratio form; from these forms it is clear that, say, a 10 per cent change in all prices *and the amount of money and the monetary value of all other assets* will leave the excess demands unchanged. Equation (14.2) represents the excess demands for bonds. Here it is assumed that this is homogeneous of degree 1 in $p_1, \cdots, p_{n-2}$, *and* p, A, M. That is, a 10 per cent change in *all* these variables will cause a corresponding 10 per cent change in the amount of bonds purchased. By a simple extension of the theorems developed in section 12 it can be shown that under these assumptions the excess-demand equation for money—(14.3)—must be homogeneous of degree one in $p_1, \cdots, p_{n-2}$, and p, A, M. In particular, the excess-demand equation

(14.5) $$pL(r, Y) - M = 0,$$

where $L(r, Y)$ is any function of r and Y, meets these conditions; for a 10 per cent change in p and M [and, of course, $p_1, \cdots, p_{n-2}$, and A, which do not appear in (14.5),[24] and hence can be ignored] will cause a corresponding 10 per cent change in the excess demand for money. As a particular case of (14.5) we can assume $L(r, Y)$ to have the form

(14.6) $$L(r, Y) = KY,$$

in which case (14.5) reduces to the familiar Cambridge equation (above, section 10).

The reader might ask whether these assumptions have not again brought us within the scope of the inconsistency theorems of sections 9 and 10. The answer is no. True that we have a system of n independent equations, each of which is homogeneous; but the equations are not

[23] In what follows it should be kept in mind that $Y = \sum_{j=1}^{m} p_j S_j / p$—where $S_j (j = 1, \cdots, m)$ are the amounts supplied of the *finished* goods of the economy.

[24] Except indirectly through Y, as in the preceding footnote. Clearly, a constant percentage change in all prices will leave Y, as defined in the preceding footnote, unchanged. Hence, we need not consider it here.

homogeneous in the dependent (or endogenous) variables of the system; the homogeneity holds only with respect to the independent (exogenous) and dependent variables taken together. Under these conditions the reader can establish for himself that there is no analogue to the basic inconsistency theorem of section 9.

The distinction can be seen intuitively as follows: A and M are the independent variables of the system: they are stipulated *numbers* determined by factors outside the economic system. Once these numbers are given, our system of n independent equations, (14.1)–(14.4), will then determine the values of the n dependent variables, $p_i(i = 1, \cdots, n - 1)$ and p. If the reader will again consult section 9, he will see that the crucial step in the proof of the inconsistency there is justified by our ability to rewrite equations (9.4) in the form (9.5). In other words, the inconsistency derives from the fact that the homogeneity assumptions enable us to reduce the number of dependent variables by one: instead of depending on the individual value of each of these variables, the functions depend on their ratios. For example, instead of depending on $y_1, \cdots, y_m$, the function will depend on the $m - 1$ ratios $y_1/y_m, \cdots, y_{m-1}/y_m$. But in the case of the system (14.1)–(14.4) it is impossible to do this. For assume that the values of the independent variables are specified as $M = 100$ and $A = 1{,}000$—or any other numbers, for that matter. Then the last two ratios of (14.1) become $100/p$ and $1{,}000/p$. Clearly, then, it is no longer possible to consider the functions of (14.1) as depending only on the *ratios* of p and the p_i.

Let us now summarize the properties of the system (14.1)–(14.4). First of all, the absolute price level appears everywhere in the system. It is, in general, impossible to break down the system into two distinct parts: one to determine relative prices, and one absolute. Both these sets of prices are determined in a truly general-equilibrium manner—by the system as a whole. In particular, it is impossible to say, as Cassel and others did, that a proportionate change in all prices, including the price of assets, will leave the real part of the system completely unaffected; for inspection of equations (14.1) shows that there will be a change in M/p, the real value of cash balances, which in turn will change the excess demands. In this way we have completely freed ourselves of the troublesome classical dichotomies.

Nevertheless, the effects of an increase in the amount of money are completely classical in nature. Assume that this increase (say, a doubling) results in a proportionate change in all prices, including that of assets. (Throughout this discussion it is assumed that the *real* value of nonmonetary assets remains fixed.) We shall show that in this case no other change (except for a doubling in the number of bonds) will occur in the economy. Examine first equations (14.1). Here it is clear that the flows of real goods and services are not affected at all: the doubling of money, value of assets, and all prices cancels out everywhere. Furthermore, this increase has absolutely no effect on the rate of interest. Equation (14.2) shows that there will be a doubling in the number of bonds. Equation (14.5)—which, for simplicity, we take as the form of (14.3)—is completely consistent with our assumptions: r and Y are constant, as was shown in the discussion of (14.1), and the sole effect of the doubling of M is to double p. Equation (14.4) is also obviously satisfied: the doubling of all prices corresponds to the doubling of the general price level.

One more point must be noted in connection with the rate of interest. As we have seen, despite the fact that it appears in the excess-demand equation for money, the rate of interest remains constant after a change in the money supply. Thus, we can simultaneously accept both the Keynesian proposition, that the rate of interest influences the amount of money people want to hold, and the classical proposition, that changing the amount of money will not affect the rate of interest. This seeming paradox is explained by the assumption of equation (14.5), on the one hand, that variations in the rate of interest will affect only the amount of *real* balances people want to hold; and by the proof, on the other, that changes in the amount of money will cause a proportionate change in all prices, thus leaving *real* balances constant. Finally, it should be emphasized that, in contrast to both the classical and (radical) Keynesian position, the rate of interest is not determined by any one particular equation, but, just like any other variable, by the system as a whole.

In this way, I believe it is possible to reconstitute the classical theory. As frequently in such cases, once the reformulation is completed, it is possible to go back to the texts and show that it is really closer to what the original propounders had in mind. What we have in essence done is to solve the problems arising from the classical homogeneity assumptions by assuming still more homogeneity. That is, we argued that the excess-demand functions were homogeneous of degree zero in the monetary value of assets as well as in prices. This is really closer to the classical position, which, when arguing for the lack of any effect of an increase in the amount of money, assumed that the prices of all things—including assets as well as currently produced goods—changed proportionately. Until assets are explicitly introduced into the demand functions of (14.1), this classical proviso with regard to the price of assets has no counterpart in the mathematical analysis.

System (14.1)-(14.4) is closer to the original classical formulation in yet another way. A re-examination of the utility-maximization theory developed by Walras and Pareto will make it obvious that the excess-demand equations derived from this theory must depend also on the amounts of assets in the economy.[25] However, since it was usually assumed that these assets were held constant, there was no point including them in the excess-demand equations. Now that we are interested in the effects of an increase in these assets (i.e., an increase in the amount of money), it is necessary to re-introduce them into our equations.

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[25] Cf. "The Demand for Money," section 1, especially the parenthetical remark at the bottom of p. 137.