RELATIVE PRICES, SAY'S LAW, AND THE DEMAND FOR MONEY*

By Don Patinkin

SUMMARY

Mathematical economists of the classical school fall roughly into two classes—one consisting of those economists who took as their starting point the theory of individual behavior, and built up market demand and supply relations from this theory (Walras, Pareto, Divisia); the other, of those who began with the market relations themselves (Cassel) in another article,1 certain aspects of the Casselian system were examined. In the present one it is proposed to subject the microeconomic theory of Walras, Pareto, and Divisia to a similar examination. The results of this examination are summarized in the following section.

The classical general-equilibrium analysis of a nonmonetary economy has achieved a firm standing in the accepted body of economic theory. This is not true, however, for its attempted generalization to a monetary economy. The basic postulate of the classical monetary theory is that people do not derive any utility from holding money, and consequently it does not enter the utility function. But this postulate stands in complete contradiction to the avowed purpose of formulating a monetary theory. For if money does not enter the utility function, people will certainly not hold money. Hence we could not possibly have any realistic monetary economy. In fact, it will be shown (Theorem VI) that the classical discussion of a "monetary economy" is involved in a contradiction unless there are no stocks of money! Thus the classical theory deals with money only as a counting unit. There is no treatment of its far more important functions as a medium of exchange or store of value. Furthermore, it will be shown that the Walrasian-Paretian system does not determine absolute prices.

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1 D. Patinkin, "The Indeterminacy of Absolute Prices in the Casselian System," a forthcoming publication. This will be referred to henceforth as "The Casselian System."
What does it mean for money to enter the utility function? It has been shown that if we eliminate the cost of making an investment, people will hold money only because of uncertainty as to the future.² Hence monetary theory is inseparably tied up with a theory of dynamic economics. The entrance of money into the utility function (which is a necessary condition for a monetary theory) thus represents the satisfaction derived by individuals from holding money as a means of dealing with uncertainty. This shows in another way the error of the classical school in trying to “splice on” a monetary equation (to determine absolute prices) to a theory of a static barter economy.

The admission of money into the utility function has several interesting implications. It will be shown (Theorem XIV) that under these circumstances it is impossible that all the demand functions should depend only on relative prices. This may be seen intuitively in the following way: The absolute price level is the price of money. If the entire system depends only on relative prices, this means that the price of money, and hence money itself, plays no role in the economy. But this is inconsistent with the original assumption that money fulfills a fundamental task in the economy by meeting the problem of uncertainty.

Similarly it will be shown that in a monetary economy, it is impossible for Say’s law to hold (Theorem XIII and Corollary I). The mathematical proof of this proposition also has its intuitive counterpart. If Say’s law holds, people will retain the same amount of cash balances regardless of the absolute price level. But if money plays a real role in the economy, then the desire to hold cash must depend to some extent on the price of money (i.e., the absolute price level).

In fact, it will be proved that dependence on relative prices and Say’s law are characteristic of economic systems in which money does not enter the utility function. These are essentially nonmonetary economies; for money there can only serve the function of a counting unit (Theorems IV, VII, and VIII). Finally, it will be shown that Say’s law and the dependence of demand functions on relative prices only are equivalent properties: if one holds, the other must hold; if one does not hold, the other cannot hold (Theorem XVI).

I shall now proceed to a rigorous mathematical presentation of the propositions just stated. The classical nonmonetary analysis will first be examined, to be followed by an examination of the classical attempt

to extend the theory to a monetary economy. The assumption of perfect competition will be maintained throughout.

\* \* \*

1. THE CLASSICAL THEORY

Consider first an isolated pure exchange economy consisting of \( m \) individuals (denoted by \( a = 1, \cdots, m \)) and \( n \) goods (denoted by \( i = 1, \cdots, n \)). The \( a \)th good is arbitrarily taken as a numéraire—a standard of value. At this point we assume that this numéraire is a good like all other goods in that it yields satisfaction directly (e.g., the numéraire may be cows, horses, etc.). Let \( p_i \) = number of units of the numéraire necessary to obtain one unit of the \( i \)th commodity. Then \( p_{na} = 1 \).

Let \( Z_{ia} (i = 1, \cdots, n; a = 1, \cdots, m) \) represent the stock of the \( i \)th good in the hands of the \( a \)th individual at the beginning of the period—i.e., his initial stocks. In the case of labor and the services of assets, \( Z_{ja} \) represents the total number of potential service hours available for disposal during the period. Thus in the case of labor, \( Z_{ja} = 24D \) where \( D \) is the number of days in the period. This is the total number of labor hours that must be allocated between leisure (\( Z_{ja} \)) and work during the period.

Let \( Z_{ia} \) represent the consumption flow of the \( i \)th good to the \( a \)th individual during the period. Let \( u^a(Z_{ia}, \cdots, Z_{na}) \) represent the utility derived from these flows. The individual maximizes this utility subject to the budget identity

\[
(1.1) \quad \sum_{i=1}^{n} p_i Z_{ia} = \sum_{i=1}^{n} p_i \bar{Z}_{ia} \quad (a = 1, \cdots, m),
\]

so that he maximizes

\[
(1.2) \quad u^a(Z_{ia}, \cdots, Z_{na}) = \lambda_a \left( \sum_{i=1}^{n} p_i Z_{ia} - \sum_{i=1}^{n} p_i \bar{Z}_{ia} \right)
\]

to yield

\[
(1.3) \quad \frac{u^a(Z_{ia}, \cdots, Z_{na})}{u^a(Z_{ia}, \cdots, Z_{na})} = p_j \quad (j = 1, \cdots, n - 1),
\]
\[
(1.3) \quad \frac{u^a(Z_{ia}, \cdots, Z_{na})}{u^a(Z_{ia}, \cdots, Z_{na})} = p_j \quad (a = 1, \cdots, m),
\]

where \( u^a_b \) is the partial derivative of \( u^a \) with respect to its \( b \)th argument.

From (1.1) and (1.3) we have \( mn \) equations in the \( mn + n - 1 \) variables \( Z_{ia}, p_j \). Therefore we can express the \( mn \) variables \( Z_{ia} \) as functions of the \( n - 1 \) variables \( p_j \). (I ignore the \( Z_{ia} \) since they remain constant throughout the discussion). This gives us
\[ Z_{ia} = Z_{ia}(p_1, \cdots, p_{n-1}) \]

\[ (i = 1, \cdots, n; a = 1, \cdots, m). \]

From this we can get our excess-demand functions

\[ X_{ia} = Z_{ia} - \bar{Z}_{ia} = X_{ia}(p_1, \cdots, p_{n-1}) \]

\[ (i = 1, \cdots, n; a = 1, \cdots, m). \]

(This represents demand or supply of the \( a \)th individual according as \( X_{ia} > 0 \) or \( < 0 \).) From (1.5) we get the excess-demand functions for the economy as a whole:

\[ X_i = \sum_{a=1}^{m} X_{ia} = X_i(p_1, \cdots, p_{n-1}) \]

\[ (i = 1, \cdots, n). \]

Our equilibrium conditions are stated as

\[ X_i(p_1, \cdots, p_{n-1}) = 0 \]

\[ (i = 1, \cdots, n). \]

There are only \( n-1 \) independent equations here since from (1.1), (1.5), (1.6) we have

\[ \sum_{i=1}^{n} p_i(Z_{ia} - \bar{Z}_{ia}) = \sum_{i=1}^{n} p_iX_{ia} = 0. \]

Summing over \( a \) we have

\[ \sum_{a=1}^{m} \sum_{i=1}^{n} p_iX_{ia} = \sum_{i=1}^{n} p_i \sum_{a=1}^{m} X_{ia} = \sum_{i=1}^{n} p_iX_i = 0 \]

identically in the \( p_i \). Thus the classical economists considered prices in the exchange economy as completely determined.\(^3\)

Now introduce production into this economy. Assume that there are \( r \) firms (denoted by \( b = 1, \cdots, r \)), each having its transformation function

\[ F^b(X_{ib}', \cdots, X_{nb}') = 0 \]

\[ (b = 1, \cdots, r), \]

where \( X_{ib}' \) represents the output of the \( i \)th good from the \( b \)th firm. (If the \( i \)th good is an input, \( X_{ib}' < 0 \).) For convenience assume that the \( n \)th good (the numéraire) is completely in the hands of consumers at the beginning of the period. Assume further that the numéraire is not producible by firms and that firms maximize their profits in terms of this numéraire. Consequently we replace (1.10) by

\[ \phi^b(X_{ib}', \cdots, X_{n-1,b}') = 0 \]

\[ (b = 1, \cdots, r) \]

and firms maximize

\[(1.12) \quad \sum_{i=1}^{n-1} p_{i}X_{i}^{b} - \mu_{s}\phi_{i}^{b}(X_{n}^{b}, \ldots, X_{n-1}^{b})\]

to yield

\[(1.13) \quad \frac{\phi_{i}^{b}}{\phi_{n-1}^{b}} = \frac{p_{i}}{p_{n-1}} \quad (k = 1, \ldots, n - 2), \quad (b = 1, \ldots, r).\]

In addition the profits of the \(b\)th firm (i.e., the net inflow of the numéraire) are

\[(1.14) \quad -X_{b}^{i} = \sum_{j=1}^{n-1} p_{j}X_{i}^{j} \quad (b = 1, \ldots, r),\]

so that

\[(1.15) \quad \sum_{i=1}^{n} p_{i}X_{i}^{b} = 0 \quad (b = 1, \ldots, r).\]

For the firms (1.11), (1.13), and (1.14) are \(2r + r(n-2) = nr\) equations in \(nr + n - 1\) variables \(X_{i}^{b}\) and \(p_{i}\). Therefore we can express the \(X_{i}^{b}\) as functions of the \(p_{i}\) to get

\[(1.16) \quad X_{i}^{b} = g_{i}(p_{1}, \ldots, p_{n-1}) \quad (i = 1, \ldots, n; b = 1, \ldots, r).\]

These are supplies or demands according as \(X_{i}^{b} > 0\) or \(<0\). The aggregate supply of firms is

\[(1.17) \quad X^{i} = \sum_{b=1}^{r} X_{b}^{i} = g_{i}(p_{1}, \ldots, p_{n-1}) \quad (i = 1, \ldots, n).\]

The excess-demand functions (1.6) are now rewritten as

\[(1.18) \quad X_{i}^{*} = X_{i} - X^{i} = X_{i}^{*}(p_{1}, \ldots, p_{n-1}) \quad (i = 1, \ldots, n),\]

and the equilibrium conditions as

\[(1.19) \quad X_{i}^{*}(p_{1}, \ldots, p_{n-1}) = 0 \quad (i = 1, \ldots, n).\]

Again there are only \(n - 1\) independent equations here since by (1.9) and (1.15)

\[(1.20) \quad \sum_{i=1}^{n} p_{i}X_{i}^{*} = \sum_{i=1}^{n} p_{i}(X_{i} - X^{i}) = 0\]

identically in the \(p_{i}\). Thus the determination of prices in an economy with both exchange and production is complete.
The process until now does not provide for any way in which the profits of the firm are returned to the consumers. Therefore as this process continues for period after period, the accumulated profits of the firms become indefinitely larger. Classical economists got around this difficulty by assuming zero profits. The vanishing of profits was provided for either through special homogeneity assumptions about the transformation functions, or through the inflow of new firms. Another possible way (but one not usually taken by the classical theory) is to assume that in every period firms distribute their profits as dividends.

2. MONEY IN THE CLASSICAL SYSTEM

The preceding section presented the classical theory (with some indicated modifications) as it relates to a nonmonetary economy. We saw that the classical theory explains this problem in a consistent manner. In this section I shall discuss the attempts of the classical economists to extend their theories to a monetary economy. For simplicity I shall confine my remarks to the exchange economy without production.

First I must make clear that by a monetary economy I mean one in which paper money is the prevailing means of exchange. Though this definition does violence to the history of money, it is very useful in sharpening the theoretical issues and avoiding the complication that arises for most other types of money—i.e., the difficulty of distinguishing between the money commodity as a commodity and as money.

Let paper money be the $n$th commodity; and let $Z_{na}$ be the stock of money the $a$th individual plans to have on hand at the end of the period. The fundamental classical proposition is that people derive no ("direct") utility from paper money, and therefore $Z_{na}$ does not enter the utility function. An immediate implication of this assumption is that since the individual derives no satisfaction from money, he will not plan to hold any money at the end of the period, regardless of what the commodity prices may be. That is, he will not use his limited resources to hold money, which yields him no satisfaction, when he can use these same resources to obtain goods, from which he does derive enjoyment. Thus the budget restraint (1.1) must be restated as

$$\sum_{j=1}^{n-1} p_j Z_{ja} = \sum_{i=1}^{n} p_i Z_{ia} \quad (a = 1, \ldots, m).$$


(The summation on the right-hand side runs up to $n$, since there is no reason why the individual’s initial stocks should not include money.) This modification of the budget restraint is the crux of the entire argument. Consequently, though it is intuitively quite obvious, I shall prove it rigorously.

Assume first that even though $Z_{mn}$ does not enter the utility function, the left-hand number of (2.1) is summed up to and including $n$. Then the individual maximizes

$$u^a(Z_{i1}, \ldots, Z_{i,n,a}) - \lambda_a \left( \sum_{i=1}^n p_i Z_{ia} - \sum_{i=1}^n p_i Z_{ia} \right)$$

(a = 1, \ldots, m)

(2.2) to yield

$$u_j^a - \lambda_a p_j = 0 \quad (a = 1, \ldots, m; j = 1, \ldots, n - 1),$$

(2.3) $$- \lambda_a p_n = 0.$$ (2.4)

Since $p_n = 1$, (2.4) implies $\lambda_a = 0$ so that we replace (2.3) by

$$u_j^a = 0 \quad (a = 1, \ldots, m; j = 1, \ldots, n - 1).$$ (2.5)

If we assume that the marginal utility of a finite amount of any good is always positive, (2.5) implies that the individual will consume an infinite amount of each good. That is,

$$Z_{ja} = + \infty \quad (a = 1, \ldots, m; j = 1, \ldots, n - 1).$$ (2.6)

If we substitute this in the budget restraint (1.1) we get

$$Z_{na} = - \infty \quad (a = 1, \ldots, m).$$ (2.7)

Let us now consider the $Z_{ia}(i = 1, \ldots, n; a = 1, \ldots, m)$. By their very nature it is clear that

$$Z_{ia} \geq 0 \quad (a = 1, \ldots, m; i = 1, \ldots, n).$$ (2.8)

It is clear that the optimum position reached in (2.6) – (2.7) violates this additional restraint for $i = n$. Consequently the inequality restraints (2.8) must be explicitly introduced into the maximization procedure. Consider (2.8) for $i = n$ and rewrite it in the equivalent equation

$$Z_{na} = y_{na} \quad (a = 1, \ldots, m),$$ (2.9)

where $y_{na}$ is some real variable. Then individuals must maximize their utility subject to this additional restraint. That is, they maximize$^6$

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$^4$ In reality the maximization should be carried out using all the restrictions of (2.8), and not just the one for $i = n$. However, since this is the only one violated by (2.6) – (2.7), for simplicity I have neglected the others. This does not affect the results (2.14) in any way.
\begin{equation}
\tag{2.10}
u^a(Z_{1a}, \cdots, Z_{n-1,a}) - \lambda_a \left( \sum_{i=1}^{n} p_i Z_{ia} - \sum_{i=1}^{n} p_i \tilde{Z}_{ia} \right)
- \mu_a (Z_{na} - y_{na})^2
\end{equation}

with respect to the $Z_{ia}$ and $y_{na}$ to yield

\begin{align*}
\tag{2.11}
u_i^a - \lambda_a p_i &= 0 & (a = 1, \cdots, m), \\
\tag{2.12}
- \lambda_a p_n - \mu_a &= 0 & (j = 1, \cdots, n - 1), \\
\tag{2.13}
\mu_a y_{na} &= 0.
\end{align*}

In (2.13) there are three possibilities: (a) $\mu_a = 0, y_{na} \neq 0$; (b) $\mu_a \neq 0, y_{na} = 0$; (c) $\mu_a = y_{na} = 0$. Cases (a) and (c) can immediately be rejected. For if $\mu_a = 0$ we are immediately back to the maximizing situation (2.2). But we have seen that this leads to negative values for $Z_{na}$—a contradiction to (2.9). Therefore alternative (b) must be true. That is [substituting into (2.9)], the solution of the maximizing procedure (2.10) will lead to

\begin{equation}
\tag{2.14}
Z_{na} = 0 \\
(a = 1, \cdots, m)
\end{equation}

identically in the $p_i$. That is, no matter what the prices are, every individual will plan to hold zero cash balances at the end of the period.

Thus we have proved the following

**Theorem:** If $Z_{na}$ does not enter the utility function, the maximization procedure (2.10) yields the same equations as those yielded by the maximization of

\begin{equation}
\tag{2.15}
u^a(Z_{1a}, \cdots, Z_{n-1,a}) - \lambda_a \left( \sum_{i=1}^{n-1} p_i Z_{ia} - \sum_{i=1}^{n} p_i \tilde{Z}_{ia} \right)
(a = 1, \cdots, m)
\end{equation}

plus the identities (in the $p_i$)

\begin{equation}
\tag{2.16}
Z_{na} = 0 \\
(a = 1, \cdots, m).
\end{equation}

The reader can check this quite readily by noting that the maximization of (2.15) yields (2.11), while we have just shown that (2.13) implies (2.16). (2.12) is an additional equation involving an additional variable $\mu_a$ on which there are no other restrictions; consequently it cannot affect the results.

From equations (2.11) we obtain

\begin{equation}
\tag{2.17}
\frac{u_r^a}{u_{n-1}^a} = \frac{p_r}{p_{n-1}} & (r = 1, \cdots, n - 2), \\
\text{(a = 1, \cdots, m).}
\end{equation}

(2.1), (2.16), and (2.17) are $2m + m(n - 2) = mn$ equations in the
\( mn+n-1 \) variables \( Z_{ia}, p_j \). As in (1.4) and (1.5), we can then obtain the excess-demand functions and equilibrium conditions for the classical monetary economy:

\[
X_i = \sum_{a=1}^{m} X_{ia} = \sum_{a=1}^{m} (Z_{ia} - Z_{ia}) = \sum_{a=1}^{m} X_{ia}(p_{i1}, \ldots, p_{in-1}) = 0 \quad (i = 1, \ldots, n).
\]

Is (2.18) a consistent system? Consider the excess-supply curve of the \( n \)th commodity, money. Substituting from (2.16) into (2.18), and using the convention that supply is negative demand, we obtain

\[
-X_n = -\sum_{a=1}^{m} X_{na} = -\sum_{a=1}^{m} (Z_{na} - Z_{na})
\]

\[
= -\sum_{a=1}^{m} (0 - Z_{na}) = Z_n,
\]

i.e., regardless of the prices, every individual will want to supply his entire initial stock of money. Thus in order for equilibrium to hold in (2.19) [i.e., for the equilibrium conditions to be satisfied in (2.18) for \( i = n \)] we must have \( Z_{na} = 0 \). In other words, if there are initial stocks of money, there is no set of prices that will satisfy (2.18) for \( i = n \). Thus we have proved the following

**Theorem:** A necessary condition for the consistency of (2.18) is

\[
Z_{na} = 0 \quad (a = 1, \ldots, m).
\]

When this condition is met, the budget restraint (2.1) is then replaced by

\[
\sum_{j=1}^{n-1} p_j(Z_{ja} - Z_{ja}) = 0.
\]

This is the emasculated monetary system assumed by the classical analysis. There are no initial stocks of money, and no one has any demand for money. Transactions take place not with goods against (non-existent) money, but with goods against goods, with money in some way acting only as a counting unit.

3. **The Error of Walras and Pareto**

Before continuing, it will be interesting to examine the exact errors committed by Walras and Pareto which caused them to overlook the rather simple argument of the preceding section.

Walras states that the excess-demand functions depend on the absolute price level. Walras' argument is as follows: One of the terms
in the budget equation is \( o_u p_u \), where \( o_u \) is the amount of money supplied and \( p_u \) is the price of money in terms of some commodity (i.e., some measure of the price level). Consequently when we solve out for the excess-demand functions, they will be functions of \( p_u \). But Walras has assumed that money is not in the utility function;\(^7\) therefore, by the first theorem of the preceding section, \( o_u p_u \) is not in the budget equation; therefore the excess-demand functions (on Walras' assumptions) cannot be functions of \( p_u \).

Pareto, as contrasted with Walras, realized very well the modifications that had to be introduced into the budget equation if one excluded money from the utility function. Thus he writes: "It may be that the possessor of a good . . . derives no satisfaction from it; then we can say that he will offer the entire quantity at his disposal." ("Il peut arriver que le possesseur d'une marchandise . . . ne s'en serve pas pour satisfaire ses gouts; on dit alors qu'il offre toute la quantité à sa disposition.")\(^9\) This is precisely the content of equation (2.19). But in the very next paragraph he fails to apply this principle correctly. He states\(^{10}\) that for some individual, money may have no utility, and therefore "he will use the entire quantity that he receives to procure [other] goods" ("Il emploie toute la quantité qu'il reçoit à se procurer des [autres] biens"). Consequently, he concludes,

\[
X_{na} = Z_{na} - Z_{na} = 0
\]

identically in the \( p_u \); that is, the individual will decide (regardless of the price level) to maintain his initial stock of money constant.\(^{11}\) But if Pareto had applied his first statement correctly, he should have concluded: "and therefore he will use the entire quantity at his disposal plus his original stock to procure other goods."

Pareto's error is due to a confusion between the \( Z_{na} \) and the \( X_{na} \). His first statement says that if the \( n \)th good provides no satisfaction to the \( a \)th individual, then

\[
Z_{na} = 0
\]

so that he will offer

\[
X_{na} = -Z_{na} + Z_{na} = Z_{na}.
\]

\(^1\) Walras, op. cit., p. 311.
\(^2\) Ibid., p. 303.
\(^3\) Pareto, op. cit., p. 593. Italics in original.
\(^4\) Ibid., p. 593. Italics my own.
\(^5\) As Divisia correctly points out (op. cit., p. 411) this is equivalent to the assumption of Say's law. Cf. the beginning of the next section.
In his following paragraph Pareto confuses the statement (3.2) with the statement (3.1). Owing to this error, as we have shown in the preceding section, he fails to realize that his system is not in equilibrium and cannot be in equilibrium for any set of prices.

4. RELATIVE PRICES AND SAY'S LAW

At many points the development of Section 2 has suggested interrelationships between excluding money from the utility function, homogeneity assumptions, Say's law, and the consistency of the systems considered. In this section I shall develop several theorems bearing on these relationships.

I shall consider two systems, A and B, which differ only in the respect that in A money is included in the utility function, while in B it is excluded. Let system A be the system (1.7), based on the development (1.1) – (1.6), where \( Z_a \) is now interpreted as paper money. The meaning of including paper money in the utility function has been discussed above, in the Summary.\(^{13}\) System B is the system (2.18), derived from (2.1), (2.15) – (2.17), in which \( Z_a \) is assumed not to enter the utility functions.

Two properties of these systems will be discussed below. One is the property of “being true”; the other is the property of “being consistent.” The statement “A is true” means money enters the utility functions; the statement “B is true” means money does not enter the utility functions. Consequently, if A cannot be true, then B must be true; and conversely. The statement “A (B) is consistent” means that the system of excess-demand functions developed on the assumption that A (B) is true and set equal to zero, possesses a solution, i.e., the system (1.7) [(2.18)] possesses a solution. In some theorems we shall be able to show that truth of A (B) implies certain properties as necessary conditions. In others we shall have to be satisfied with showing that the consistency

\(^{13}\) It is interesting to note that if money enters the utility function, then \( \lambda_a \) in (1.2) is the marginal utility of money. For differentiating (1.2) with respect to \( Z_a \), remembering that \( p_a = 1 \), we obtain

\[ u_a^a = \lambda_a. \]

This marginal utility of money represents the satisfaction derived from holding an additional dollar’s worth of cash balances. For the case in which the individual is presented with a fixed income which is to be spent, \( \lambda_a \) also represents the marginal utility of an additional unit of income; that is, “the derivative of utility by income when the utility for any income is the maximum obtainable with that income (at a fixed set of prices)” (E. H. Wilson, “Notes on Utility Theory and Demand Equations,” Quarterly Journal of Economics, Vol. 60, 1946, pp. 453–454). Thus at equilibrium the marginal utilities of money (cash balances) and income must be equal.
of $A$ ($B$) implies these properties. Obviously, theorems of the first type are more powerful.

I make the usual assumptions about utility maximization stressing\(^{13}\)

**Assumption I:** The marginal utility of every commodity is always positive and finite. That is,

\[
(4.1) \quad 0 < u_i^a(z_{i1}, \ldots, z_{ina}) < \infty \quad (a = 1, \ldots, m; i = 1, \ldots, n)
\]

identically in the $Z_{ia}$. (In system $B$, $i$ goes only to $n - 1$.)

**Assumption II:** The following sufficient conditions for a true maximum are satisfied:

\[
(4.2) \quad (-1)^K \begin{vmatrix}
0 & u_i^a & \cdots & u_K^a \\
u_i^a & u_{i1}^a & \cdots & u_{iK}^a \\
\cdots & \cdots & \cdots & \cdots \\
u_K^a & u_{K1}^a & \cdots & u_{KK}^a
\end{vmatrix} > 0
\]

(a = 1, \ldots, m),

(K = 2, 3, \ldots, n),

identically in the $Z_{ia}$ (where $u_{ij}^a$ is the partial derivative of $u_i^a$ with respect to its $j$th argument).

**Assumption III:** The $u_i^a$ and $u_{ij}^a$ ($a = 1, \ldots, m$) are continuous functions of the $Z_{ia}$.

I state in addition the following definitions:

**Definition I:** Say's law for the individual holds when

\[
(4.3) \quad \sum_{j=1}^{n} p_j (Z_{ja} - \overline{Z}_{ja}) = 0 \quad (a = 1, \ldots, m)
\]

identically in the $p_j$.\(^{14}\)

**Definition II:** Say's law for the market holds when

\[
(4.4) \quad \sum_{j=1}^{n-1} p_j (Z_j - \overline{Z}_j) = 0
\]

identically in the $p_j$, where

\[
(4.5) \quad Z_j = \sum_{a=1}^{m} Z_{ja}, \quad \overline{Z}_j = \sum_{a=1}^{m} \overline{Z}_{ja}.
\]


DEFINITION III: Homogeneity of degree \( t \) for the individual holds when

\[
Z_{ja} - Z_{ja} = X_{ja}(p_1, \cdots, p_{n-1})
\]

(4.6)

\( (a = 1, \cdots, m; j = 1, \cdots, n - 1) \)

is homogeneous of degree \( t \) in a designated subset of prices.

DEFINITION IV: Homogeneity of degree \( t \) for the market holds when

\[
Z_j - Z_j = X_j(p_1, \cdots, p_{n-1})
\]

(4.7)

\( (j = 1, \cdots, n - 1) \)

is homogeneous of degree \( t \) in a designated subset of prices.

For convenience I use the following abbreviations:

- S.L.I. = Say's law for the individual,
- S.L.M. = Say's law for the market,
- h.d.i. \( t \) = homogeneous of degree \( t \) for the individual,
- h.d.m. \( t \) = homogeneous of degree \( t \) for the market.

THEOREM I: S.L.I. holds if and only if

\[
Z_{na} - \overline{Z}_{na} = 0
\]

(1.8)

\( (a = 1, \cdots, m) \)

identically in the \( p_i \).

THEOREM II: S.L.M. holds if and only if

\[
Z_{n} - \overline{Z}_{n} = 0
\]

(4.9)

identically in the \( p_i \).

Theorem I [II] is established by substituting (4.4) [(4.5)] into (1.1) [(1.9)] and noting that we obtain (4.8) [(4.9)]. Conversely, substituting (4.8) [(4.9)] into (1.1) [(1.9)] yields (4.4) [(4.5)].

THEOREM III: S.L.I. is a sufficient condition for S.L.M.

This is true since summing (4.3) over \( a \) yields (4.4). I have not been able to determine if in general this condition is necessary or not. Consider, for example, an economy with two individuals. It may be that

\[
Z_{n1} - \overline{Z}_{n1} = f(p_1, \cdots, p_{n-1}),
\]

(4.10)

\[
Z_{n2} - \overline{Z}_{n2} = -f(p_1, \cdots, p_{n-1}),
\]

so that

\[
\sum_{a=1}^{n} (Z_{na} - \overline{Z}_{na}) = 0
\]

(4.11)

identically in the \( p_i \). Thus S.L.M. would be satisfied, while S.L.I. would not be. However, this is not a valid counterexample until we show that equations (4.10) do not violate the usual assumptions made
about the properties of the utility functions. This question needs further study.

From Theorem III it follows that if S.L.I. is a necessary condition for some propositions to be true, then S.L.M. is also a necessary condition for this proposition. If S.L.M. is proved impossible, then from the contrapositive of Theorem III it follows that S.L.I. is impossible. But if under certain specified conditions S.L.I. is proved impossible, it does not follow that S.L.M. is impossible. However, for S.L.M. to be true while S.L.I. were not, the utility functions would have to be interrelated in very restricted ways in order to have the individuals exactly offset each other [cf. (4.10)]. Consequently, for all practical purposes, proof that S.L.I. is impossible creates a very strong presumption that S.L.M. is impossible.

The reader can quickly ascertain that Theorem III and the preceding paragraph will hold true if throughout we replace S.L.I. by h.d.i. $t$, and S.L.M. by h.d.m. $t$.

**Theorem IV:** A necessary condition for $B$ to be true (i.e., for money not to enter the utility functions) is

$$Z_{na} = 0 \quad (a = 1, \ldots, m)$$

identically in the $p_i$.

This is the first theorem of Section 2 [cf. (2.14)].

If $B$ and S.L.M. hold, then (4.12) and (4.9) imply

$$Z_n = \sum_{a=1}^{m} Z_{na} = 0.$$  

Since $Z_{na} \geq 0 \quad (a = 1, \ldots, m)$ this implies

$$Z_{na} = 0 \quad (a = 1, \ldots, m).$$

Combining (4.12) and (4.14) we have (4.8)—S.L.I. (In reality this is a special case of S.L.I., where not only excess demand for money is identically zero, but the demand and supply are each identically zero for every individual.) Using Theorem III we can then state.

**Theorem V:** In system $B$ (i.e., if money does not enter the utility functions) S.L.I. is a necessary and sufficient condition for S.L.M.

Because of this theorem, any result proved for $B$ which holds for S.L.I. also holds for S.L.M., and conversely.

**Theorem VI:** A necessary condition for the consistency of $B$ is

$$Z_{na} = 0 \quad (a = 1, \ldots, m).$$

This was already proved at the end of Section 2.
Substituting (4.15) into the budget restraint for \( B \), (2.1), we obtain (4.3)—S.L.I. Using Theorem V we have

**Theorem VII:** A necessary condition for the consistency of \( B \) is S.L.I. (S.L.M.).

Note that the S.L.I. required by Theorem VII is of the special type discussed immediately preceding Theorem V.

If \( B \) is consistent, then it is derived from the equations (2.17) and (2.21). It is obvious that if in any of these equations the \( p_j \) \( (j = 1, \ldots, n-1) \) are multiplied by an arbitrary factor \( \xi \), nothing will change. Therefore we have proved

**Theorem VIII:** A necessary condition for the consistency of \( B \) is that the \( Z_{ja} \) \( (j = 1, \ldots, n-1; a = 1, \ldots, m) \) be h.d.i. 0 in the \( p_j \) \( (j = 1, \ldots, n-1) \).

The reader can easily prove that the \( Z_{ja} \) are h.d.i. 0 in the \( p_j \) if and only if the \( X_{ja} \) are similarly homogeneous, since they differ only by a constant \( Z_{ja} \). Therefore the above theorem can be restated with \( Z_{ja} \) replaced by \( X_{ja} \). By Theorem III (stated for h.d.i. and h.d.m.) we can replace h.d.i. 0 in Theorem VIII by h.d.m. 0.

Consider \( B \) and assume S.L.I. to hold. Then the budget restraint is (4.3). Using exactly the same reasoning as with the preceding theorem we have proved

**Theorem IX:** In system \( B \), a necessary condition for the existence of S.L.I. is that the \( X_{ja} \) \( (j = 1, \ldots, n-1; a = 1, \ldots, m) \) be h.d.i. 0 in the \( p_j \).

Note that we have proved this without assuming \( B \) to be consistent.

It has been shown elsewhere\(^{15}\) that if the \( X_j \) \( (j = 1, \ldots, n-1) \) are h.d.m. 0 in the \( p_j \), then \( X_1 \) is h.d.m. 1. Thus the number of variables in \( B \) is effectively reduced by 1.\(^{16}\) Since there are \( n-1 \) of the \( p_j \), this means that there can be only \( n-2 \) independent equations in \( B \) if \( B \) is to be consistent. One degree of independence is removed by Walras' law.\(^{17}\) The other one must be removed by some functional relationship between the \( X_j \). This establishes

**Theorem X:** If the \( X_j \) \( (j = 1, \ldots, n-1) \) are h.d.m. 0 in the \( p_j \), a necessary condition for the consistency of \( B \) is the existence of a function

\[
F(X_1, \ldots, X_{n-1}) = 0
\]

identically in the \( p_j \) relating some or all of the \( X_j \), and such that when all but one of the argu-

\(^{16}\) Ibid., §9.
\(^{17}\) Ibid., equation (6.6).
ments of $F$ are $0$, the remaining one is also $0$. As a special case of a function satisfying these conditions we have Say's law (4.4).

Even if (4.16) holds, there are only $n - 2$ independent equations to determine $n - 1$ prices. This will be true for any system of equations $X_i = 0$ $(i = 1, \ldots, n)$. Hence

**Theorem XI:** If the $X_j$ $(j = 1, \ldots, n-1)$ are h.d.m. 0 in the $p_j$, then at the most we can solve for the relative prices.

By Theorem III and the subsequent discussion, we can replace $X_j$ by $X_{ja}$ and h.d.m. 0 by h.d.i. 0. Forming the contrapositive of Theorem XI modified in this way we see that the possibility of solving for the absolute prices implies that not all the $X_{ja}$ are h.d.i. 0 in the $p_j$. Forming the contrapositive of Theorem VIII (since solubility for absolute price implies consistency) we see that nonhomogeneity implies that system $B$ is not true. Thus we have proved

**Theorem XII:** A necessary condition for solubility in terms of absolute prices is that $A$ is true (i.e., money enters the utility function).

I shall now examine the relationship between S.I.I. and the assumption that money enters the utility function. If S.I.I. holds, then (4.8) is true and therefore

\[
\frac{\partial(Z_{na} - \bar{Z}_{na})}{\partial p_r} = 0 \quad (r = 1, \ldots, n - 1)
\]

(4.17)

identically in the $p_r$. This partial derivative can be expressed in terms of the marginal utilities as

\[
\frac{\partial(Z_{na} - \bar{Z}_{na})}{\partial p_r} = \lambda_a Z_r - \bar{Z}_r \frac{U_{ra}^*}{U^*} + \lambda_a \frac{U_{ra}^*}{U^*},
\]

(4.18)

where

\[
U^* = \begin{vmatrix}
0 & u_1^a & \cdots & u_n^a \\
u_1^a & u_{11}^a & \cdots & u_{1n}^a \\
\cdots & \cdots & \cdots & \cdots \\
u_n^a & u_{n1}^a & \cdots & u_{nn}^a
\end{vmatrix}
\]

(4.19)

and $U_{ra}^*$ is the cofactor of the element of the $i$th row and $j$th column [counting the first row (column) as the zero-th row (column)]. Expression (4.18) must be identically 0 for all $p_r$. Multiplying through by $p_r$ and summing up over $r$ we have

\[\text{Cf. Hicks, *op. cit.*, p. 309.}\]
\[ \frac{\lambda_n U_{n*} \sum_{r=1}^{n-1} p_r (Z_r - Z_r)}{U^*} + \frac{\lambda_n \sum_{r=1}^{n-1} p_r U_{n*}}{U^*} = 0 \]

identically in the \( p_r \). On the assumption of S.L.I., using (4.3), the first term is identically 0, so this reduces to

\[ \frac{\sum_{r=1}^{n-1} p_r U_{r*}}{U^*} = 0 \quad \text{identically in the } p_r. \] (4.21)

We have also\(^{11}\)

\[ \frac{\sum_{r=1}^{n} p_r U_{r*}}{U^*} = 0 \quad \text{identically in the } p_r. \] (4.22)

Subtracting (4.22) from (4.21) and eliminating \( \lambda_n \) and \( U^* \) we obtain

\[ p_n U_{n*} = 0 \quad \text{identically in the } p_r. \] (4.23)

Since \( p_n = 1 \) this reduces to

\[ U_{n*} = 0 \quad \text{identically in the } p_r. \] (4.24)

Conversely, if we assume (4.24) to hold, then substituting in (4.22) we obtain (4.21) which upon substitution into (4.20) yields (4.3). We have proved the following

**Theorem XIII**: A necessary and sufficient condition for S.L.I. to hold in system \( A \) is that \( U_{n*} = 0 (a = 1, \cdots, m) \) identically in the \( p_r \).

Comparing (4.24) with Assumption II we have

**Corollary I**: On the basis of Assumption II, if \( A \) is true (if money enters the utility function) then S.L.I. cannot hold.

As pointed out after Theorem III, this corollary creates a very strong presumption that S.L.I.M. is also inconsistent with \( A \).

I will now consider the relationship between \( A \) and homogeneity for the individual.\(^{12}\) For convenience I shall denote the row vector \((p_1, \cdots, p_{n-1}), \bar{P}\), and the row vector \((\theta p_1, \cdots, \theta p_{n-1})\) by \(\theta \bar{P}\), where \(\theta\) is any finite number. Substituting the solutions (1.4) into (1.3) we have, by definition of a solution,


\(^{12}\) The proof of the following theorem is due to Kenneth J. Arrow (Cowles Commission).
(4.25) \[
\frac{u_f^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}(\tilde{\rho})]}{u_a^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}(\tilde{\rho})]} \equiv p_j
\]

identically in the \( p_j \). Assume now that the \( Z_{j,a} (j = 1, \cdots, n-1) \) are h.d.i. 0 in the \( p_j \). Then
\[
\frac{u_f^a[Z_{1a}(\theta\tilde{\rho}), \cdots, Z_{n-1,a}(\theta\tilde{\rho}), Z_{na}(\theta\tilde{\rho})]}{u_a^a[Z_{1a}(\theta\tilde{\rho}), \cdots, Z_{n-1,a}(\theta\tilde{\rho}), Z_{na}(\theta\tilde{\rho})]} = \frac{u_f^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}(\tilde{\rho})]}{u_a^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}(\tilde{\rho})]} = \theta p_j
\]

identically in the \( p_j \) and \( \theta \). Consider a fixed set of prices. Then

(4.27) \[
Z_{j,a}(\theta\tilde{\rho}) \equiv Z_{j,a}(\tilde{\rho}) \quad (j = 1, \cdots, n-1)
\]

identically in \( \theta \). Consequently, by definition of the limiting process,

(4.28) \[
\lim_{\theta \to 0} Z_{j,a}(\theta\tilde{\rho}) = Z_{j,a}(\tilde{\rho}) \quad (j = 1, \cdots, n-1),
\]

where the right-hand term is generally finite. From (1.1) and (4.28)

(4.29) \[
\lim_{\theta \to 0} Z_{na}(\theta\tilde{\rho}) - Z_{na} = \lim_{\theta \to 0} \sum_{j=1}^{n-1} \theta p_j [Z_{j,a}(\theta\tilde{\rho}) - Z_{j,a}] = 0.
\]

Since (4.26) holds identically in \( \theta \) for a fixed set of \( p_j (j = 1, \cdots, n-1) \) it must also hold in the limit. Taking the limit of both sides at \( \theta = 0 \) we obtain, by virtue of (4.28) – (4.29) and Assumption III,

(4.30) \[
\frac{u_f^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}]}{u_a^a[Z_{1a}(\tilde{\rho}), \cdots, Z_{n-1,a}(\tilde{\rho}), Z_{na}]} = 0.
\]

By Assumption I, the left-hand member of (4.30) cannot be zero. Thus the assumption that all the \( Z_{j,a} (j = 1, \cdots, n-1; a = 1, \cdots, m) \) are h.d.i. 0 in the \( p_j \) leads to a contradiction. Hence

**Theorem XIV:** A necessary condition for \( \Lambda \) to be true is that not all the \( Z_{j,a} (j = 1, \cdots, n-1; a = 1, \cdots, m) \) are h.d.i. 0 in the \( p_j \). That is, if money enters the utility function, then not all the \( Z_{j,a} \) can be h.d.i. 0 in the \( p_j \).

From Theorems XII and XIV we can derive

**Theorem XV:** A necessary condition for solvability in terms of absolute prices is that at least one of the \( Z_{j,a} (j = 1, \cdots, n-1; a = 1, \cdots, m) \) not be h.d.i. 0 in the \( p_j \).
Let us now consider all the possibilities with reference to S.L.I. and h.d.i. 0. Either (a) $A$ is true; (b) $B$ is true and $Z_{aa} = 0$; or (c) $B$ is true and $Z_{aa} \neq 0$. This exhausts all the possibilities. If $A$ is true, it follows from Corollary I and Theorem XIV that neither S.L.I. nor h.d.i. 0 can be true. Now assume $B$ is true. Then it is developed from (2.1) and (2.17). If $Z_{aa} = 0$, then (2.1) reduces to (2.21), which is identical with (4.3) — S.L.I. It is also clear that (2.21), (2.17) are h.d.i. 0 in the $p_j$. If $Z_{aa} \neq 0$, it is obvious from (2.1), (2.17) that neither S.L.I. nor h.d.i. 0 holds. Thus in every case, either S.L.I. and h.d.i. 0 are both true, or both false. This means

**Theorem XVI:** A necessary and sufficient condition for h.d.i. 0 in the $p_j$ for $Z_{ja} (j = 1, \ldots, n-1; a = 1, \ldots, m)$ is that S.L.I. be true.\(^{21}\)

If we make the very reasonable assumption (cf. the discussion subsequent to Theorem III) that the utility functions are such that S.L.I. (h.d.i. 0) holds if and only if S.L.M. (h.d.m. 0) holds, then the above theorem can be restated with S.L.I. replaced by S.L.M. and h.d.i. 0 by h.d.m. 0.\(^{22}\)

6. CONCLUSIONS

Let us now briefly summarize the more important implications of this chapter. Since the assumptions made here (namely, utility maximization) are more restrictive than those of “The Cassel System,” the theorems are correspondingly more detailed. The Walrasian-Paretian “monetary” model is the system $B$ just analyzed—where money does not enter the utility function. This model is consistent only if there are no stocks of money. There is an interesting difference between this model and the Casselian model. It was shown there that the assumption of homogeneity makes the Casselian system inconsistent.\(^{23}\) If there are no initial stocks of money, such a situation cannot arise for the Walrasian-Paretian system. For, as shown by Theorem XVI, the damaging assumption of homogeneity simultaneously brings with it the antidote of Say’s law. Just as the former reduces the number of

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\(^{21}\) Thus Modigliani’s discussion of the relationship between Say’s law and homogeneity (as well as Lange’s concession to him on this point) is incorrect. (F. Modigliani, “Liquidity Preference and the Theory of Interest and Money,” *Econometrica*, Vol. 12, January, 1944, pp. 45-88, esp. pp. 67-70.)

\(^{22}\) In “The Casselian System,” §12, it was shown that the system Oscar Lange presents in his *Price Flexibility and Employment* (Bloomington, Indiana: Principia Press, 1945) is inconsistent. This inconsistency is again due to the failure to put money in the utility function. (*Ibid.*, pp. 15-16, especially footnote 6, p. 16.) As can be seen from this section, this assumption is clearly inconsistent with the liquidity preference theory Lange follows in the rest of his book.

\(^{23}\) “The Casselian System,” §10.
effective variables by one, the latter reduces the number of independent
equations by one, leaving the system consistent. In the Casselian sys-
tem, however, the assumption of homogeneity can be made independ-
ently of Say’s law. This is due to the fact that the Casselian system is
not built up from the assumption of utility maximization. Hence its
excess-demand functions are not restricted in the manner that those of
the Walrasian-Paretian system are by Theorem XVI.

The analysis of the preceding section (especially Theorems XI, XV,
and XVI) makes the issue unmistakably clear. Ours is a dynamic,
uncertain world—and therefore a world in which money has very im-
portant functions to perform. Hence any realistic attempt to de-
scribe this world must provide for the inclusion of money in the utility
function. With this inclusion there must simultaneously come the
abandonment of Say’s law, and the recognition that at least one of the
excess-demand functions cannot depend solely on relative prices.34
Until these things are done, our analysis will deal not with a real mone-
tary economy, but with an emasculated one in which money merely
performs the function of a counting unit.

34 Various ways of introducing this nonhomogeneity are discussed in “The
Casselian System,” §14.