

A Constant-Utility Index of the Cost of Living.¹

Attempts have been made to construct a true cost-of-living index by assuming knowledge of the Engel curves or by setting limits to the index in terms of Paasche and Laspeyres type indexes. The goal in all these attempts was to express the index of the cost of living in terms of measurable phenomena which are independent of the subjective concepts of utility. In this paper, we shall construct a cost-of-living index which depends only upon observable prices and properties of demand functions.

Let u = utility index ;
 x_i = quantity of the i -th commodity ;
 p_i = price of the i -th commodity ;
 r = income.

We assume the following propositions from the theory of consumer behaviour :

Utility function :

$$u = u(x_1, x_2, \dots, x_n) ;$$

Budget constraint² :

$$\sum_{i=1}^n p_i x_i = r ;$$

Conditions of utility maximization :

$$\frac{\partial u}{\partial x_1} / p_1 = \frac{\partial u}{\partial x_2} / p_2 = \dots = \frac{\partial u}{\partial x_n} / p_n ;$$

Slutsky equation :

$$\frac{\partial x_i}{\partial p_j} = -x_j \frac{\partial x_i}{\partial r} + s_{ij} ;$$

where s_{ij} = the substitution term, which is symmetrical in i and j .

We may solve the budget constraint and the maximization equations for each of the x_i as functions of the p_i and r . These are the demand equations :

$$x_i = x_i(p_1, p_2, \dots, p_n, r), \quad i = 1, 2, \dots, n.$$

Substitute the demand equations into the utility index to get :

$$u = u[x_1(p_1, p_2, \dots, p_n, r), \dots, x_n(p_1, p_2, \dots, p_n, r)] \quad (1)$$

The true cost-of-living index is defined as the ratio of two incomes. The denominator of this ratio is the actual base period income. The numerator is the smallest income required in order to buy, at current prices, that complex of goods which would leave one on the same level of utility as was experienced in the base period. This definition implies: $u = \text{constant}$, and $du = 0$.

In terms of equation (1) this definition leads to :

$$du = \sum_i \sum_j \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial p_j} dp_j + \sum_i \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial r} dr = 0 \quad (2)$$

Everywhere along the constant level of utility we have :

$$\frac{\partial u}{\partial x_1} / p_1 = \dots = \frac{\partial u}{\partial x_n} / p_n \quad (3)$$

¹ Included in Cowles Commission Paper, New Series, No. 26.

² In some versions of the theory r = expenditures. In our formulation r = income. Some of the X_i may be future commodities (savings), and the corresponding p_i are the prices of the commodities properly discounted to the present.

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These maximization conditions hold because we have defined the numerator of our index to be the *smallest* income which would leave the consumer at the constant level of utility.

On substituting (3) into (2), we get :

$$\sum_i \sum_j p_i \frac{\partial x_i}{\partial p_j} dp_j + \sum_i p_i \frac{\partial x_i}{\partial r} dr = 0 \quad (4)$$

$$\text{or } dr = - \frac{\sum_i \sum_j p_i \frac{\partial x_i}{\partial p_j} dp_j}{\sum_i p_i \frac{\partial x_i}{\partial r}}$$

This is a partial differential equation in r and the p_i , which is independent of the utility index.

Multiply both side of the Slutsky equation by p_i and sum both sides over all i . The result is :

$$\sum_i p_i \frac{\partial x_i}{\partial p_j} = -x_j \sum_i p_i \frac{\partial x_i}{\partial r} + \sum_i p_i s_{ij}.$$

It is a well established¹ theorem that :

$$\sum_i p_i s_{ij} = 0; \text{ therefore we get :}$$

$$\sum_i p_i \frac{\partial x_i}{\partial p_j} = -x_j \sum_i p_i \frac{\partial x_i}{\partial r} \quad (5)$$

On substituting (5) into (4), our differential equation becomes :

$$dr = \sum_i x_j (p_1, \dots, p_n, r) dp_j \text{ along } u = \text{constant} \quad (6)$$

If both sides of (6) are divided by $r = \sum_j p_j x_j$, the differential equation becomes the familiar Divisia-type price index. However, an index of the cost of living cannot be calculated from (6) by the usual methods² used to approximate Divisia-type indexes because the $x_j (p_1, \dots, p_n, r)$ which appear as weights, must be taken along the constant level of utility, and these quantities are not, in general, directly observable.

First, we must examine the integrability conditions for the partial differential equation (6). If there is to exist a function :

$$r = r (p_1, p_2, \dots, p_n) \quad (7)$$

as an integral of (6), we must have :

$$\left[\frac{\partial x_j (p_1, \dots, p_n, r)}{\partial p_i} \right]_{u=\text{const.}} = \left[\frac{\partial x_i (p_1, \dots, p_n, r)}{\partial p_j} \right]_{u=\text{const.}} \quad (8)$$

Equation (8) always holds as a consequence of the symmetry properties of the Slutsky equations. We have :

$$\left[\frac{\partial x_j (p_1, p_2, \dots, p_n, r)}{\partial p_i} \right]_{u=\text{const.}} = \frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial r} \frac{\partial r}{\partial p_i} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial r} = s_{ji}$$

since equation (6) requires that : $\frac{\partial r}{\partial p_i} = x_i$.

¹ See, e.g. J. R. Hicks, *Value and Capital*, Oxford Press, 1939, p. 310-11. This proposition can very easily be proved by using the Slutsky equation and the condition that the demand equations are homogeneous of order zero in prices and income.

² See G. C. Evans, *Mathematical Introduction to Economics*, McGraw Hill, 1930, p. 103.

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Similarly we have $\left[\frac{\partial x_i (p_1, \dots, p_n, r)}{\partial p_j} \right]_{u=\text{const.}} = s_{ij}$.

Thus the integrability conditions for (6) hold if the Slutsky equation holds, and equation (7) exists as the constant-utility level of income for the numerator of the cost-of-living index. If we know the demand functions, and if the Slutsky equation holds, we can always compute the true cost-of-living index by integrating (6). We shall next proceed to obtain an explicit form for (7) in a simple case.

Suppose that the demand equations are all of the form :

$$x_i = \sum_j \alpha_{ij} \frac{p_j}{p_i} + \beta_i \frac{r}{p_i}, \quad i = 1, 2, \dots, n. \tag{9}$$

The α 's and β 's in (9) are not unrestricted. If we multiply both sides of (9) by p_i and sum over all i , we get :

$$\sum_i p_i x_i = \sum_i \sum_j \alpha_{ij} p_j + r \sum_i \beta_i = r.$$

Two sets of restrictions are :

$$\sum_i \beta_i = 1 \text{ and } \sum_i \sum_j \alpha_{ij} p_j = 0. \tag{10}$$

The symmetry properties of the Slutsky equation imply :

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial r} = \frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial r}, \tag{11}$$

or $\frac{\alpha_{ij}}{p_i} + x_j \frac{\beta_i}{p_i} = \frac{\alpha_{ji}}{p_j} + x_i \frac{\beta_j}{p_j}, \quad i \neq j$

On substituting (9) into (11), we obtain :

$$p_j \alpha_{ij} + \beta_i \sum_k \alpha_{jk} p_k = p_i \alpha_{ji} + p_j \sum_k \alpha_{ik} p_k, \quad i \neq j. \tag{12}$$

If the coefficients of p_k on both sides of (12) are to be equal we must have :

$$\beta_i \alpha_{jk} = \beta_j \alpha_{ik}; \quad i \neq j \neq k \tag{13}$$

hence the coefficients of the price ratios in (9) must be of the form :

$$\alpha_{ij} = \beta_i \gamma_j, \quad i \neq j. \tag{14}$$

For the more general case including $i = j$, we have :

$$\alpha_{ij} = \beta_i \gamma_j + \delta_{ij} \epsilon_i \tag{15}$$

where $\delta_{ij} = 1$ for $i = j$, and 0 for $i \neq j$. From (10) we see that :

$$\sum_i \sum_j \beta_i \gamma_j p_j + \sum_i \sum_j \delta_{ij} \epsilon_i p_j = 0 \text{ and } \sum_i \beta_i = 1; \text{ hence it follows that :}$$

$$\epsilon_i = -\gamma_i \dots \dots \dots \tag{16}$$

If we substitute (14) and (16) into (9), we obtain :

$$x_i = \sum_j \beta_i \gamma_j \frac{p_j}{p_i} - \gamma_i + \beta_i \frac{r}{p_i}, \quad i = 1, 2, \dots, n. \tag{17}$$

The demand functions (17) are now constructed to satisfy the budget constraint and the Slutsky equation. If these functions are substituted into the differential equation (6), we find the integral for r to be :

$$r = C \prod_i p_i^{\beta_i} - \sum_i \gamma_i p_i \tag{18}$$

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where C is a constant of integration. The cost-of-living index for this case is :

$$\frac{r}{r_0} = \frac{C \prod_i p_i \beta_i - \sum_i \gamma_i p_i}{C \prod_i (p_i)_0 \beta_i - \sum_i \gamma_i (p_i)_0} \quad (19)$$

In (19), the $(p_i)_0$ and r_0 are the observed prices and income of the base period.

The actual statistical estimates of r from (18) will not be exact. In the estimation of the β 's and γ 's from samples of observations on prices, quantities and incomes, there will be sampling errors. At best, we shall be able to estimate confidence intervals for the β 's and γ 's. The intervals for these parameters will imply, in turn, specific intervals for r . These intervals have one advantage over the intervals usually developed for the cost-of-living index from indexes of the Paasche and Laspeyres type. The latter intervals provided only an upper limit for the index on one base and a lower limit for the index on another base ; they do not provide upper and lower bounds for a given index simultaneously. On the other hand, our intervals give upper and lower bounds for the index on any base ; and, furthermore, it is possible to calculate the probability that the true value of the index will be covered by our interval in repeated samples.

It should be pointed out, however, that our index is subject to the restrictions that apply to all indexes based on the theory of consumer behaviour. In addition to the fulfilment of the obvious restriction that the base period indifference map remains invariant in other periods, there are more subtle restrictions connected with the transformation from individual to market data. What the economic theory of the true cost-of-living index really gives is a cost-of-living index for an individual. This means that to use our method of calculating the index on the basis of parameters of the demand functions, we must have a knowledge of the properties of the demand equations of an individual. The demand equations that we customarily estimate statistically are demand curves based on market data that refer to an entire group of individuals.

The following interpretation of our results, and, for that matter, the results of other methods of calculating the true cost of living, seems to be the most satisfactory solution. Regard the indifference map as pertaining to an average individual, say, a family head earning an average wage in a typical urban area. This is specifically the situation that is claimed for many official cost-of-living indexes. Estimate the demand equations from *per capita* market data, and regard these statistical demand equations as pertaining to the average individual whose cost-of-living is being measured. This is the usual interpretation of *per capita* demand equations. Then use the estimated parameters of the *per capita* demand equation to calculate the cost-of-living index for the average individual whose utility level is to remain constant.

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