

ON MATHEMATICS FOR ECONOMISTS<sup>1</sup>

J. MARSCHAK

IN a recent paper, John Maurice Clark pleaded for greater communicability between mathematical economists and others:<sup>2</sup> the results obtained by mathematical economists "would be far more useful if they became part of the common equipment of economists in general;" the general economist should have these results put for him "in a form that makes it possible for him to appraise their significance." Professor Clark advocates, as it were, *translating* from one language into another to establish communication. Professors Crum and Schumpeter give an alternative: to *learn* the foreign language, or at least its rudiments.

In 1938, Professor Crum published "Rudimentary Mathematics for Economists and Statisticians" as a supplement to the *Quarterly Journal of Economics*. In the preface, the author remarked:

. . . in offering the book to those economists who may hope to benefit by its study, I shall not relax advocacy of every effort directed to extending and strengthening the mathematical training of prospective economists. I surely express the sincere wish of many colleagues in venturing to hope a later generation of economists will be so well prepared in mathematics that occasion for a book of this sort may disappear.

The fairly large printing of the 1938 volume was rapidly exhausted, and the book under review is a second, enlarged edition. Why the first edition was exhausted, and why the need for the second edition has been felt, must be clear to any observer of recent developments in economic teaching, economic research, and even the practical application of economics in government and business.

To begin with an example from "micro-economics" (theory of a firm or household): Of the

young college students who honestly try to understand why, say, minimum average cost "must" equal marginal revenue and marginal cost, a large proportion soon gives up and takes to parrot work for examinations. But members of an energetic minority realize — sometimes helped by students in science departments — that by learning a little calculus on the side, one can escape the cramming of a book full of elusive words. Two pages of clear reasoning suffice to set out clearly the assumptions of maximized profits, and to deduce the implications for a specified set of institutional assumptions (of perfect competition or otherwise).

Less widely recognized is the usefulness of mathematical habits of thinking in the field of macro-economics — the field which J. M. Clark describes in his address as

. . . the study of relations between tangible aggregates in the economy at large. There mathematics is at its best, verification is most nearly practicable, and the difficulties of other methods are at their greatest.

This view of Clark's is confirmed by the experience of recent discussion on predicting unemployment from economic models. How much sooner would the authors understand each other if each had taken the trouble of writing down the system of equations he had in mind! It would come out that Keynesians as well as non-Keynesians (whatever these labels may mean operationally), governmental pessimists and industrial optimists have used essentially the same system of equations. Disregarding encumbering details and some more sophisticated variants, every estimator talked, unwittingly or not, the prose of the following three equations (two identities and one behavior equation):

- (1) Unemployment equals labor force minus the quotient of net output and labor productivity;
- (2) Net output is used up in consumption, private net investment (including undesired inventories), and government purchases of goods and services;
- (3) Consumption depends on net output (income) minus personal taxes, and possibly on past income and available liquid assets.

<sup>1</sup> Review of *Rudimentary Mathematics for Economists and Statisticians* by W. L. Crum and Joseph A. Schumpeter, New York and London: McGraw Hill Company, 1946, IX, 183 pages. This review will be included in Cowles Commission Papers, New Series, No. 25.

<sup>2</sup> "Mathematical Economists and Others: A Plea for Communicability," *Econometrica*, Vol. 15, No. 2, pp. 75-78 (a section of J. M. Clark's address on "Some Cleavages among Economists," *American Economic Review*, Vol. 37, No. 2, *Proceedings*, May, 1947, pp. 1-11).

Thus three variables — net output, unemployment, consumption — are determined for any set of values of the remaining variables, regarded as parameters. The latter are assumed to be determined by sociological or political forces. The economist estimates the parameters of the behavior equation (3) from experience (time series, family budgets, attitude surveys). He then posits plausible values, or a range of alternative plausible values, for each of the other parameters — labor force and productivity, government expenditures and taxes, private investment, possibly liquid assets and past incomes — using all the sociological, technological, and political information available. The three equations can then be solved for net output, consumption, and unemployment. To be sure, most estimators treated the system not by solving it in a straightforward way but by trial and error. Nevertheless, the logical structure was essentially the same. They obtained divergent results because they inserted different values for the parameters. But it is exceedingly difficult to compare the results and uncover the reason for divergences because few authors set out their assumptions and unknowns with sufficient clarity. This is typical of many economic discussions.

The phenomenon known in the classroom as “the fear of using the blackboard” is not fear of unfamiliar symbols (words are also symbols!). It is the fear of committing oneself to a set of clearly stated propositions: it is easier to hide confusion in a long sentence. Another example of this psychological defect is suggested by a fact pointed out in J. M. Clark’s address (with reference to the substitution of production factors in response to change in relative costs): “Essential factors of time are terribly easy to ignore.” I think they would not be so easy to ignore if every author were required to write his assumed relations distinctly, with time-lags specified as to their length (sometimes zero). Lagged relations (expressed as difference equations) are, in fact, at the very basis of the modern theory and measurements of market fluctuations and adjustments. Far from concealing the “essential factors of time,” mathematical economics treats them most explicitly and makes them amenable to verification.

Similarly, much confusion would be avoided if students acquired the habit of assigning to each variable its unit of measurement. It does happen that economists get impressed by the numerical closeness between the “multiplier” (say, 3) and the income-velocity of circulation (say, 3 per year; yet 300 per century!); or that the effect of issuing a billion paper dollars is compared with the effect of government spending at a rate of 1 billion dollars a year!

Another elementary yet useful habit is to count the variables and the (independent) equations of a system as a step (though not always a sufficient one) in testing the consistency and completeness of a theory; and to see, in particular, how many degrees of freedom (number of variables that can be simultaneously given independent values) are left for the policymaker.

The *ceteribus paribus* clause is a common source of embarrassment to economic students. For example, demand for food depends on food price, with money income constant; and on money income, with food price constant. But money income includes farm income and is therefore itself not independent of food price. A student who has understood the logic of partial and total derivatives has rid himself of this type of embarrassment once and for all.

Also, economic argument often fails to indicate whether a proposition A implies B (if A then B), or whether A is implied by B (only if A then B). In a mathematical textbook the economist will learn the important distinction between sufficient and necessary conditions.

Many other good logical habits are often absent in economic learning and teaching, analysis and action, but are present in the most elementary mathematical study. Since habits are acquired by exercise, not exhortation, the usefulness of mathematical studies is obvious. In particular, the student should develop a skill in formulating economic problems mathematically. The book under review is too short to achieve this successfully, although it gives proper attention to a few “bad habits” of economists.<sup>3</sup>

The mathematics contained in the book under review is of greatly varying degree of diffi-

<sup>3</sup> Especially in its persuasive footnotes, such as that on p. 84, on non-temporal variations.

culty. Almost one-quarter of the text — its first two chapters of 40 pages — is used to get the student slowly acquainted with simple graphs. The next two chapters deal with limits, and with rates and derivatives, including the case of two or more independent variables. Maxima and minima are treated next. Of the two final chapters, on differential equations and on determinants, the former seeks to acquaint the reader with a variety of cases and can hardly be understood by the reader whom the first 40 pages were intended to reach, unless he has done much hard work in between. This hard work would be presumably based on exercises beyond those discussed in the rest of the book itself. The lack of exercises for the student (as usually provided in mathematical textbooks, including R. G. D. Allen's *Mathematical Analysis for Economists*) is the greatest gap in this book. May we hope that this need will be supplied in a later edition of the book, or in a companion volume?

The present volume has about 25 pages more, of text and footnotes, than its predecessor. Except in one case (a footnote explaining logarithms) the changes indicate that the authors' experience — perhaps in particular that of the new author, Professor Schumpeter — made them feel the students' need for more rather than less mathematics. They have added sections on Taylor expansion; homogeneous functions; Lagrange multipliers; the concept of integral; operator notation for derivatives; dynamic economic equations; partial differential equations; and a new chapter of eight pages on determinants. Because of the very limited space, and of the absence of exercises, the student is given not more than the first, very general acquaintance with these subjects; in some cases, not much more than a hint.<sup>4</sup>

Opinions on the choice of mathematical subject matter will differ. It would seem to the present reviewer that the introductory chapters might be shortened considerably, and the

space so gained might be used for other elementary subjects like the dimensionality of variables and the determinateness (or, more generally, rank) of equation systems. I have already mentioned above both these subjects as essential to an economist's sound thinking. The second of them requires treating determinants rather early in the course. This helps, in the experience of the reviewer, when, at a later stage, the highly important technique of implicit differentiation of equation systems is introduced. Economic functions are seldom fully specified as to their form. Therefore, economic equations often have to remain unsolved with respect to any of the variables. To study the effects of a change in a given variable (regarded as a parameter) one has to differentiate, with respect to this variable, the relevant equation or system "as it stands." The resulting equation or system is linear in the unknown derivative or derivatives and can be solved with respect to them. This is how some of the most important theorems of modern economics are obtained: e.g., the Slutsky-Allen-Hicks relation between income-elasticity and price-elasticity of demand; or statements on the effect of changing wage-rates or interest-rates upon national income.

All examples used to illustrate the mathematical theories contained in the book are taken from economics, mostly from the economics of a single firm (cost functions), single household (utility function), or a single commodity market (demand curve). One might wish a greater emphasis on relationships between aggregates. This desire of the reviewer overlaps another one: the multi-equational nature of economics (whether micro- or macro-economics) ought to be more strongly emphasized. The student should learn to express in equations any "theory of international trade," or "tax incidence," or "money and capital markets," or any other hypothesis of his own making or of some writer's making; to show whether such a system is consistent and complete; and to discuss the effects of a given change in its parameters.

The suggested kind of economic material might usefully replace certain traditional relics of doctrine. One such relic is the linear homogeneous production function which Wicksteed

<sup>4</sup> Many students will feel dissatisfied with the treatment of second order conditions for absolute and for restricted maxima of functions of  $n$  variables: the chapter on maxima and minima gives these conditions very incompletely (pp. 128, 131), to be supplemented somewhat hastily in the chapter on determinants (p. 178). Yet the subject has bearing upon the important question of economic stability, and of economic functions compatible with stability.

introduced and linked with a theorem of Euler. Even though, in the words of Crum and Schumpeter, "the partiality of many theorists to this type of production function is amply accounted for by the interesting results that follow from it" (p. 107), the yield is hardly worth the economic student's effort.<sup>5</sup>

<sup>5</sup> Upon closer examination, the properties of linear homogeneous functions should rather deter economists from using them as (approximations to) production functions of firms. If prices are independent of the firm's output and of the amount of factors used, its profit is a linear homogeneous function of production factors if the production function is linear homogeneous. Profit has, in this case, no proper maximum with respect to the production factors, though it has a proper maximum with respect to their ratios. Those who introduced into the production theory Euler's theorem on homogeneous functions did so to overcome the overdeterminateness of a system in which maximum profit is zero while relative prices are rigid. Actually, they made the system either inconsistent with given prices or indeterminate with respect to the factors of production. (This becomes particularly clear when there is only one variable factor of production, so that a homogeneous linear production function implies that average and marginal costs are constant.) If the book would state the second order conditions for the maximum more fully than it has done (pp. 131; 178, n. 2) these implications of the homogeneous production function would come out flagrantly. (See Marschak and Andrews, *Econometrica*, 1944; especially §§ 38, 41 *et seq.*). It must be added that the theoretical tradition as rendered in R. G. D. Allen's *Mathematical Analysis for Economists* — in the text as well as in some exercises — also fails to achieve clarity in this field.

Nor is the following remark of the authors helpful, although it repeats a statement occasionally encountered in literature: "Fortunately for the so-called marginal-productivity theory of distribution, it is not necessary that this relation [sc. Euler's equation for homogeneous functions of the first degree.] hold identically (for all values of the variable). It is sufficient that it holds in an equilibrium point" (p. 109). This would seem to assign to the equilibrium point (i.e., to certain amounts of production factors) the property of making profit both a maximum and zero. If such a point exists, the marginal returns to scale are indeed easily shown to be constant in its neighborhood and thus reminiscent of linear homogeneous production functions. But such a point can exist only for a certain set of relative prices; and unless all firms have identical production functions this set of prices (the one consistent with maximum-at-zero profits) will vary from firm to firm, which is ruled out by perfect competition. If, on the other hand, firms are assumed to have identical production functions, it is difficult to see how those surviving the "free entry and exit" are selected. Nor is much help obtained by giving positive profits the name of rent.

All this points to the doubtfulness of zero-profits as a workable assumption. The inconsistency which it implies is not overcome either by assuming linear homogeneous production functions; or by making statements about the neighborhood of the equilibrium point; or by allowing prices to adjust themselves; or by letting the number of firms vary; or by the semantic subterfuge of "rent." A satisfactory static theory of production is still to be developed. It will have to start with the equilibrium of  $n$  firms, each

Many economic problems require mathematical techniques which are not taught in the usual courses of college mathematics, mostly designed for beginners in physical sciences or engineering. Such courses often give early and detailed attention to, say, conic sections; but postpone the partial and total derivatives until the very end, and suppress entirely the discussion of equation systems and their implicit differentiation. This leaves the economist ill-equipped. Another mathematical chapter of much use to economists (as a tool for dealing with time-lags) is the difference equations. Dynamic economics is touched upon by Professors Crum and Schumpeter in illustrating linear differential equations of second order,<sup>6</sup> giving rise to exponential or periodic solutions. This might be the place to introduce (linear) difference equations, usual in the current theory of business cycles. Or, to name two other important aspects of economics: The modern economic theory of risk and uncertainty — applied, for example, in the discussion of effects of profit taxes on investment of various types — requires the introduction of simple probability distributions and the construction of stochastic equations. And the recent fundamental revision of economics as a *Theory of Games and Economic Behavior* (von Neumann and Morgenstern) has shown the usefulness, for economists, of symbolic logic and the theory of sets.

It is doubtful whether the full title of the

---

having, in general, a different production function and some having zero-outputs of certain products.

<sup>6</sup> On p. 162, "Sales being denoted by  $S$ , the time rate of sales was  $dS/dt$ , and the rate of increase of this rate of sales was  $d^2S/dt^2$ ." The economic meaning of "sales" as distinct from "rate of sales" is not clear. Perhaps  $S$  should denote inventories, and  $dS/dt$  the excess of sales over production? But then the role of the second derivative of  $S$  in the description of human behavior becomes rather artificial. Nor is human behavior (the very basis of deductions of economic theory) too well related to the only other dynamic economic example of the book, that on p. 164. Would it not be more instructive to confront the student, instead, with some of the outstanding existing dynamic theories, e.g., with Schumpeter's own business-cycle theory? Being based on the assumed behavior of several distinct social groups such a theory would presumably consist of a set of equations, at least one for each group. But just this is an essential characteristic of economic reality, and the student can only benefit by being faced with it as early as possible

book is justified. It contains rudimentary mathematics for economists, but statisticians need a different kind of mathematical instruction.

It seems that even when the desire expressed in Professor Crum's preface and quoted at the beginning of this review is fulfilled, and the mathematical preparation of economists is considerably improved — e.g., made to approach in sheer bulk that of some engineers or biologists, or even of physicists — there will be a need to survey selected mathematical chapters in special application to economics.<sup>7</sup> This is confirmed by class experience with students who switched from mathematics or physics to economics. Therefore, books which, like that of Crum and Schumpeter (or, for that matter, like the more extensive book of R. G. D. Allen), present economic problems in the order of mathematical difficulty will long retain their usefulness if properly supplemented and expanded. As a follow-up, a different type of book is also needed: a systematic account of the present state of theoretical micro- and macro-economics, both static and dynamic, under conditions of certainty as well as uncertainty. In such a book, mathematics would be used as freely and on as advanced a level as is required for a tolerably logical presentation. Such a book would discuss economic problems for their own sake and in their own logical order, rather than as illustrations for mathematical theorems and techniques. We need both

kinds of textbooks: on mathematics for economists, and on mathematical economics.

With the present student body, the economists' training in mathematics and mathematical economics, beginning where Professors Crum and Schumpeter begin, challenging students' thought by numerous exercises and covering all major fields of economic application, should occupy considerably more time than is now usual. As a minimum one might visualize a sequence of three courses: (1) mathematics for economists: functions of one variable; (2) mathematics for economists: functions of  $n$  variables; (3) mathematical economics. Experience indicates that with the present selection of students, such a sequence would require in general somewhat more than an academic year of 3-4 weekly class hours and 5 or more weekly hours of home work. However, the following alternative would be much more satisfactory. To discipline himself in independent and reliable use of mathematics, the student should start in a department of mathematics with elementary training in calculus and, say, the theory of equations, during a full academic year. He can then proceed directly to the courses (2) and (3) just mentioned. For such a plan, R. G. D. Allen's textbook (of which the first 10 chapters or so, dealing with functions of one variable, can then be skipped except for the "economic exercises" contained in them) is better suited than the book under review; although Allen's excellent book — and especially its examples — might also be improved by some shift in emphasis toward the fields indicated in the earlier parts of this review.

Universities have begun to realize the need for better mathematical training of economists. Yet some time will elapse before the full extent of the necessary re-tooling of future economic theorists is recognized.

<sup>7</sup> One has to disagree with Professors Crum and Schumpeter when (as on pp. 8; 90, n. 1; 96; 102) they repeatedly tell the reader that certain more complicated mathematical cases "are not needed in elementary economic work." Who can tell? Derivatives of periodic functions, treated on p. 96 as mere luxury, might usefully be applied on p. 105 when a dynamic differential equation is discussed! Implicit functions, discussed on p. 8 as "non-elementary," are both easy to handle and of great importance to economists.