

Cowles Foundation Paper 24

Reprinted from
ECONOMETRICA, Journal of the Econometric Society, Vol. 15, No. 2, April, 1947
The University of Chicago, Chicago 37, Illinois, U.S.A.

STATISTICAL ANALYSIS OF THE DEMAND FOR FOOD: EXAMPLES OF SIMULTANEOUS ESTIMATION OF STRUCTURAL EQUATIONS*¹

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I. INTRODUCTION: THE SIMULTANEOUS-EQUATIONS APPROACH

IN ECONOMIC theory it is shown that the demand for a commodity can be considered as a function of the price of the commodity, the prices of other commodities, and the disposable income of the consumer. By analogy, we are led to the hypothesis that the total demand for the commodity may be considered a function of all prices and of total disposable income of all consumers. The ideal method of verifying this hypothesis and of obtaining a picture of the demand function involved would be to conduct a large-scale experiment, imposing alternative prices and levels of income on the consumers and studying their reaction. If we could produce a large number of observations in this way we should probably find that the data would not satisfy, exactly, any simple functional relationship. Let x_t be the t th observation of the quantity consumed, $p_{1t}, p_{2t}, \dots, p_{nt}$ the corresponding prices of the n commodities in the market, and y_t income, and let $F(p_{1t}, p_{2t}, \dots, p_{nt}, y_t; \alpha_1, \alpha_2, \dots, \alpha_k)$ be a function containing k parameters $\alpha_1, \alpha_2, \dots, \alpha_k$. Then we could write, in a purely formal way,

$$(1.1) \quad x_t = F(p_{1t}, p_{2t}, \dots, p_{nt}, y_t; \alpha_1, \alpha_2, \dots, \alpha_k) + u_t,$$

where u_t is a "residual." In order to operate with a reasonably simple function, F , with a finite number of parameters, α , it would be necessary to admit nonzero values of the u 's. The relation (1.1) would be only a useless rewriting of the facts unless we could say something more about the properties of the u 's. The u 's must have some properties that are predictable *on the average*. A rational way of expressing this hypothesis is to assume that the u 's are stochastic variables having certain char-

* This paper will be reprinted as Cowles Commission Papers, New Series, No. 24.

¹ The present paper contains some preliminary, and highly tentative, results of a project of quantitative research in agricultural economics, of the Department of Economics of the University of Chicago, and the Cowles Commission for Research in Economics. The authors gratefully acknowledge their indebtedness to colleagues at the U. S. Bureau of Agricultural Economics and the University of Chicago for helpful suggestions and constructive criticism. Special thanks are extended to Miss Selma Schweitzer and Mrs. Lois N. Shores. They have carried out most of the numerical work and have also helped to improve the exposition of the statistical methods involved. Needless to say, the authors assume all responsibility for the results presented.

acteristic distribution properties. The economic meaning of such a model is that, given a certain set of values of prices and income, consumers do not always behave in exactly the same way, perhaps because of the influence of other, neglected, variables or simply because the individuals are not absolutely consistent in their behavior.

If we were able to conduct an experiment as described we might take the values of the p 's and of y to be a set of fixed, predetermined numbers, while the observations of x would be random variables, the stochastic properties of which would be defined implicitly by the stochastic properties assumed for the u 's. On the basis of a set of observations and an assumed known form of the function F we might then be able to estimate the parameters α and also the parameters in the distribution of the u 's.

Implicit in this statement is the assumption that there are no errors of measurement in the p 's and y , in other words, that the observed values of the prices are the prices actually paid by the consumers and that the observations of y are correct measurements of their income.

But suppose it is not possible to carry out a rational experiment of the type described. Could we not assume that an "experiment" of a similar type is being carried out automatically by the market mechanism of the economy? Most of the numerical studies of demand functions have in fact been founded on this basis. It has been assumed that observed series of simultaneous values of consumer purchases, prices, and income represent data that are statistically of the same nature as those that one would obtain by an experiment of the type described. On this basis various types of demand functions have been fitted to observed data by choosing quantities consumed as the "dependent" variables and the prices and income as "independent" variables. There is a fundamental error in this approach. It leads to a logical contradiction: The demand function should, theoretically, be independent of the manner in which the prices and income are being fixed. But the "demand" function obtained by fitting the function (1.1) to market data for quantities purchased, prices, and income—as if they were the results of our hypothetical experiment—will depend on the nature of the other economic relations that, together with the demand function, determine the observable quantities, prices, and incomes. It is not always easy to see this intuitively. A simple example might be helpful.

Let x_t be a time series of the quantity consumed of a certain commodity and let p_t be the corresponding price series. And suppose that the demand function to be estimated is of the simple form

$$(1.2) \quad x_t = \alpha p_t + \beta + u_t, \quad t = 1, 2, \dots$$

Suppose it is known that the u 's are normally and independently dis-

tributed with $E(u_t) = 0$, $E(u_t^2) = \sigma_u^2$ for all values of t . We assume that x and p are observed without errors of measurement. Suppose that we try to estimate α and β by a least-squares regression of x on p . We shall show that the result of this procedure *will depend on the form of the supply function*. Let us assume two different alternatives for the supply function:

$$(1.3a) \quad x_t = h_1 p_t + k_1 + v_t$$

or

$$(1.3b) \quad x_t = h_2 p_{t-1} + k_2 + w_t,$$

where the v 's and the w 's are random residuals while the h 's and the k 's are constants.

Suppose, first, that (1.3a) is the true supply function. And let us assume that the v 's are normally and independently distributed with $E(v_t) = 0$ and $E(v_t^2) = \sigma_v^2$ for all values of t . Let $E(u_t v_t) = \sigma_{uv}$ be the covariance between u and v . Let, further, m_{xx} , m_{xp} , etc. denote second-order moments about the mean, and m_x , m_p , etc. the means of the variables, over the range $t = 1, 2, \dots, T$. Then the regression of x on p yields a regression equation

$$x^{(\text{est})} = ap + b,$$

where

$$(1.4) \quad a = \frac{m_{xp}}{m_{pp}},$$

$$(1.5) \quad b = \frac{m_x m_{pp} - m_p m_{xp}}{m_{pp}}.$$

Let us consider the estimate, a , that would be obtained for an infinite sample, so that we do not have the extra complications of sampling variations. In order to see what the formula (1.4) would mean under this condition we solve the two equations (1.2) and (1.3a) for x_t and p_t , obtaining

$$(1.6) \quad x_t = A + U_t,$$

$$\text{where } A = \frac{\alpha k_1 - \beta h_1}{\alpha - h_1}, \quad \text{and} \quad U_t = \frac{\alpha v_t - h_1 u_t}{\alpha - h_1}$$

and

$$(1.7) \quad p_t = B + V_t,$$

$$\text{where } B = \frac{k_1 - \beta}{\alpha - h_1}, \quad \text{and} \quad V_t = \frac{v_t - u_t}{\alpha - h_1}.$$

Equations (1.6) and (1.7) are called the *reduced form* of the system (1.2) and (1.3a). For an infinite sample we can replace the moments m_{xp} and m_{pp} by the covariance σ_{xp} and the variance σ_{pp} respectively, where, from (1.6) and (1.7),

$$(1.8) \quad \sigma_{xp} = \frac{\alpha\sigma_v^2 - (\alpha + h_1)\sigma_{uv} + h_1\sigma_u^2}{(\alpha - h_1)^2},$$

$$(1.9) \quad \sigma_{pp} = \frac{\sigma_v^2 - 2\sigma_{uv} + \sigma_u^2}{(\alpha - h_1)^2}.$$

From this we obtain a value for a that approaches, in the probability sense, the limit

$$(1.10) \quad \frac{\alpha\sigma_v^2 - (\alpha + h_1)\sigma_{uv} + h_1\sigma_u^2}{\sigma_v^2 - 2\sigma_{uv} + \sigma_u^2}.$$

Whether this is an estimate of α in the sense that, apart from chance fluctuations arising in samples of finite size, it equals α , depends not only on α but on the value of the σ 's and h_1 . In fact, if the supply equation (1.3a) is true, and we have no other a priori information, it is impossible to estimate α or β , by any method whatsoever. This is seen as follows: Let us multiply equation (1.2) by an arbitrary factor c and equation (1.3a) by $(1-c)$ and add the two equations. This gives

$$(1.11) \quad x_t = [h_1 + c(\alpha - h_1)]p_t + [k_1 + c(\beta - k_1)] + [cu_t + (1-c)v_t].$$

This equation is of the same form as any one of the original equations, and the residual term $[cu_t + (1-c)v_t]$ has exactly the same general properties as u or v , which are in any case not observable. By varying c we get an infinity of equations, any one of which can replace (1.2) or (1.3a) without any observable effect on the x 's and the p 's. Obviously the data do not contain any information by which to identify the particular equation (1.2).

Thus, if (1.3a) is the supply equation there exists no formula for estimating the parameters of the demand function (1.2). This cannot be seen from the specification of the demand function alone.

Let us now assume that (1.3b) is the true supply function instead of (1.3a). And let us again solve the system for x_t and p_t . We then obtain, as the *reduced form* of the system (1.2) and (1.3b), the following system:

$$(1.12) \quad x_t = h_2 p_{t-1} + k_2 + w_t \quad [\text{Equation (1.3b)}],$$

$$(1.13) \quad p_t = \frac{h_2}{\alpha} p_{t-1} + \frac{k_2 - \beta}{\alpha} + \frac{w_t - u_t}{\alpha}.$$

Obviously p_{t-1} does not depend on u_t and w_t . It depends only on u_{t-1}, u_{t-2}, \dots , and w_{t-1}, w_{t-2}, \dots . By fitting each of the two equations (1.12) and (1.13) to the data by the method of least squares using p_{t-1} as the independent variable we can obtain estimates of the coefficients of the reduced form, viz., $h_2, k_2, h_2/\alpha$, and $(k_2 - \beta)/\alpha$. We can get these estimates as accurately as we please by taking a sufficiently large sample, supposing the assumptions made to be valid over a sufficiently long period. From these four estimates we can in turn calculate the corresponding values of α and β . *But this method of estimating α and β could not have been deduced from the specification of (1.2) alone.*

Suppose that, in this second model, we should consider the regression of x on p , that is, the regression coefficients (1.4) and (1.5). Let us calculate what $a = m_{xp}/m_{pp}$ would be in the present model, assuming an infinite sample. From (1.12) and (1.13) we obtain

$$(1.14) \quad m_{xp} = \frac{h_2^2}{\alpha} m_{p_{t-1}, p_{t-1}} + \frac{1}{\alpha} (\sigma_w^2 - \sigma_{uw}),$$

$$(1.15) \quad m_{pp} = \frac{h_2^2}{\alpha^2} m_{p_{t-1}, p_{t-1}} + \frac{1}{\alpha^2} (\sigma_w^2 - 2\sigma_{uw} + \sigma_u^2).$$

Therefore,

$$(1.16) \quad a = \frac{m_{xp}}{m_{pp}} = \frac{\alpha(h_2^2 m_{p_{t-1}, p_{t-1}} - \sigma_{uw} + \sigma_w^2)}{h_2^2 m_{p_{t-1}, p_{t-1}} + \sigma_u^2 - 2\sigma_{uw} + \sigma_w^2}.$$

Obviously, this expression is not, in general, equal to α , nor is it, in general, equal to the expression (1.10).

These illustrations should be sufficient to show that it is not possible to devise estimation formulae for the estimation of a demand function on the basis of specification of this function alone. *That is, it is impossible to derive statistically the demand functions from market data without specification of the supply functions involved.* More generally, if we wish to estimate any particular economic relationship on the basis of market data we are forced to consider, simultaneously, the whole system of economic relations that together represent the mechanism that produces the data we observe in the market. Our examples above already indicate some of the tools of statistical technique that are available for dealing with problems of this nature. We shall attempt to set out the principles in somewhat more general terms.

Let $y_1(t), \dots, y_n(t)$ denote n observable economic time series. Assume that these n series satisfy n linear stochastic lag relations involving m observable "exogenous" time series $z_1(t), \dots, z_m(t)$, and n random, nonobservable residuals $u_1(t), \dots, u_n(t)$ expressing the

stochastic nature of economic behavior. To simplify the exposition we shall assume that there is only one type of lag terms involved, namely $y_i(t-1)$, $i=1, 2, \dots, n$. But our results can easily be generalized to the case where other lags are involved. Let this system of relations be

$$(1.17) \quad \sum_{j=1}^n \alpha_{ij} y_j(t) + \sum_{j=1}^n \beta_{ij} y_j(t-1) + \sum_{j=1}^m \gamma_{ij} z_j(t) = u_i(t),$$

$$i = 1, 2, \dots, n,$$

$$t = 1, 2, \dots, T.$$

It is assumed that the variables y and z are observable *without significant errors of measurement*, in other words, that these variables are measured according to their definition in our economic theory.

We make the following assumptions about the random elements $u_i(t)$:

$$(1.18) \quad E u_i(t) = 0, \quad i = 1, 2, \dots, n; t = 1, 2, \dots, T;$$

$$(1.19) \quad E[u_i(t)u_j(t)] = \sigma_{ij}, \quad i, j = 1, 2, \dots, n; t = 1, 2, \dots, T;$$

$$(1.20) \quad E[u_i(t)u_j(t-\theta)] = 0, \quad i, j = 1, 2, \dots, n; \theta \neq 0$$

$$(1.21) \quad z_j(t) \text{ stochastically independent of } u_i(t'),$$

$$j = 1, 2, \dots, m; i = 1, 2, \dots, n; t, t' = 1, 2, \dots, T.$$

(1.19) says that simultaneous values of the n u 's have a certain unknown n -by- n matrix of variances and covariances that do not depend on t . (1.20) means that there is no serial correlation in the u 's. It is easily seen that the assumptions (1.20) and (1.21) imply that $y_j(t-1)$ and $u_i(t)$ are stochastically independent for all values of i and j .

The problem is to estimate the α 's, the β 's, the γ 's, and the σ 's from the observations of the y 's and the z 's at successive points of time $t=0, 1, 2, \dots, T$. From the form of the system (1.17), however, we see immediately that there is, in each equation, an arbitrary proportionality factor that we can never estimate from the data, because we cannot observe the u 's and hence we do not know the scale of the σ 's. One way of disposing of this arbitrariness is to impose a *rule of normalization*, e.g., by assuming that one of the α 's in each equation is =1. After this normalization there remain $n(n-1)$ α 's, n^2 β 's, nm γ 's, and $n(n+1)/2$ σ 's to be estimated from the data.

Now, let us consider the *reduced form* of the system (1.17). This is the system of equations obtained by considering the system (1.17), for any given value of t , as a system of n linear equations in $y_1(t), \dots, y_n(t)$ and solving for $y_i(t)$ in terms of the lagged y 's, the z 's, and the u 's.

Obviously these solutions will be linear expressions in the variables $y(t-1)$, the $z(t)$'s, and the $u(t)$'s. That is, the reduced form of (1.17) will be

$$(1.22) \quad y_i(t) = \sum_1^n \pi_{ij} y_j(t-1) + \sum_1^m \pi_{ij}' z_j(t) + v_i(t), \quad i = 1, 2, \dots, n, \\ t = 1, 2, \dots, T,$$

where the π 's are constants that depend on the α 's, the β 's, and the γ 's in (1.17), while the v 's are new random residuals that are simply linear combinations of the u 's in (1.17). The v 's therefore have stochastic properties that are exactly similar to those of the u 's as given by (1.18)–(1.21). The variances and covariances of the v 's will depend upon the variances and covariances of the u 's and upon the α 's. In particular, the v 's will be stochastically independent of the variables $y(t-1)$ and the variables $z(t)$. This follows from our assumptions about the u 's. It can then be shown that by fitting each of the equations (1.22) to the data by the method of least squares, considering $y_i(t)$ as the dependent variable, the estimates p_{ij} and p_{ij}' obtained for π_{ij} and π_{ij}' , respectively, will possess certain optimal properties of "best estimates."

Obviously the system (1.22) is equivalent to the system (1.17) as far as the observations of the y 's and the z 's are concerned. Knowledge of the values of the π 's and the parameters of the distribution of the v 's would cover exhaustively all the implications of our theory as far as its observable consequences are concerned. This raises the following fundamental problem: To every system (1.17) there generally corresponds one and only one reduced form (1.22), that is, one and only one set of values of the π 's and the variances and covariances of the v 's. But does the converse also hold? That is, will the knowledge of the values of the π 's and the variances and covariances of the v 's determine uniquely the values of the α 's, the β 's, etc.? *This is the problem of identification.*

There are, altogether, $n(n+m)$ π 's and $n(n+1)/2$ variances and covariances of the v 's. These parameters can be estimated from the data. Since these parameters are functions of the α 's, the β 's, the γ 's, and the σ 's our estimates provide us with $[n(n+m) + n(n+1)/2]$ equations by which to calculate estimates of the original parameters. But from our counting above we found that, without additional restrictions upon the original parameters, there were $[n(n-1) + n(n+m) + n(n+1)/2]$ unknown parameters to be estimated. That is, there would be $n(n-1)$ more unknown parameters than could be estimated from the data. We then say that there is *lack of identification* in the system. The only way of getting around this difficulty is to assume that we have some

additional a priori information about the unknown parameters. The most frequently used type of assumptions of this kind is that some of the α 's, β 's, and γ 's are known to be equal to zero, which means that some of the variables do not actually occur in all of the equations. By adding a sufficient number of such restrictions the system may become "*just identified*"; that is, having estimates of the π 's and the variances and covariances of the v 's we have exactly enough independent equations to determine the original, unknown parameters. If we have even more a priori restrictions than are required for this purpose, the system may become *overidentified*, that is, the estimates of the π 's and the variances and covariances of the v 's give us more equations by which to derive the original parameters than there are such unknown parameters. This happens, of course, only if we *neglect the extra restrictions* in estimating the π 's and the variances and covariances of the v 's. To take account of side restrictions upon the π 's or the variance-covariance matrix of the v 's in the process of estimating these parameters often leads, however, to very complicated computational problems. To get around this difficulty, certain short-cut methods have been worked out, whereby part of the a priori restrictions upon the coefficients are neglected, and therefore, some statistical efficiency given up, in return for simpler computational procedure. These methods will be explained in Section III.²

The notions of "lack of identification," "just identified," and "overidentified" can obviously be applied also to a single equation in the system (1.17), by considering whether or not the knowledge of the π 's and the variance-covariance matrix of the v 's permit us to derive uniquely the parameters of that particular equation. Some of the equations in the system may be identifiable while others are not.³

It will be noticed that these properties of identifiability are formal properties of the system of equations that one is considering in each case. A careful analysis of these properties is necessary before one attempts to derive actual, numerical estimates.

In the following sections we shall discuss a tentative application of the various principles discussed above to a model of the connection between agriculture and the rest of the economy.

² See also T. Koopmans, "Statistical Estimation of Simultaneous Economic Relations," *Journal of the American Statistical Association*, Vol. 40, December, 1945, pp. 448-466.

³ For a more exhaustive and rigorous treatment of the problems of identification see *Statistical Inference in Dynamic Economic Models*, to be published as Cowles Commission Monograph No. 10.

II. MACRODYNAMIC MODELS EXPLAINING THE DEMAND FOR FOOD

We have pointed out that if one wants to study economic relations within one particular sector of the economy, one usually has to consider also the economic relations that govern the other sectors. But for practical reasons some simplification of the general theory is unavoidable. By methods of aggregation one has to try to reduce the number of the relations in which one is not directly interested to a minimum. The models below illustrate this principle.

A. *The Demand for Food*

If one divides total consumption into two groups, food and nonfood, one could say, by analogy from the microtheory of consumers' choice, that the per capita demand for food is a function of the price of food, the price of nonfood, and the per capita disposable income. Let c_1 denote annual per capita expenditure for food, and let p_1 be the price of food, p_2 the price of nonfood, P the total cost-of-living index, and r , per capita disposable income. We may take $y_1 = c_1/p_1$ as an index of the volume of food consumption. Since P is a function of p_1 and p_2 we might consider the variables p_1 and P instead of p_1 and p_2 . Let $y_2 = p_1/P$ denote the relative price of food and $y_3 = r/P$ the deflated per capita disposable income. Assuming the demand function to be a linear function of the relative price of food, y_2 , and the "real income," y_3 , we are then led to the hypothesis:

$$(2.1a) \quad y_1(t) = \alpha_{12}y_2(t) + \alpha_{13}y_3(t) + \alpha_{10} + u_1(t),$$

where the α 's are constants and $u_1(t)$ is a random residual (a random "shift") for each value of t . The demand for food may be subject to a trend due to changes in tastes, eating habits, etc. so that, alternatively, we might consider the demand

$$(2.1b) \quad y_1(t) = \alpha_{12}y_2(t) + \alpha_{13}y_3(t) + \gamma_{11}t + \alpha_{10} + u_1(t).$$

Still another alternative would be that the consumption of food also depends, to some extent, on past income $y_3(t-1)$, in which case we would write

$$(2.1c) \quad y_1(t) = \alpha_{12}y_2(t) + \alpha_{13}y_3(t) + \gamma_{11}t + \gamma_{12}y_3(t-1) + \alpha_{10} + u_1(t).$$

The α 's, the γ 's, and the u 's would of course have different interpretations in each of these three alternative equations. A similar remark applies to the various alternatives with respect to the other equations discussed below.

It will be noticed that in each of the three alternative demand equa-

tions there are three *simultaneous, or jointly dependent, variables*, that is, variables that have to be "explained" by other relations in the economic system, while the variables t and $y_3(t-1)$ may be considered as *given* or predetermined in the sense that they are stochastically independent of the random variable $u_1(t)$. We assume that $E[u_1(t)u_1(t-\tau)] = 0$ for $\tau \neq 0$, in other words, that there is no serial correlation in the residuals $u_1(t)$.

B. The Income Equation

In statistical studies of demand functions it has, usually, been assumed that income could be considered as a *given*, or independent, variable. In particular, it has been argued that, if the commodity in question represents only a small part of the budget, the repercussions upon income of variations in the demand for the commodity could be neglected. This hypothesis is obviously false. We could always split up total consumption into small subgroups by a sufficiently detailed specification of the various types of consumer goods. Obviously such a regrouping could not alter the fact that changes in the total consumer expenditures have a direct effect on income, income being the sum of consumers' expenditures and investment expenditures. We must therefore assume that income $y_3(t)$ depends to some extent on the random shifts $u_1(t)$ in the demand for food.

To arrive at an equation for income we must first derive the demand for all consumer goods. By our definition above this total demand is the sum of the demand for food and the demand for nonfood. Instead of considering the two equations: "demand for food," and "demand for nonfood," we might, however, equally well consider the pair: "demand for food," and "demand for all consumer goods." For the latter we may adopt the commonly accepted hypothesis that total per capita consumers' expenditure, deflated, is a linear function of real income, subject to random shifts. If $c(t)$ denotes per capita consumers' expenditure we therefore assume that

$$(2.2) \quad \frac{c(t)}{P(t)} = \alpha y_3(t) + \alpha_0 + u(t),$$

where $u(t)$ is a random residual that is the sum of the residual $u_1(t)$ in the demand equation for food and the corresponding residual in the demand function for nonfood. It will be noted that the prices p_1 and p_2 do not appear explicitly in this equation. This is equivalent to assuming that if the prices of individual commodities change there is no change in the relative allocation of income between present and future expected consumption (i.e., savings) except that which is brought about by the (real) income effect of the price change.

In order to derive an equation for income, y_3 , we now adopt the hypothesis that investment expenditures measured in constant dollars represent an *autonomous variable*, an impressed force. We define the investment expenditures, $z(t)$ as

$$(2.3) \quad z(t) = \frac{I(t)}{P(t)} = y_3(t) - \frac{c(t)}{P(t)},$$

where $I(t)$ is per capita investment expenditures in current dollars. From (2.2) and (2.3) we then derive

$$(2.4a) \quad y_3(t) = \gamma_{31}z(t) + \alpha_{30} + u_3(t),$$

where

$$\gamma_{31} = \frac{1}{1 - \alpha}, \quad \alpha_{30} = \frac{\alpha_0}{1 - \alpha}, \quad u_3(t) = \frac{u(t)}{1 - \alpha}.$$

This might be called "the multiplier theory of income." There might be a trend in the consumption function (2.2), in which case we would obtain

$$(2.4b) \quad y_3(t) = \gamma_{31}z(t) + \gamma_{32}t + \alpha_{30} + u_3(t).$$

Furthermore, it is possible that the consumption function also depends on lagged income, in which case we would obtain

$$(2.4c) \quad y_3(t) = \gamma_{31}z(t) + \gamma_{32}t + \gamma_{33}y_3(t - 1) + \alpha_{30} + u_3(t).$$

C. Supply of Food in the Retail Market

In order to arrive at an approximate model for the marketing chain for food products we propose to split the supply mechanism into two steps: namely the "supply by the farmers, demand by the commercial sector," and "supply by the commercial sector, demand by the consumers." In other words, we consider the whole commercial sector between the farmers and the consuming public as a "factory" buying raw materials from the farmers and supplying finished food products to the public.

Consider the supply of finished food products. In general, one might assume that this supply would depend on the retail price and the prices paid to farmers for crude foodstuffs. As an alternative to the variable "prices paid to farmers" in this supply function one might instead consider farm output of foodstuff, assuming that the farmer *has to sell*, once the foodstuff has been produced, and that, therefore, prices paid to farmers may be considered as a residual share. One might also expect a trend in the supply equation, due to gradual change in processing and marketing technique. Since the commercial sector has

the alternative of exporting instead of selling on the home market, and of importing food instead of buying from domestic farmers, the export and import prices might also enter the supply equation. In that case we assume, as an approximation, that the export and import prices are proportional, so that the "foreign" price can be represented by one price, the export price.

Let $y_4(t)$ denote the per capita supply of food by farmers to the commercial sector. (For simplicity we may consider—somewhat artificially perhaps—the farmers' own food consumption as also going through the commercial channels.) Let, further, $y_5(t)$ denote prices paid to farmers, and let $p_e(t)$ be the price of food in foreign markets. Both these prices should be considered as "normalized" by deflating them by the general cost-of-living index. Then we might consider the following alternative hypotheses regarding the retail supply of food:

$$(2.5a) \quad y_1(t) = \alpha_{22}y_2(t) + \alpha_{24}y_4(t) + \gamma_{21}t + \alpha_{20} + u_2(t),$$

$$(2.5b) \quad y_1(t) = \alpha_{22}y_2(t) + \alpha_{25}y_5(t) + \gamma_{21}t + \alpha_{20} + u_2(t),$$

$$(2.5c) \quad y_1(t) = \alpha_{22}y_2(t) + \alpha_{25}y_5(t) + \gamma_{21}t + \gamma_{22}p_e(t) + \alpha_{20} + u_2(t).$$

D. Supply of Foodstuffs by Farmers

For many farm products one might consider current output as a result of decisions based on *past* prices and other variables not related to the current market situation, such as weather, pasture conditions, available acreage, etc. The farmers have, on the other hand, undoubtedly some possibilities of almost instantaneous adjustment to the current price situation. They can speed up or slow down the feeding of livestock, put more labor, or less labor, into harvesting crops, etc. Other products, such as vegetables, or poultry, may have a period of production much shorter than a year. When we use annual data it would, therefore, seem necessary to include current prices as a variable influencing farm output of food. A trend might account for certain technological changes in technique of production, changes in the farm population, etc. We might then consider the following equation as an approximation to the farmers' supply equation:

$$(2.6a) \quad y_4(t) = \alpha_{45}y_5(t) + \gamma_{41}t + \gamma_{42}y_5(t-1) + \alpha_{40} + u_4(t),$$

where the random residuals $u_4(t)$ might be expected to be large, particularly because we have no explicit variable accounting for the influence of the weather.

The supply equation (2.6a) does not explicitly take account of the effect of change in capacity, such as change in acreage, livestock, farm machinery, etc. One way of accounting for such changes might be

to include last year's production as an additional variable, a "scale factor," for the output of the current year. This leads us to

$$(2.6b) \quad y_4(t) = \alpha_{45}y_5(t) + \gamma_{41}t + \gamma_{42}y_5(t-1) + \gamma_{43}y_4(t-1) + \alpha_{40} + u_4(t).$$

If we were interested in a more detailed study of the determinants of food output, the equations (2.6a) or (2.6b) could probably not be considered as adequate behavioristic equations for the production policy of the farmers. It would be necessary to study production functions, principles of profit maximalization, etc. The equations (2.6a) or (2.6b) must, to some extent, be considered as "derived" equations, the parameters of which are again functions of certain behavioristic parameters.

For the purpose of studying the demand for food one might even consider a much simpler hypothesis, namely:

$$(2.6c) \quad y_4(t) = \text{a predetermined variable.}$$

That is, we might think of farm food output as being practically independent of the current market situation. Such a hypothesis is obviously not strictly true. On the other hand, noneconomic factors, such as weather, together with lagged prices and other factors that do not depend on the current market situation, might be so dominant in determining current farm output that the errors of assuming that 1) α_{45} is zero and 2) $u_4(t)$ is uncorrelated with the u 's in the other equations are not serious. Under these conditions it would be permissible, for the purpose of estimating the other equations in our system, to consider $y_4(t)$ as a statistically predetermined, or fixed, variable, without *explaining* how this variable itself is being determined in the system.

E. The Demand for Farm Food Products by the Commercial Sector

If $y_4(t)$ is not considered as a predetermined variable, it is necessary to study not only the supply function for farm food products but also the demand function for these products. We shall assume that the commercial sector demands farm food products for three purposes: 1) for processing and sale in the domestic retail market, 2) for export, and 3) for maintenance of, or changes in, commercial stocks. If we assume that the demand for stocks depends only on current prices, we may write this demand function as

$$(2.7a) \quad y_4(t) = \alpha_{52}y_2(t) + \alpha_{55}y_5(t) + \gamma_{51}t + \gamma_{52}p_c(t) + \alpha_{50} + u_5(t).$$

If we were to consider (2.5a) as the supply function for the retail market, we might even consider a much simpler demand function for farm food products, namely

$$(2.7b) \quad y_5(t) = \alpha_{52}y_2(t) + \gamma_{51}t + \alpha_{50} + u_5(t),$$

that is, we might assume that the price received by farmers is a "residual" which is a linear function of the prices obtained in the retail market. This then would have to be considered as a somewhat "superficial," institutional, equation, rather than a structural equation.

Above we have considered 5 groups of equations, namely [(2.1a), (2.1b), (2.1c)], [(2.4a), (2.4b), (2.4c)], [(2.5a), (2.5b), (2.5c)], [(2.6a), (2.6b), (2.6c)], [(2.7a), (2.7b)]. Choosing one equation from each of these groups, we have a system of 5 equations. It will be noticed that these five equations involve 5 simultaneous random residuals, denoted by u , and 5 simultaneous, observable variables, denoted by y . In addition, there are certain other variables, namely t , $z(t)$, and $p_s(t)$. These latter variables are statistically different from the y 's, in the sense that, stochastically, they do not depend on the random residuals u . The same is, by assumption, true of the various lagged values of the variables y that occur in some of the equations. The variables $y(t-1)$ are stochastically independent of the variables $u(t)$. Statistically, the variables $y(t-1)$ can, therefore, be grouped together with t , $z(t)$, and $p_s(t)$ under the category of "predetermined variables,"⁴ while the 5 variables $y(t)$ [except in the particular case (2.6c)] may be called "jointly dependent" variables, because their stochastic properties depend on the stochastic properties assumed for the random variables $u(t)$.

We may consider any one of the possible systems of five equations as a system determining the five jointly dependent variables $y(t)$ as functions of the five random variables $u(t)$ and the predetermined variables. This means that, for any given set of values of the predetermined variables, the joint distribution of the five jointly dependent variables $y(t)$, for any value of t , is given implicitly by the joint probability distribution of the five variables $u(t)$. *It is this joint probability distribution that must form the basis for the estimation of the unknown parameters, the α 's and the γ 's.* The statistical procedure involved will be explained in subsequent sections.

Suppose that, among the various alternative systems of five equations discussed above, there is one for which there exists a set of values of the parameters such that, for the assumed distribution of the u 's, the resulting joint probability distribution of the y 's is identical with the true distribution of the observable y 's, for all values of the predetermined variables. Then we may say that the model is "true," in the sense that it is consistent with observations. Suppose, for example, that the model (2.1a), (2.4a), (2.5a), (2.6a), and (2.7a), together with a

⁴ *Op. cit.* in footnote 3, Article II, section II.

certain assumption concerning the joint probability distribution of the u 's, represents a true model, by appropriate choice of the values of the parameters. In that case it is obvious that there exist an infinity of equivalent system of equations that also represent "true" models. For example, by an arbitrary linear combination of two or more of the original equations we can derive a new equation that, together with four of the old equations, forms a true model. Why is it that we are interested in one particular member of this infinite set of true systems? It is because, in setting up the original model, we believe that there is one particular system of equations that is a system of *autonomous*, or *structural* equations, that is, equations such that it is possible that the parameters in any one of the equations could *in fact* change, e.g., by the introduction of some new economic policy, *without* any change taking place in any of the parameters of the other equations. If there is one system for which this is true, the other systems that can be derived from it will not have this property. The parameters of equations in derived systems will be functions of the parameters in two or more of the equations in the original system.

Suppose that we should succeed in deriving the *structural* equations of a model that is "true" in the sense discussed above and suppose also that these equations are identifiable, so that we could measure statistically the parameters involved. What could be the use of this knowledge? It would, first of all, help to satisfy a justified scientific curiosity. But we believe that such knowledge also could be of more immediate practical importance. The results could be used to judge, in advance, the effects of various policies that might be considered. If the policy considered represents a *known change* in the structure, e.g., a known absolute or relative change in one or more of the parameters or variables involved and if the structure *before* the change is known, then obviously the structure *after* the change is also known, and we can compare the two. A variety of practical policies with regard to taxation, subsidies, etc., are precisely of this type.

It is clear, then, that the fundamental objective of statistical inference with respect to economic models is to derive estimates of the structural parameters. Knowing the structural parameters, all the relations implied by the model can be derived. In a sense these structural parameters play a role similar to that of the elements in chemistry.

III. GENERAL COMPUTATIONAL PROCEDURES—

THE REDUCED-FORM METHOD

In the preceding sections we showed that, for statistical purposes the observable variables occurring in systems of economic relations can be divided into two groups: the "jointly dependent variables" and the

“predetermined variables.” We mentioned that, in a complete system, it is possible to solve for each of the jointly dependent variables in terms of the predetermined variables and the random residuals. [Equations (1.12) and (1.13) are examples of such solutions.] These solutions are called the reduced form of the system. Thus each jointly dependent variable in a given equation is expressible as a linear function of all the predetermined variables occurring in the system. If the equation under consideration is identified, the structural coefficients can be estimated from estimates of the coefficients of the reduced form. We shall here present a brief discussion of the theory involved in this method of estimation and describe the computational procedures in some detail.⁵

We are given a system of simultaneous economic relationships defined by G linear equations,

$$(3.1) \quad \sum_{j=1}^G \beta_{ij}y_j + \sum_{j=1}^K \gamma_{ij}z_j = u_i, \quad i = 1, 2, \dots, G,$$

where the y 's are mutually dependent economic variables, the z 's are predetermined variables (including lagged variables of the y 's), and the u 's are random disturbances that have a joint probability distribution. We desire to estimate the coefficients of one of these equations on the basis of T observations obtained on each y involved in the equation and each z involved in the system. Without loss of generality we assume that the equation to be estimated is given by

$$(3.2) \quad \sum_{j=1}^H \beta_j y_j + \sum_{j=1}^{K^*} \gamma_j z_j^* = u,$$

where $H \leq G$, $K^* \leq K$ and the z^* 's represent the predetermined variables effectively present in the equation.

Since (3.1) is a complete system, the rank of the matrix (β_{ij}) is G . Hence we can solve for the y 's in terms of z 's and u 's. The resulting system of equations is known as the reduced form. In the reduced form the y 's of equation (3.2) are given by an expression that can be written symbolically as

$$(3.3) \quad y_j = \sum_{i=1}^{K^*} \pi_{ji}^* z_i^* + \sum_{i=1}^{K^{**}} \pi_{ji}^{**} z_i^{**} + v_j, \quad j = 1, 2, \dots, H,$$

⁵ For a complete mathematical treatment see T. W. Anderson and H. Rubin “Estimation of the Parameters of a Single Stochastic Difference Equation in a Complete System,” to be published shortly in the *Annals of Mathematical Statistics*.

where the z^{**} 's represent the predetermined variables that are effectively present in the system but not in the equation. If we multiply equation (3.3) by β_j ($j=1, 2, \dots, H$), sum with respect to j , and compare the result with (3.2), we see that

$$(3.4) \quad \sum_{j=1}^H \beta_j \pi_{ji}^{**} = 0, \quad i = 1, 2, \dots, K^{**},$$

$$(3.5) \quad \sum_{j=1}^H \beta_j \pi_{ji}^* = -\gamma_i, \quad i = 1, 2, \dots, K^*.$$

It is known that if the equation under consideration is identified without any restrictions imposed on the covariance matrix of the u 's, the number of predetermined variables that are in the system but not in the equation must be at least $H-1$. That is, $K^{**} \geq H-1$. Moreover, if the equation is identified in this manner, the rank of the matrix (π_{ji}^{**}) in (3.4) must equal $H-1$. Thus, if the coefficients of the reduced form are known, the β 's can be computed from equations (3.4) except for a factor of proportionality. The γ 's can then be computed from (3.5). The β 's can be normalized by setting one of them equal to unity, or more generally, by setting $\sum_{i=1}^H \sum_{j=1}^H \beta_i \xi_{ij} \beta_j = 1$, where ξ_{ij} are predetermined constants that are independent of the β 's.

In practice, the π_{ji}^{**} 's and π_{ji}^* 's are seldom known and have to be estimated from the observations. If $K^{**} = H-1$, these estimates are given directly by least squares. However, if $K^{**} > H-1$, the restrictions implied in (3.4) make the least-squares estimates inconsistent. Consistent estimates of all the parameters involved in (3.4) and (3.5) can be obtained by the method of maximum likelihood. The computational procedures involved in obtaining these estimates are outlined below.

It should be pointed out that although the above discussion revolved around a single equation, the coefficients of any number of equations of the system can be estimated by this method provided that these equations are identified in the manner described above.⁶ With this in mind, the computational procedures are divided into two parts, (1) those which are applicable to more than one or all equations of the system and (2) those which are applicable to a single equation under consideration.

⁶ In Cowles Commission Monograph No. 10 (cf. footnote 3), a maximum-likelihood method is presented for estimating simultaneously all the structural coefficients of a complete system.

1. *Computations Applicable to Some or All of the Equations to be Estimated*

Step 1.1. Compute:

M_{yy} = an L -rowed square matrix with elements $a_{ij} = \sum_{t=1}^T (y_{it} - \bar{y}_i)$
 where $L \leq G$ is the number of y 's in all the equations to be
 estimated and $\bar{y}_i = (1/T) \sum_{\alpha=1}^T y_{i\alpha}$;

M_{yz} = a rectangular matrix composed of L rows and K columns with
 elements $b_{ij} = \sum_{t=1}^T (y_{it} - \bar{y}_i)(z_{jt} - \bar{z}_j)$;

M_{zz} = a K -rowed square matrix with elements $c_{ij} = \sum_{t=1}^T (z_{it} - \bar{z}_i)$
 $(z_{jt} - \bar{z}_j)$.

Step 1.2. Compute the regression of each of the L jointly dependent variables on all of the predetermined variables and let P be the matrix of partial-regression slopes. Symbolically, P is given by

$$P = M_{yz} M_{zz}^{-1}.$$

Step 1.3. Compute the sample variance and covariance matrix W of residuals of each y on all of the z 's:

$$T^*W = M_{yy} - PM_{yz}' \quad (\text{check by symmetry}),$$

where M_{yz}' is the conjugate of M_{yz} and $T^* = T - K - 1$.

2. *Computations Applicable to a Single Equation*

Let the single equation to be estimated be given by (3.2) or, in matrix notation, by

$$(3.2') \quad \beta y_t' + \gamma z_t^{*'} = u_t',$$

where y_t and u_t are row vectors of $H (\leq G)$ elements and $z_t^{*'}$ is a row vector of K^* elements. It is assumed here that $K = K^* + K^{**}$, where K^* is the number of predetermined variables in the given equation, and K^{**} is the number of predetermined variables in the complete system that do not occur in the given equation.

There are two possible cases that may arise:

CASE I. Equation (3.2) is exactly identified, that is, $K^{**} = H - 1$.

Step 2.1'. Estimate the β 's by solving the set of equations $\hat{\beta} P^{**} = 0$, (check by substituting the $\hat{\beta}$'s in the equations), where P^{**} is that part of the matrix P (step 1.2) which corresponds to the y 's in the given equation and to the z 's in the system but absent from the equation.

Step 2.2'. Estimate the γ 's from the set of equations $\hat{\gamma} = -\hat{\beta} P^*$ (check by recomputation), where P^* is the part of P which corresponds to the y 's and the z 's effectively present in the given equation.

It will be noticed that step 2.1' contains $(H - 1)$ homogeneous equations in H unknowns. Hence, all solutions will be proportional. The

equations may be normalized by setting the leading coefficients equal to unity.

CASE II. Equation (3.2) is overidentified, that is, $K^{**} > H - 1$.

Step 2.1. Compute N , the residual variance and covariance matrix of the predetermined variables z^{**} not occurring in the equation (3.2) on the predetermined variables z^* that do occur in the equation. This matrix N is given by

$$N = M_{z^{**}z^{**}} - M_{z^{**}z^*}(M_{z^*z^*})^{-1}M_{z^*z^{**}}' \text{ (check by symmetry).}$$

Step 2.2. Compute $P^{**}NP^{**'}$ (check by symmetry), where P^{**} is defined in step 2.1'.

Step 2.3. Obtain the latent vector associated with the smallest root of the determinantal equation

$$| P^{**}NP^{**'} - \lambda T^*W_e | = 0,$$

where T^*W_e is the part of T^*W which corresponds to the y 's in the particular equation.

The elements of the latent vector, except for a factor of proportionality, are the estimates ($\hat{\beta}$) of the β 's in equation (3.2).

There are two possible procedures for computing the latent vector in step (2.3): (a) the iterative process and (b) the direct procedure. In practice, the first method appears to be preferable.

The steps of the iterative process are as follows:

Step 2.31. Compute

$$U = (P^{**}NP^{**'})^{-1}(T^*W_e) \quad \text{(check by recomputation).}$$

Step 2.32. Guess a set of values (say, unity) as a first approximation to the row vector desired. Call this vector b_1 . The second approximation is obtained by multiplying U by the conjugate of b_1 . That is, $B_2 = Ub_1'$. Normalize B_2 by dividing the vector by its largest element. Call this new vector b_2 . Obtain the third approximation from $B_3 = Ub_2'$. Normalize B_3 and continue the process until stability is obtained to the desired number of decimal places.

The final vector B_n is, except for a factor of proportionality, the desired estimate of the vector β in equation (3.2'). The reciprocal of the largest element in B_n is the smallest root of the determinantal equation in step 2.3.

The steps of the direct procedure are as follows:

Step 2.31'. Compute the smallest root (λ_1) of the determinantal polynomial in step 2.3. Compute the adjoint of the matrix ($P^{**}NP^{**'} = \lambda_1 T^*W$). Except for a factor of proportionality, the elements of any column or row of this adjoint will be the desired estimates of the β 's.

The detailed steps of computation for the case $H=2$ are as follows:

1. Compute the adjoint of T^*W (check by determinant).
2. Compute the coefficients of $a_0\lambda^2 - a_1\lambda + a_2 = 0$:
 $a_0 = \text{determinant } |T^*W|$,
 $a_1 = \text{trace of } (\text{adj. } T^*W)(P^{**}NP^{**'})$ where the trace is the sum of the diagonal elements,
 $a_2 = \text{determinant } |P^{**}NP^{**'}|$.

Then,

$$\lambda_1 = \frac{a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

and

$$\lambda_2 = \frac{a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

(check: determinant $|P^{**}NP^{**'} - \lambda_1 T^*W| = 0$).

The detailed steps of computation for the case $H=3$ are as follows:

1. Compute the adjoint of T^*W .
2. Compute the adjoint of $P^{**}NP^{**'}$.
3. Compute the coefficients of $a_0\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$:
 $a_0 = \text{determinant } |T^*W|$,
 $a_1 = \text{trace of } (\text{adj. } T^*W)(P^{**}NP^{**'})$,
 $a_2 = \text{trace of } (T^*W)(\text{adj. } P^{**}NP^{**'})$,
 $a_3 = \text{determinant } |P^{**}NP^{**'}|$.
4. Compute the roots $\lambda_1 < \lambda_2 < \lambda_3$ by Horner's method (check: $|P^{**}NP^{**'} - \lambda_1 T^*W| = 0$).

Step 2.4. The estimate of the vector γ is obtained by computing

$$\hat{\gamma} = -\beta\tilde{P}^* = -\hat{\beta}M_{y,z^*}(M_{z^*,z^*})^{-1} \quad (\text{check by recomputation}),$$

where M_{z^*,z^*} is that part of the matrix M_{zz} (step 1.1) which consists of the predetermined variables in the equations under consideration.

The following alternative procedure may be used to obtain $P^{**}NP^{**'}$ (step 2.2). This procedure utilizes the matrix \tilde{P}^* which is used in computing $\hat{\gamma}$ in step 2.4.

Step 2.21': Compute $\tilde{P}^* = M_{y,z^*}(M_{z^*,z^*})^{-1}$ (same as step 2.4).

Step 2.22': Compute $\tilde{P}^*M'_{y,z^*}$ (check by symmetry).

Step 2.23': Compute $P^{**}NP^{**'} = Q^{**} - \tilde{P}^*M'_{y,z^*}$, where Q^{**} is that part of the matrix $PM_{y,z}$ (step 1.3) which corresponds to the y 's in the particular equation and the z 's outside of the equation.

IV. SOME NUMERICAL RESULTS

As an illustration,⁷ the general methods described in Section 3 will be applied to the estimation of the coefficients of the following system of equations:

$$(4.1) \quad y_1(t) = \alpha_{12}y_2(t) + \alpha_{13}y_3(t) + \gamma_{18}z_8(t) + \gamma_{19}z_9(t) + \alpha_{10} + u_1(t),$$

$$(4.2) \quad y_1(t) = \alpha_{22}y_2(t) + \alpha_{23}y_4(t) + \gamma_{28}z_8(t) + \alpha_{20} + u_2(t),$$

$$(4.3) \quad y_3(t) = \gamma_{37}z_7(t) + \gamma_{39}z_9(t) + \alpha_{30} + u_3(t),$$

$$(4.4) \quad y_4(t) = \alpha_{45}y_5(t) + \gamma_{46}z_6(t) + \gamma_{48}z_8(t) + \alpha_{40} + u_4(t),$$

$$(4.5) \quad y_5(t) = \alpha_{52}y_2(t) + \gamma_{53}z_8(t) + \alpha_{50} + u_5(t).$$

These equations correspond to (2.1c), (2.5a), (2.4c) (omitting trend), (2.6a), and (2.7b), except that, for the sake of symmetry, we have here used the notations $y_5(t-1) = z_6(t)$, $z(t) = z_7(t)$, $t = z_8(t)$, and $y_3(t-1) = z_9(t)$. The following series were used for the model:

y_1 = food consumption per capita published by the Bureau of Agricultural Economics. (An adjustment has been made in the official series for 1934 to exclude the quantity of meat purchased by the Government for relief purposes and distributed through noncommercial channels.)

y_2 = Retail prices of food products (*BAE*), deflated by the Index of Consumer Prices for Moderate Income Families in Cities, published by the Bureau of Labor Statistics.

y_3 = Disposable Income per capita (Dept. of Commerce), deflated by the *BLS* Consumer Price Index.

y_4 = Production of agricultural food products per capita (*BAE*).

y_5 = Prices received by farmers for food products (*BAE*), deflated by *BLS* Consumer Price Index.

$z_6 = y_5(t-1)$ = Prices received by farmers for food products, lagged one year.

$z_7 = z(t)$ = Net investment per capita = disposable income minus consumers' expenditures, based on Dept. of Commerce data, deflated by *BLS* Consumer Price Index.

$z_8 = t$ = Time.

$z_9 = y_3(t-1)$ = Disposable Income per capita lagged one year.

All the data are expressed in terms of index numbers (1935-39 = 100) except for time, z_8 , which has the values, 1, 2, . . . , 20. The analysis covers the period 1922 through 1941. The data are given in Table 1.

⁷ The model discussed here has been chosen primarily because it presents, in a simple form, almost all the particular statistical problems that have been discussed in the foregoing sections. Actually, we have carried out numerical work of a variety of different models, some of which might be more realistic than the one presented here.

TABLE 1
DATA USED IN THIS STUDY

Calendar year	y_1	y_2	y_3	y_4	y_5	z_6	z_7	z_8	z_9
	Food Cons. per capita	Food Prices ÷ by cost of living	Disp. Income ÷ by cost of living	Food Prod. per Capita	Prices Rec'd. by farmers for food ÷ by cost of living	$y_1(t-1)$	Investment per capita ÷ by cost of living	Time	$y_1(t-1)$
1922	98.6	100.2	87.4	108.5	99.1	98.0	92.9	1	77.4
1923	101.2	101.6	97.6	110.1	99.1	99.1	142.9	2	87.4
1924	102.4	100.5	96.7	110.4	98.9	99.1	100.0	3	97.6
1925	100.9	106.0	98.2	104.3	110.8	98.9	123.8	4	96.7
1926	102.3	108.7	99.8	107.2	108.2	110.8	111.9	5	98.2
1927	101.5	106.7	100.5	105.8	105.6	108.2	121.4	6	99.8
1928	101.6	106.7	103.2	107.8	109.8	105.6	107.1	7	100.5
1929	101.6	108.2	107.8	103.4	108.7	109.8	142.9	8	103.2
1930	99.8	105.5	96.6	102.7	100.6	108.7	92.9	9	107.8
1931	100.3	95.6	88.9	104.1	81.0	100.6	97.6	10	96.6
1932	97.6	88.6	75.1	99.2	68.6	81.0	52.4	11	88.9
1933	97.2	91.0	76.9	99.7	70.9	68.6	40.5	12	75.1
1934	97.3	97.9	84.6	102.0	81.4	70.9	64.3	13	76.9
1935	96.0	102.3	90.6	94.3	102.3	81.4	78.6	14	84.6
1936	99.2	102.2	103.1	97.7	105.0	102.3	114.3	15	90.6
1937	100.3	102.5	105.1	101.1	110.5	105.0	121.4	16	103.1
1938	100.3	97.0	96.4	102.3	92.5	110.5	78.6	17	105.1
1939	104.1	95.8	104.4	104.4	89.3	92.5	109.5	18	96.4
1940	105.3	96.4	110.7	108.5	93.0	89.3	128.6	19	104.4
1941	107.6	100.3	127.1	111.3	106.6	93.0	238.1	20	110.7
Sum	2015.1	2013.7	1950.7	2084.8	1941.9	1933.3	2159.7	210	1901.0
Mean	100.755	100.685	97.535	104.240	97.095	96.665	107.985	10.500	95.050

1. Computations Applicable to All the Equations

Step 1.1.

$$M_{yy} = \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{matrix} \begin{pmatrix} 151.7295 & & & & \\ & 62.4765 & & & \\ & & 583.2285 & & \\ & & & 2659.2455 & \\ & & & & 391.6480 \\ & & & & & 3164.9495 \end{pmatrix}.$$

$$M_{yz} = \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{matrix} \begin{pmatrix} z_6 & z_7 & z_8 & z_9 \\ 257.7885 & 1793.9165 & 72.2500 & 401.6250 \\ 920.2995 & 1870.0155 & -257.7500 & 430.5750 \\ 1407.0145 & 8385.2305 & 436.8500 & 1764.4650 \\ 364.6480 & 2073.5520 & -172.7000 & 306.1900 \\ 2169.4165 & 6297.5185 & -306.8500 & 1290.2350 \end{pmatrix}.$$

$$M_{zz} = \begin{matrix} & z_6 & z_7 & z_8 & z_9 \\ \begin{matrix} z_6 \\ z_7 \\ z_8 \\ z_9 \end{matrix} & \begin{pmatrix} 3071.7255 & 3963.8095 & -415.2500 & 1714.9250 \\ & 32367.3055 & 658.1500 & 4956.0350 \\ & & 665.0000 & 317.7000 \\ & & & 2067.0700 \end{pmatrix} \end{matrix}$$

Step 1.2. Compute

$$P = M_{zz}^{-1} = \begin{matrix} & z_6 & z_7 & z_8 & z_9 \\ \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{matrix} & \begin{pmatrix} -0.059219 & 0.039928 & -0.041452 & 0.154065 \\ 0.240867 & 0.041350 & -0.253400 & -0.051727 \\ 0.199603 & 0.206909 & 0.523532 & 0.111453 \\ -0.127889 & 0.062000 & -0.487126 & 0.180446 \\ 0.648731 & 0.160701 & -0.078116 & -0.287321 \end{pmatrix} \end{matrix},$$

where

$$M_{zz}^{-1} = \begin{pmatrix} 0.0011139016 & 0.0000142602 & 0.0012295671 & -0.0011473077 \\ & 0.0000490411 & 0.0000239537 & -0.0001330942 \\ & & 0.0029815502 & -0.0015357821 \\ & & & 0.0019907811 \end{pmatrix}$$

Step 1.3. Compute

$$PM_{zz}' = \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{matrix} \begin{pmatrix} 115.242969 & 97.187553 & 505.217486 & 115.530617 & 334.480389 \\ 97.188184 & 342.036418 & 483.664372 & 201.496935 & 793.961335 \\ 505.220325 & 483.666081 & 2441.183845 & 445.533223 & 1719.196027 \\ 115.531281 & 201.497041 & 445.532272 & 221.303177 & 495.296158 \\ 334.480389 & 793.961335 & 1719.196027 & 495.296158 & 2072.643540 \end{pmatrix}$$

$$T^*W = M_{yy} - PM_{zz}'$$

$$= \begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{matrix} \begin{pmatrix} 36.486531 & -34.711053 & 34.764014 & 65.355383 & -68.360744 \\ & 241.189082 & 132.426128 & -92.664935 & 437.309462 \\ & & 218.061655 & 44.718777 & 316.343436 \\ & & & 170.344823 & -175.089998 \\ & & & & 1092.305960 \end{pmatrix}$$

Above, under step 1.2, we have calculated the coefficients p_{36} , p_{37} , p_{38} , and p_{39} in a purely formal fashion, neglecting a priori information. However, from the form assumed for equation (4.3) we know that p_{36} and p_{38} are zero. The coefficients p_{37} and p_{39} obtained under this restriction (that is, the coefficients in the regression of y_3 on z_7 and z_9) are $p_{36} = 0$, $p_{37} = 0.20282$, $p_{38} = 0$, $p_{39} = 0.36732$. These values of the coefficients are used in the following calculations, instead of those in the third row of the matrix P above.

2. *Computations Applicable to Single Equations.* (In the notations below α_{ij} corresponds to $-\beta_{ij}$ and γ_{ij} to $-\gamma_{ij}$ in section 3.)

Equation (4.1): $y_1 = \alpha_{12}y_2 + \alpha_{13}y_3 + \gamma_{18}z_8 + \gamma_{19}z_9 + \alpha_{10} + u_1$, where for the

sake of brevity, $y_1 = y_1(t)$, $y_2 = y_2(t)$, etc. This equation is exactly identified. Therefore, we follow the computation outlined in Case I.

Step 2.1. The reduced forms for the y 's involved in equation (4.1) are:

$$\begin{aligned} y_1 &= p_{16}z_6 + p_{17}z_7 + p_{18}z_8 + p_{19}z_9 + p_{10} + v_1, \\ y_2 &= p_{26}z_6 + p_{27}z_7 + p_{28}z_8 + p_{29}z_9 + p_{20} + v_2, \\ y_3 &= \quad \quad p_{37}z_7 \quad \quad + p_{39}z_9 + p_{30} + v_3. \end{aligned}$$

Multiplying y_1 by 1, y_2 by $-\alpha_{12}$, and y_3 by $-\alpha_{13}$, we obtain

$$\begin{aligned} y_1 - \alpha_{12}y_2 - \alpha_{13}y_3 &= (p_{16} - \alpha_{12}p_{26})z_6 + (p_{17} - \alpha_{12}p_{27} - \alpha_{13}p_{37})z_7 \\ &\quad + (p_{18} - \alpha_{12}p_{28})z_8 + (p_{19} - \alpha_{12}p_{29} - \alpha_{13}p_{39})z_9 \\ &\quad + p_0' + v'. \end{aligned}$$

Comparing coefficients with equation (4.1), we see that

$$\begin{aligned} p_{16} - \alpha_{12}p_{26} &= 0, \\ p_{17} - \alpha_{12}p_{27} - \alpha_{13}p_{37} &= 0, \\ p_{18} - \alpha_{12}p_{28} &= \gamma_{18}, \\ p_{19} - \alpha_{12}p_{29} - \alpha_{13}p_{39} &= \gamma_{19} \quad (\text{the } p\text{'s were computed in step 1.2}). \end{aligned}$$

That is:

$$\begin{aligned} \hat{\alpha}_{12} &= \frac{p_{16}}{p_{26}} = \frac{-0.059219}{0.240867} = -0.245858, \\ \hat{\alpha}_{13} &= \frac{p_{17} - \alpha_{12}p_{27}}{p_{37}} = \frac{0.039929 - (-0.245858)(0.041350)}{0.20282} \\ &= \frac{0.050094}{0.20282} = 0.246987, \\ \hat{\gamma}_{18} &= -0.041452 - (-0.245858)(-0.253400) = -0.103752, \\ \hat{\gamma}_{19} &= 0.154065 - (-0.245858)(-0.051727) - (0.246987)(0.36732) \\ &= 0.050625, \\ \hat{\alpha}_{10} &= \hat{y}_1 + 0.245858\hat{y}_2 - 0.246987\hat{y}_3 + 0.103752\hat{z}_8 - 0.050625\hat{z}_9 \\ &= 97.677. \end{aligned}$$

Equation (4.1), therefore, becomes

$$(4.1) \quad y_1 = -0.246y_2 + 0.247y_3 - 0.104z_8 + 0.051z_9 + 97.677 + u_1.$$

Equation (4.2): $y_1 = \alpha_{22}y_2 + \alpha_{24}y_4 + \gamma_{28}z_8 + \alpha_{20} + u_2$. This equation is overidentified and we can employ the procedure outlined for Case II.

Step 2.1.

$$M = M_{z^*z^*} - M_{z^*z^*}(M_{z^*z^*})^{-1}M_{z^*z^*}'$$

$$M_{z^*z^*} = 665.0000, \quad M_{z^*z^*}^{-1} = 0.0015037594,$$

$$M_{z^*z^*}(M_{z^*z^*})^{-1}$$

$$= \begin{matrix} & z_8 \\ z_6 \\ z_7 \\ z_8 \end{matrix} \begin{pmatrix} -415.2500 \\ 658.1500 \\ 317.7000 \end{pmatrix} (0.0015037594) = \begin{pmatrix} -0.6244360909 \\ 0.9896992491 \\ 0.4777443614 \end{pmatrix},$$

$$M_{z^*z^*}(M_{z^*z^*})^{-1}M_{z^*z^*}'$$

$$= \begin{pmatrix} 259.297087 & -410.972613 & -198.383346 \\ & 651.370561 & 314.427451 \\ & & 151.779841 \end{pmatrix},$$

$$M_{z^*z^*} = \begin{matrix} & z_6 & z_7 & z_8 \\ z_6 \\ z_7 \\ z_8 \end{matrix} \begin{pmatrix} 3071.7255 & 3963.8095 & 1714.9250 \\ & 32367.3055 & 4956.0350 \\ & & 2067.0700 \end{pmatrix} \quad \begin{matrix} \text{(obtained from} \\ \text{step 1.1),} \end{matrix}$$

$$N = \begin{pmatrix} 2812.428413 & 4374.782113 & 1913.308346 \\ & 31715.934939 & 4641.607549 \\ & & 1915.290616 \end{pmatrix} \quad \begin{matrix} [N \text{ remains the same} \\ \text{for equations (4.2)} \\ \text{and (4.5), since } z_8 \\ \text{is the only prede-} \\ \text{termined variable} \\ \text{appearing].} \end{matrix}$$

Step 2.2. Compute $P^{**}MP^{**}'$.

$$P^{**} = \begin{matrix} & z_6 & z_7 & z_8 \\ y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} -0.059219 & 0.039928 & 0.154065 \\ 0.240867 & 0.041350 & -0.051727 \\ -0.127889 & 0.062000 & 0.180446 \end{pmatrix}.$$

The matrix P^{**} is obtained from step 1.2.

$$P^{**}N = \begin{pmatrix} 302.902868 & 1722.392895 & 367.105148 \\ 759.348734 & 2125.098119 & 553.711076 \\ 256.806671 & 2244.460972 & 388.695107 \end{pmatrix}.$$

$$P^{**}NP^{**'} = \begin{pmatrix} 107.392153 & 125.191003 & 134.293070 \\ 125.190542 & 242.133046 & 134.558682 \\ 134.295315 & 134.558682 & 176.452309 \end{pmatrix}.$$

Step 2.31. Compute $U = (P^{**}MP^{**'})^{-1}(T^*W_e)$.

$$(P^{**}NP^{**'})^{-1} = \begin{pmatrix} 0.68244015 & -0.11143354 & -0.4344099 \\ & 0.02536299 & 0.0654677 \\ & & 0.2863605 \end{pmatrix}$$

$$T^*W_e = \begin{matrix} & y_1 & y_2 & y_4 \\ \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} & \begin{pmatrix} 36.486531 & -34.711053 & 65.355383 \\ & 241.189082 & -92.664935 \\ & & 170.344823 \end{pmatrix} \end{matrix}.$$

This matrix is obtained from step 1.3.

$$U = (P^{**}NP^{**'})^{-1}(T^*W_e) = \begin{pmatrix} 0.376823 & -10.310203 & -19.072360 \\ -0.667527 & 3.918683 & 1.519058 \\ 0.592637 & 4.333359 & 14.322444 \end{pmatrix}.$$

Step 2.32.

$$\begin{aligned} b_1 &= (1. & 1. & 1. &), \\ B_2 &= (-29.005740 & 4.770214 & 19.248440), \\ b_2 &= (- 1.5069 & 0.2478 & 1.0000 &), \\ B_3 &= (-22.1951 & 3.4960 & 14.5032 &), \\ b_3 &= (- 1.5304 & 0.2411 & 1.0000 &), \\ B_4 &= (-22.1348 & 3.4854 & 14.4602 &), \\ b_4 &= (- 1.5307 & 0.2410 & 1.0000 &), \\ B_5 &= (-22.1339 & 3.4852 & 14.4596 &), \\ b_5 &= (- 1.5507 & 0.2410 & 1.0000 &), \end{aligned}$$

$$\hat{\alpha}_{22} = -\frac{3.4852}{-22.1339} = 0.15746,$$

$$\hat{\alpha}_{24} = -\frac{14.4596}{-22.1339} = 0.65328.$$

Step 2.4.

$$\begin{aligned}
 \hat{\gamma} &= -\hat{\alpha}M_{y,z}(M_{z,z})^{-1} = -\hat{\alpha}\tilde{p}^*. \text{ Therefore} \\
 \hat{\gamma}_{28} &= (\tilde{p}_{18}^*)(\hat{\alpha}_{21}) + (\tilde{p}_{28}^*)(-\hat{\alpha}_{22}) + (\tilde{p}_{48}^*)(-\hat{\alpha}_{24}) \\
 &= \left(\frac{72.5000}{665.0000}\right)(1) + \left(\frac{-257.7500}{665.0000}\right)(-0.15746) \\
 &\quad + \left(\frac{-172.7000}{665.0000}\right)(-0.65328) \\
 &= 0.33934.
 \end{aligned}$$

We therefore have

$$(4.2) \quad y_1 = 0.157y_2 + 0.653y_4 + 0.339z_8 + 13.319 + u_2.$$

Equation (4.3): $y_3 = \gamma_{37}z_7 + \gamma_{39}z_9 + \hat{\alpha}_{30} + u_3$.

This equation can be obtained directly by the method of least squares which gives

$$(4.3) \quad y_3 = 0.203z_7 + 0.367z_9 + 40.731 + u_3.$$

Equation (4.4): $y_4 = \alpha_{45}y_5 + \gamma_{46}z_6 + \gamma_{48}z_8 + \alpha_{40} + u_4$.

This equation is overidentified and we proceed as in equation (4.2).

Step 2.1.

$$\begin{aligned}
 M_{z^*z^*} &= \begin{matrix} & z_6 & z_8 \\ z_6 & \begin{pmatrix} 3071.7255 & -415.2500 \\ & 665.0000 \end{pmatrix} \\ z_8 & \end{matrix} \\
 M_{z^*z^*}^{-1} &= \begin{pmatrix} 0.0003555646 & 0.0002220274 \\ & 0.0016424013 \end{pmatrix}.
 \end{aligned}$$

$$M_{z^*z^*} - M_{z^*z^*}M_{z^*z^*}^{-1}M_{z^*z^*} = N,$$

$$\begin{aligned}
 \begin{matrix} z_7 \\ z_9 \end{matrix} \begin{pmatrix} 32367.3055 & 4956.0350 \\ & 2067.0700 \end{pmatrix} - \begin{pmatrix} 7456.421511 & 3290.612426 \\ 3290.612444 & 1453.412115 \end{pmatrix} \\
 = \begin{pmatrix} 24910.883989 & 1665.422574 \\ 1665.422574 & 613.657885 \end{pmatrix}.
 \end{aligned}$$

Step 2.2.

$$P^{**} = \begin{matrix} & z_7 & z_9 \\ y_4 & \begin{pmatrix} 0.062000 & 0.180446 \\ & 0.160701 & -0.287321 \end{pmatrix} \\ y_5 & \end{matrix}.$$

$$P^{**}N = \begin{pmatrix} 1844.993649 & 213.988310 \\ 3524.693089 & 91.318276 \end{pmatrix}.$$

$$P^{**}NP^{**'} = \begin{pmatrix} 153.002941 & 235.008989 \\ 235.008989 & 540.184046 \end{pmatrix}.$$

Step 2.31.

$$(P^{**}NP^{**'})^{-1} = \begin{pmatrix} 0.01969999 & -0.00857055 \\ & 0.00557987 \end{pmatrix}.$$

$$T^*W_e = \begin{matrix} y_4 \\ y_6 \end{matrix} \begin{pmatrix} 170.344823 & -175.089998 \\ & 1092.305960 \end{pmatrix}.$$

$$U = (P^{**}NP^{**'})^{-1}(T^*W_e) = \begin{pmatrix} 4.856409 & -12.810934 \\ -2.436928 & 7.595543 \end{pmatrix}.$$

Step 2.32.

$$b_1 = \begin{pmatrix} 1. & 1. \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -7.954525 & 5.158615 \end{pmatrix},$$

$$b_2 = \begin{pmatrix} -1.5420 & 1.0000 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} -20.2995 & 11.3533 \end{pmatrix},$$

$$b_3 = \begin{pmatrix} -1.7880 & 1.0000 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} -21.4942 & 11.9528 \end{pmatrix},$$

$$b_4 = \begin{pmatrix} -1.7983 & 1.0000 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} -21.5442 & 11.9779 \end{pmatrix},$$

$$b_5 = \begin{pmatrix} -1.7987 & 1.0000 \end{pmatrix},$$

$$B_6 = \begin{pmatrix} -21.5462 & 11.9788 \end{pmatrix},$$

$$b_6 = \begin{pmatrix} -1.7987 & 1.000 \end{pmatrix},$$

$$\hat{\alpha}_{45} = -\frac{11.9788}{-21.5462} = 0.55596.$$

Step 2.4. $\hat{\gamma} = -\hat{\alpha}M_{y_4}(M_{z_6 z_8})^{-1} = -\hat{\alpha}\tilde{p}^*$. We solve for \tilde{p}^* from the equation $\tilde{p}^* = M_{y_4}(M_{z_6 z_8})^{-1}$ by the method of least squares.

$$\begin{matrix} y_4 \\ z_6 \\ z_8 \end{matrix} \begin{pmatrix} & & & \\ 391.6480 & 364.6480 & -172.7000 & \\ & 3071.7255 & -415.2500 & 2169.4165 \\ & & 665.0000 & -306.8500 \end{pmatrix}$$

1	-0.135185	0.118711	0.706253
1	-1.601445	0.415894	0.738952
	1.466260	-0.297183	-0.032699

$$\begin{aligned} \tilde{p}_{48.6} &= -0.20268, & \tilde{p}_{58.6} &= -0.02230, \\ \tilde{p}_{46.8} &= -0.09131, & \tilde{p}_{56.8} &= 0.70324, \end{aligned}$$

$$\begin{aligned} \hat{\gamma}_{46} &= \tilde{p}_{46.8}\hat{\alpha}_{41} + \tilde{p}_{56.8}\hat{\alpha}_{45}, \\ \hat{\gamma}_{46} &= 0.09131(1) + (0.70324)(-0.55596) = -0.29966, \\ \hat{\gamma}_{48} &= \tilde{p}_{48.6}\hat{\alpha}_{41} + \tilde{p}_{58.6}\hat{\alpha}_{45}, \\ \hat{\gamma}_{48} &= -0.20268(1) + (-0.02230)(-0.55596) = -0.19028. \end{aligned}$$

Hence,

$$(4.4) \quad y_4 = 0.556y_5 - 0.300z_6 - 0.190z_8 + 81.250 + u_4.$$

Equation (4.5): $y_5 = \alpha_{52}y_2 + \gamma_{58}z_8 + u_5$.

$$P^{**} = \begin{matrix} y_2 & \begin{pmatrix} 0.240867 & 0.041350 & -0.051727 \\ 0.648731 & 0.160701 & -0.287321 \end{pmatrix} \\ y_5 & \end{matrix}$$

$$N = \begin{matrix} z_6 & \begin{pmatrix} 2812.428413 & 4374.782113 & 1913.308346 \\ & 31715.934939 & 4641.607549 \\ z_9 & & 1915.290616 \end{pmatrix} \\ z_7 & \\ z_9 & \end{matrix}$$

$$P^{**}N = \begin{pmatrix} 759.348734 & 2125.098119 & 553.711076 \\ 1977.807690 & 6601.207913 & 1436.830196 \end{pmatrix}.$$

$$(P^{**}NP^{**'})^{-1} = \begin{pmatrix} 0.1621036149 & -0.0566654657 \\ -0.0566654657 & 0.0203260159 \end{pmatrix}.$$

$$T^*W_s = \begin{pmatrix} 241.189082 & 437.309462 \\ 437.309462 & 1092.305960 \end{pmatrix}.$$

$$U = (P^{**}NP^{**'})^{-1}T^*W_s = \begin{pmatrix} 14.317278 & 8.993419 \\ -4.778333 & -2.578118 \end{pmatrix}.$$

Step 2.32.

$$\begin{aligned} b_1 &= (1. & & 1. & &), \\ B_2 &= (23.310697 & & -7.356451), \\ b_2 &= (1.0000 & & -0.3156 & &), \end{aligned}$$

$$B_3 = (11.478955 \quad -3.9647 \quad),$$

$$b_3 = (1.0000 \quad -0.3454 \quad),$$

$$B_4 = (11.2110 \quad -3.8879 \quad),$$

$$b_4 = (1.0000 \quad -0.3468 \quad),$$

$$B_5 = (11.1984 \quad -3.8842 \quad),$$

$$b_5 = (1.0000 \quad -0.3469 \quad),$$

$$B_6 = (11.1975 \quad -3.8840 \quad),$$

$$\alpha_{62} = - \frac{11.1975}{-3.8840} = 2.88298,$$

$$\begin{aligned} \gamma_{63} &= \frac{-257.7500}{665.0000} (2.88298) + \frac{-306.8500}{665.0000} (1) \\ &= -0.38759(-2.88298) - 0.461428 \\ &= 0.65599. \end{aligned}$$

$$(4.5) \quad y_6 = 2.883y_2 + 0.656z_3 - 200.068.$$

V. SUMMARY OF RESULTS

Using the original notations of the predetermined variables involved, our findings can be summarized as follows:

A. Reduced Form Equations

$$(4.6) \quad \text{Est. of } y_1(t) = -0.059y_5(t-1) + 0.040z(t) + 0.154y_3(t-1) \\ - 0.041t + 87.932,$$

$$\text{Multiple correlation: } R^2 = 0.7546,$$

$$(4.7) \quad \text{Est. of } y_2(t) = 0.241y_5(t-1) + 0.041z(t) - 0.052y_3(t-1) \\ - 0.253t + 80.560,$$

$$R^2 = 0.5865,$$

$$(4.8) \quad \text{Est. of } y_3(t) = 0.203z(t) + 0.367y_3(t-1) + 40.731,$$

$$R^2 = 0.8833,$$

$$(4.9) \quad \text{Est. of } y_4(t) = -0.128y_5(t-1) + 0.062z(t) + 0.180y_3(t-1) \\ - 0.487t + 97.923,$$

$$R^2 = 0.5651,$$

$$(4.10) \quad \text{Est. of } y_5(t) = 0.649y_5(t-1) + 0.161z(t) - 0.287y_3(t-1) \\ - 0.078t + 45.072,$$

$$R^2 = 0.6549.$$

B. The Final System of Structural Equations

(4.1 est.) $y_1(t) = -0.246y_2(t) + 0.247y_3(t) + 0.051y_3(t - 1) - 0.104t + 97.677 + u_1(t),$

(4.2 est.) $y_1(t) = 0.157y_2(t) + 0.653y_4(t) + 0.339t + 13.319 + u_2(t),$

(4.3 est.) $y_3(t) = 0.203z(t) + 0.367y_3(t - 1) + 40.731 + u_3(t),$

(4.4 est.) $y_4(t) = 0.556y_5(t) - 0.300y_5(t - 1) - 0.190t + 81.250 + u_4(t).$

{This equation may also be written as

$$y_4(t) = 0.556[y_5(t) - y_5(t - 1)] + 0.256y_5(t - 1) - 0.190t + 81.250 + u_4(t)\},$$

(4.5 est.) $y_5(t) = 2.883y_2(t) + 0.656t - 200.068 + u_5(t).$

A theory of confidence intervals for the parameters has not yet been worked out. Such a theory is essential in order to judge the reliability of the estimates.

The residuals are given in Table 2.

TABLE 2
DATA USED IN THIS STUDY
Estimates of the Residuals, u(t)

Year	$u_1(t)$	$u_2(t)$	$u_3(t)$	$u_4(t)$	$u_5(t)$
1922	0.14	-1.64	-0.60	1.74	9.64
1923	0.16	-0.64	-4.22	3.86	4.94
1924	0.90	0.19	-0.15	4.46	7.26
1925	0.53	1.48	-3.15	-8.12	2.64
1926	2.21	0.22	0.31	-0.02	-8.39
1927	0.78	0.31	-1.50	-0.56	-5.88
1928	0.29	-1.24	3.84	-1.49	-2.34
1929	-0.52	1.06	0.19	-3.83	-8.42
1930	-0.35	-0.20	-2.55	-0.16	-9.39
1931	0.30	0.60	-7.10	9.89	-1.11
1932	-0.22	1.86	-8.89	6.20	6.02
1933	0.33	0.42	0.39	1.89	0.74
1934	0.24	-2.40	2.58	-0.77	-9.31
1935	-1.75	0.30	2.86	-16.75	-1.75
1936	-1.86	0.95	5.92	-8.39	0.59
1937	-1.72	-0.55	1.89	-7.05	4.56
1938	-0.92	-0.81	1.14	6.00	1.77
1939	1.16	1.47	6.06	4.67	1.37
1940	0.65	-0.45	5.55	5.94	2.68
1941	-0.36	-0.92	-2.59	2.48	4.38

C. Comments

The main purpose of reproducing the calculations above has been to illustrate, explicitly, the application of certain methods and principles set forth in the preceding sections of this paper. Much careful research, both in the economic theory and in the statistics involved, is yet to be carried out before one can draw final, practical, conclusions from the results obtained.

To those familiar with multiple-correlation results obtained from agricultural data the most striking features of the results above are, probably, the relatively low values of the multiple-correlation coefficients (given under *A*), and the relatively large residuals (given in Table 2). A refinement of the economic model involved might improve these correlations. We should like to point out, however, that when one is searching for structural economic relations one cannot in general expect to find as high correlations as those obtained from a mechanical application of the method of multiple correlation to the variables in the structural equations. For the correlations obtained by the latter procedure are due not only to the occurrence of the same predetermined variables in the equations of the reduced form, but also to the inter-correlations between the residuals in these equations. The method of multiple correlation would produce higher correlations at the expense of a bias in the estimates of the structural coefficients involved.

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and the University of Chicago*