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REMARKS ON THE THEORY OF AGGREGATION*

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THE contributions of Kenneth May and Shou Shan Pu,[1] presented elsewhere in this issue, raise some fundamental questions for the theory of macroeconomics. This theory is in a formative stage, and the basic objectives must be settled in order that a rigorous development may be forthcoming.

The aggregate equations of May and Pu have a very important common characteristic, namely, that they depend on the satisfaction of all the equilibrium conditions (profit maximization) in their systems. The same characteristic is found in the first formulation of index number theory in terms of rational behavior by Dresch.[2] In the theory of microeconomics, demand and supply equations are, of course, derived from these equilibrium conditions. We should expect that the aggregate demand and supply equations will also depend upon the fact that the equilibrium conditions hold. But there are certain equations in microeconomics that are independent of the equilibrium conditions and we should expect that the corresponding equations of macroeconomics will also be independent of the equilibrium conditions. The principal equations that have this independence property in microeconomics are the technological production functions. The aggregate production function should not depend upon profit maximization, but purely on technological factors.

Consider, for example, the αth individual firm producing goods $x_{1\alpha}, \cdots, x_{m\alpha}$, at prices $p_1, \cdots, p_m$, using the labor input $n_{1\alpha}, \cdots, n_{r\alpha}$, at wages $w_1, \cdots, w_r$, and using the capital input $z_{1\alpha}, \cdots, z_{s\alpha}$ at prices $q_1, \cdots, q_s$. The model for this firm under perfect competition will be:

$$(1)\quad f_\alpha(x_{1\alpha}, \cdots, x_{m\alpha}, n_{1\alpha}, \cdots, n_{r\alpha}, z_{1\alpha}, \cdots, z_{s\alpha}) = 0, \quad \alpha = 1, 2, \cdots, A,$$

$$(2)\quad \frac{\partial x_{i\alpha}}{\partial n_{j\alpha}} = \frac{w_j}{p_i}, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A,$$

$$(3)\quad \frac{\partial x_{i\alpha}}{\partial z_{j\alpha}} = \frac{q_j}{p_i}, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A.$$

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[1] Kenneth May, "The Aggregation Problem for a One-Industry Model," ECONOMETRICA, Vol. 14, October, 1946, pp. 285–298; Shou Shan Pu, "A Note on Macroeconomics," ECONOMETRICA, Vol. 14, October, 1946, pp. 299–302.

[2] Francis W. Dresch, "Index Numbers and the General Economic Equilibrium," *Bulletin of the American Mathematical Society*, Vol. 44, February, 1938, pp. 134–141.

Equation (1) is the production function and (2) and (3) are the marginal-productivity or profit-maximizing equations. In the system (1)–(3) there are $m+r+s$ independent equations for each firm,[3] involving $2(m+r+s)$ variables $\{x_{i\alpha}\}$, $\{n_{i\alpha}\}$, $\{z_{i\alpha}\}$, $\{p_i\}$, $\{w_i\}$, $\{q_i\}$. We can solve this system for each of the $x_{i\alpha}$, $n_{i\alpha}$, and $z_{i\alpha}$ in terms of the p_i, w_i, and q_i to get

$$(4) \qquad x_{i\alpha} = x_{i\alpha}(p_1, \cdots, p_m, w_1, \cdots, w_r, q_1, \cdots, q_s), \quad i = 1, 2, \cdots, m; \alpha = 1, 2, \cdots, A;$$

$$(5) \qquad n_{i\alpha} = n_{i\alpha}(p_1, \cdots, p_m, w_1, \cdots, w_r, q_1, \cdots, q_s), \quad i = 1, 2, \cdots, r; \alpha = 1, 2, \cdots, A;$$

$$(6) \qquad z_{i\alpha} = z_{i\alpha}(p_1, \cdots, p_m, w_1, \cdots, w_r, q_1, \cdots, q_s), \quad i = 1, 2, \cdots, s; \alpha = 1, 2, \cdots, A.$$

Equations (4), (5), (6) are the supply equations of output and the demand equations for input for the individual firm; they show how much of each commodity will be offered to the market and how much of each productive factor will be demanded corresponding to any price-wage situation. These equations depend upon the technological possibilities of production as given by (1) and upon economic decision as given by (2) and (3). The analogues for (4), (5), (6) in macroeconomics are

$$(7) \qquad X = X(P, W, Q),$$

$$(8) \qquad N = N(P, W, Q),$$

$$(9) \qquad Z = Z(P, W, Q),$$

which state that aggregate supply, X, aggregate labor demanded, N, and aggregate capital demanded, Z, are functions of the price and wage aggregates P, W, Q. These equations too should depend upon economic decisions and upon technological possibilities.

But the equation (1) does not involve prices of products or factors.

[3] The equations (2) and (3) can be written as

$$\lambda_\alpha \frac{\partial f_\alpha}{\partial x_{i\alpha}} + p_i = 0, \qquad i = 1, 2, \cdots, m, \quad \alpha = 1, 2, \cdots, A,$$

$$\lambda_\alpha \frac{\partial f_\alpha}{\partial n_{i\alpha}} - w_i = 0, \qquad i = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A,$$

$$\lambda_\alpha \frac{\partial f_\alpha}{\partial z_{i\alpha}} - q_i = 0, \qquad i = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A,$$

where λ_α is the Lagrange multiplier. The equations (2) and (3) of the text are obtained from these, which are $m+r+s$ in number, by eliminating the variable λ_α; hence (2) and (3) represent $m+r+s-1$ independent equations.

It is a purely technological phenomenon and not an economic decision.[4] The analogues of (1), (2), and (3) in macroeconomics are

$$F(X, N, Z) = 0, \tag{10}$$

$$\frac{\partial X}{\partial N} = \frac{W}{P}, \tag{11}$$

$$\frac{\partial X}{\partial Z} = \frac{Q}{P}. \tag{12}$$

If our system of macroeconomics is to be (10), (11), (12), the aggregates of (10) should be defined so that they do not depend on the equilibrium conditions in (2) and (3). On the other hand, if the business-firm side of the macroeconomic system is to be represented by (7), (8), (9), there is no reason to make the aggregates independent of the equilibrium conditions. The supply-demand equations are the result of a combination of technological and economic equations; consequently the aggregates should be constructed under the assumption that both types of equations hold. The decision as to what type of aggregate to construct (i.e., independent of or dependent upon the equilibrium conditions) can only be decided on the basis of the type of macroeconomic system that we want to develop. The latter decision can be made if we state our goals clearly. If our purpose is only to forecast aggregate output, we may dispense with the technological production function in the macrosystem and define our aggregates in terms of the supply-demand equations. An example of a system that is suitable for this purpose can easily be devised. Let us define the following variables:

X_1^S = supply of consumer goods,
X_2^S = supply of producer goods,
P_1 = price of consumer goods,
P_2 = price of producer goods,
N^S = supply of labor,
W = wage rate,
X_1^D = demand for consumer goods,
X_2^D = demand for producer goods,
N^D = demand for labor,
Y = income.

[4] It is an economic decision in the sense that less output than that shown by the production can always be obtained from any factor input. The rational entrepreneur operates with a production function that gives the maximum output corresponding to any input. However market variables do not enter into this maximization decision. The engineer rather than the economist deals with this aspect of the production process.

All variables are endogenous except N^S which is taken as given by demographic and other sociological forces. A model system is

(13)	$X_1^S = X_1^S(P_1, P_2, W)$,	supply of consumer goods,
(14)	$X_2^S = X_2^S(P_2, W)$,	supply of producer goods,
(15)	$X_1^D = X_1^D(P_1, Y)$,	demand for consumer goods,
(16)	$X_2^D = X_2^D(P_1, P_2, W)$,	demand for producer goods,
(17)	$N^D = N^D(P_1, P_2, W)$,	demand for labor,
(18)	$\dot{P}_1 = \dot{P}_1[\int_{-\infty}^{t}(X_1^S - X_1^D)d\theta]$,	price-adjustment equation for consumer-goods market,
(19)	$\dot{P}_2 = \dot{P}_2[\int_{-\infty}^{t}(X_2^S - X_2^D)d\theta]$,	price-adjustment equation for producer-goods market,
(20)	$\dot{W} = \dot{W}(N^S - N^D)$,	wage-adjustment equation,
(21)	$Y = P_1X_1^S + P_2X_2^S$,	definition of income.

The equation system (13)–(21) can be solved for Y, X_1^S, X_2^S, or N^D in terms of the exogenous variables and can thus be used for purposes of prediction of aggregate output or employment on the assumption that the parameters of these equations do not change. There are, however, no purely technological functions in this system; they have been, so to speak, solved out of the system. Every one of the equations in this simple system involves an economic decision or market behavior. Some are based on profit maximization, some on utility maximization, and some on market interactions. All the aggregates should account for these phenomena. The output aggregates should depend upon prices and wages via the profit-maximizing equations. The indexes of Dresch, May,[5] and Pu are appropriate for a model of this type. It should be remarked, however, that the basis for the construction of their indexes must be extended to cover consumer and market behavior as well as firm behavior if they are to be used in a complete system like (13)–(21).

There is a very informative analogy in economic theory to the procedure of eliminating the production function from the equation systems. In the theory of consumer behavior, the usual practice is to start out with the utility function (or a monotonic transformation of the same), derive the utility-maximization equations, and finally solve for the demand equations in terms of prices and income. Utility never appears as a variable in the system because this variable, along with one equation, is eliminated in the process of solving for the consumer-demand equations. In a similar way the supply equations combine the production function and the profit-maximizing equations to obtain the

[5] May constructed his model so that all variables can be expressed in terms of N, aggregate employment. By similar methods, we can also express every variable in terms of price and wage aggregates like P_1, P_2, W. If we follow the latter procedure, May's macroeconomic system resembles (13)–(21) very closely.

supply equations as functions of wages and prices. The supply and demand equations hold only if the corresponding maximization equations hold; hence it is legitimate to construct the aggregates of supply and demand under the assumption that the maximization equations hold.

Let us now consider the conditions under which it is desirable to make some of the aggregates independent of the maximizing conditions. Suppose that we want to be able to forecast national product or total employment as a function of exogenous variables but also to forecast the effect upon the system of changes in fundamental parameters or controlled exogenous variables or of any other autonomous action. If the structural change in question is a change in parameters of the production function, we shall not be able to forecast the influence of the change without a knowledge of that function. To make the discussion concrete, suppose that marginal productivity of labor in the producer-goods industry changes by the amount ϵ. What is the influence of this change upon the system? To answer this question, we should need to work with a system like the following:

(22)	$X_1^S = F_1(N_1^D, X_{2,1}^D, K_1, t),$	production of consumer goods,
(23)	$\dfrac{\partial X_1^S}{\partial N_1^D} = \dfrac{W}{P_1},$	marginal productivity of labor in consumer-goods industry,
(24)	$\dfrac{\partial X_1^S}{\partial X_{2,1}^D} = \dfrac{P_2}{P_1},$	marginal productivity of capital in consumer-goods industry,
(25)	$X_2^S = F_2(N_2^D, X_{2,2}^D, K_2, t),$	production of producer goods,
(26)	$\dfrac{\partial X^S}{\partial N_2^D} = \dfrac{W}{P_2},$	marginal productivity of labor in producer-goods industry,
(27)	$\dfrac{\partial X_2^S}{\partial X_{2,2}^D} = 1,$	marginal productivity of capital in producer-goods industry,
(28)	$X_{2,1}^D + X_{2,2}^D = X_2^D,$	definition of total demand for producer goods,
(29)	$N_1^D + N_2^D = N^D,$	definition of total demand for labor,
(30)	$X_1^D = X_1^D(P_1, Y),$	demand for consumer goods,
(31)	$\dot{P}_1 = \dot{P}_1[\int_{-\infty}^{t} (X_1^S - X_1^D) d\theta],$	price-adjustment equation for consumer-goods market,

(32) $\dot{P}_2 = \dot{P}_2[\int_{-\infty}^{t}(X_2^S - X_2^D)d\theta]$, price-adjustment equation for producer-goods market,

(33) $\dot{W} = \dot{W}(N^S - N^D)$, wage-adjustment equation,

(34) $Y = P_1 X_1^S + P_2 X_2^S$, definition of income,

(35) $K_1 = \int_{-\infty}^{t}[X_{2,1}^D - D_1(X_{2,1}^D, K_1)]d\theta$, definition of stock of capital in consumer-goods industry,

(36) $K_2 = \int_{-\infty}^{t}[X_{2,2}^D - D_2(X_{2,2}^D, K_2)]d\theta$, definition of stock of capital in producer-goods industry,

where

N_1^D = demand for labor in consumer-goods industry,
$X_{2,1}^D$ = demand for new capital goods in consumer-goods industry,
K_1 = stock of existing capital in consumer-goods industry,
N_2^D = demand for labor in producer-goods industry,
$X_{2,2}^D$ = demand for new capital goods in producer-goods industry,
K_2 = stock of existing capital in producer-goods industry,
D_1 = depreciation of capital in consumer-goods industry,
D_2 = depreciation of capital in producer-goods industry,

and all other variables are as defined above.

If the marginal productivity of labor is to change by an amount ϵ through technological improvement in the producer-goods industry we should replace (25) and (26)[6] by

$$X_2^S = F_2^*(N_2^D, X_{2,2}^D, K_2, t), \tag{25*}$$

such that

$$\frac{\partial F_2^*}{\partial N_2^D} = \frac{\partial F_2}{\partial N_2^D} + \epsilon, \qquad \frac{\partial F_2}{\partial N_2^D} + \epsilon = \frac{W}{P_2} \cdot \tag{26*}$$

We can now compare the solutions to the equation systems under two hypotheses. One solution[7] will be that for the system (22)–(36), given by

$$Y = Y(N^S, t); \tag{37}$$

the other solution will be for the same system but with (25) and (26) replaced by (25*) and (26*). We shall have for this case, say,

[6] We are assuming that the marginal productivity of capital is unaffected by this change.

[7] We can solve the system for any of the endogenous variables.

$$Y = Y^*(N^S, t). \tag{38}$$

A comparison between the properties of (37) and (38) will show the influence on the level of national income of a change in the marginal productivity of labor in the producer-goods industry.

The information about the change in productivity cannot, in general, be obtained from the system (13)–(21) where the production function has been solved out of the system and where only the supply and demand equations are used. From the system (22)–(36) we can derive the forecast equations like (37), and we can also appraise the results of technological changes.

The production functions of the system (22)–(36) are (22) and (25). These are meant to be purely technological functions which show the relation between factor input and product output. When we speak of changes in productivity, as in the case above, we have in mind technological change which is independent of the economic calculations of profit maximization. These two aggregate production functions must show how much output can be obtained from any factor input whether this factor input satisfies the equilibrium conditions or not. If we construct our aggregates so that the aggregate production function exists only when the profit-maximizing equations exist, we do not obtain the technological relation between input and output; we obtain a relation between measures of input and output that satisfy certain economic criteria involving prices and wages.

If we want to develop systems like (13)–(21), the index numbers of Dresch and May can probably be adapted to such systems very elegantly. If we want to develop systems like (22)-(36), then we have to consider new types of indexes.[8]

There are some points made in Pu's paper that are not entirely well taken. These points are the following: (1) Distribution effects are not accounted for in the aggregates suggested in the present writer's earlier paper. (2) These aggregates have no economic significance. (3) The criteria put forth for the construction of these aggregates are too restrictive.

We shall now consider these points in order. If the production function for the αth firm can be written as

$$x_\alpha = C_\alpha \prod_{i=1}^{r} n_{i\alpha}{}^{a_{i\alpha}} \prod_{i=1}^{s} z_{i\alpha}{}^{b_{i\alpha}}, \qquad \alpha = 1, 2, \cdots, A, \tag{39}$$

with the aggregates defined as

[8] See L. R. Klein, "Macroeconomics and the Theory of Rational Behavior," ECONOMETRICA, Vol. 14, April, 1946, pp. 93–108.

$$X = \left(\prod_{\alpha=1}^{A} x_\alpha \right)^{1/A}, \tag{40}$$

$$N^a = \left(\prod_{\alpha=1}^{A} \prod_{i=1}^{r} n_{i\alpha}{}^{a_{i\alpha}} \right)^{1/A}, \tag{41}$$

$$Z^b = \left(\prod_{\alpha=1}^{A} \prod_{i=1}^{s} z_{i\alpha}{}^{b_{i\alpha}} \right)^{1/A}, \tag{42}$$

then it has been shown[9] that a well-defined macroeconomic system can be established from a knowledge of the microeconomic system.

The aggregates X, N, Z defined in (40), (41), and (42) are weighted geometric means of the individual x_α, $n_{i\alpha}$, and $z_{i\alpha}$, the weights being the individual production elasticities ($a_{i\alpha}$ and $b_{i\alpha}$) in the case of N and Z. Methods of calculating a and b, the exponents of N and Z respectively, have been given in the paper mentioned above. Pu claims to account for distributional effects by using the individual marginal productivities to derive his aggregates. Formulas (41) and (42) use the individual production elasticities to derive the aggregates. There is nothing to choose between using elasticities (logarithmic derivatives) and marginal productivities (ordinary derivatives) since they give approximately the same information. Elasticities are independent of units while marginal productivites are not; otherwise they are not essentially different. However, Pu is to be criticized for using equilibrium values of marginal productivities to obtain the aggregates which enter as variables in the production function. The objection to this method has already been discussed at length above. The elasticities used as weights in (41) and (42) are parameters of the individual production functions and do not depend upon the profit-maximizing equations. It is easy to see that if the distribution of the individual elasticities among firms is altered, the aggregates N and Z will be changed since the elasticities enter as weights in the calculation of N and Z. This is the sense in which these aggregates depend upon distributional characteristics.

What is the economic significance of aggregates? There is no reason to assume, as Pu does, that there is something sacred about a sum. It is true that the man in the street knows more about a sum than about many other types of aggregates, but in constructing scientific theory, we must look for useful results rather than things familiar to the layman. Any macroeconomic theory which will enable us to make people happier through an analysis of the interrelationship between aggregates of income, employment, output, etc., is a good theory regardless of the specific form of the aggregates. The economy is generally better off if

[9] *Ibid.*

the sum of all individual outputs[10] rises, and it is generally better off if the product of all outputs rises. A sum and product are often equally good for our purposes.

Pu's implied preference for his aggregates (sums) to (40), (41), and (42) (geometric means) brings to mind the lengthy discussion in the older books on statistical methods concerning the relative merits of arithmetic and geometric means. The right-hand sides of (41) and (42) are weighted geometric means of individual quantities and the exponents a and b of the left-hand side have a very simple economic interpretation; they are weighted averages of the individual elasticities in the production functions of the several firms.

There are at least two essentially different approaches to the problem of aggregation. We may accept the traditional theories of microeconomics and the commonly used aggregates such as the Federal Reserve Board production index, the Bureau of Labor Statistics price index, the Commerce Department national-income data, etc., and try to determine the structure of a macroeconomic system that is implied by these two sets of information. Such a system may be very complicated or may not even exist, but if we could construct it, we could hope to make forecasts of these well-known aggregates. Alternatively we could proceed differently by assuming the theories of micro- and macroeconomics in advance and then discovering what aggregates are consistent with these assumptions. In this case, we cannot know in advance the form of the aggregates but must accept those forms which satisfy a mathematical requirement. In the latter approach we cannot in general claim to have models that help us to predict the Federal Reserve Board index or the Bureau of Labor Statistics price index. Instead we obtain models that attempt to forecast the particular aggregates that satisfy the criteria assumed. It will often be true that in practice the correlation between these aggregates obtained by the second method and the published indexes will be so high that one set can be substituted for the other.

The matter of whether or not the criteria imposed upon aggregates in the present writer's earlier paper are too restrictive may perhaps be settled by considering the assumptions made in (39) about the production functions for the individual firm. The rest of the model follows very readily if this assumption holds. In many particular firms, statisticians have found that the Cobb-Douglas function (or simple modifications of this function) fit the output-input data very well. If we can determine a large enough sample of these individual-firm production functions, it will not be difficult to construct the aggregates that satisfy

[10] We assume a common unit of measurement.

the theory presented by the author. Thus far, there is no reason to believe that the logarithmic production function can not be applied in general as a good approximation of the output-input relationship for the individual firm. However, if another universal form of the production function were found to approximate closely the data of individual firms, we could undoubtedly construct alternative aggregates which would also lead to simple macroeconomic systems.

In closing, it may be useful to outline some of the major unsolved problems in the theory of aggregation:

1. The aggregates must be constructed so that the macroeconomic models are complete, covering the rational behavior of households and market interactions between households and firms, as well as the rational behavior of firms.

2. The aggregates should be such that they can be readily constructed or approximated from available data.

3. The assumptions of perfect competition must be dropped.[11]

4. The entire theory of aggregation must be developed for stochastic models of micro- and macroeconomics.

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[11] Dresch, *op. cit.*, has already made considerable progress in this direction.