Cowles Foundation Paper 19a
Reprinted from
ECONOMETRICA, Journal of the Econometric Society, Vol. 14, No. 4, October, 1946
The University of Chicago, Chicago 37, Illinois, U.S.A.

THE AGGREGATION PROBLEM FOR A ONE-INDUSTRY MODEL*†

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1. INTRODUCTION

THE aggregation problem arose from the introduction of simplified models as explicit mathematical formulations of economic theories. These models have justified themselves on pragmatic grounds, but their meaning and theoretical validity remains to be clarified. What economic entities correspond to the variables of these models? What is the significance of the functions and operations involved? Is it legitimate to operate with such models as if micro- and macroeconomic behavior were similar?

Some illuminating partial answers to these questions have been obtained. Dresch[1] has shown that Divisia-type indices of the variables of a general economic equilibrium satisfy relations analogous to the marginal relations holding between the variables of the two-industry simplified economic system introduced by G. C. Evans. This result was obtained independently of the existence of functional relations among the variables of the simplified system. Recently, Klein[2] has suggested more rigorous criteria for the solution of the problem: first, that the aggregates representing factors and outputs be functionally related, and second, that there should hold between the aggregates marginal relations similar to those of the general equilibrium. Klein derives conditions for the satisfaction of these criteria and gives a solution for the two-industry model based on special assumptions concerning the production functions.

In this paper we give a solution of the aggregation problem for a one-industry model of the type that underlies the theories of J. M. Keynes and others. Our solution requires no special conditions on the functions and satisfies the following criteria which include those of Klein:

* This article, together with the following ones by Messrs. Pu and Klein, and another one by Mr. May to be published in a later issue of ECONOMETRICA, will be reprinted as Cowles Commission Papers, New Series, No. 19.

† The author desires to express his indebtedness to the discussions held in Professor Griffith C. Evans' seminar on Mathematical Economics at the University of California at Berkeley.

[1] Francis W. Dresch, "Index Numbers and the General Equilibrium," *Bulletin of the American Mathematical Society*, Vol. 44, February, 1938, pp. 134–141, and Griffith C. Evans, "Maximum Production Studied in a Simplified Economic System," ECONOMETRICA, Vol. 2, January, 1934, pp. 37–50.

[2] Lawrence R. Klein, "Macroeconomics and the Theory of Rational Behavior," ECONOMETRICA, Vol. 14, April, 1946, pp. 93–108.

1. *The variables of the simplified model are defined as aggregates of those of the general model.*

2. *The functions of the simplified model are derived as functionals of the functions of the general model.*

3. *The number of degrees of freedom of the two models is the same*, i.e., the simplified model gives all the information about the aggregates that can be obtained from the general model. The significance of this criterion will become clear as the discussion proceeds.

4. *The marginal relations in the simplified model follow from those in the general model and are similar in form and economic significance.* In this paper we limit ourselves to the marginal relations of free-competition equilibrium.

2. THE ONE-INDUSTRY MODEL

We consider the following model involving total employment N, net output U, the money wage per unit employment w, the price of net output p, and money profits Π. We have the accounting relation

$$\Pi = PU - wN \tag{1}$$

and the technical relation

$$U = \psi(N) \tag{2}$$

which gives net output as a function of employment with "given resources."

If we now treat equations (1) and (2) in a manner analogous to the corresponding equations for a single enterprise and maximize profits on the assumption of free competition, we get the equilibrium condition

$$\psi'(N) = \frac{w}{P} = p \tag{3}$$

where

$$\psi''(N) < 0 \tag{3.1}$$

is a sufficient condition for the maximum.

As it stands the model (1)–(3) has two degrees of freedom, or one if we are interested only in the real wage p. To determine all the variables we may add a supply function for labor $N = S(p)$, an equation of demand[3] $U = \chi(N)$, or some other relation. A model is usually considered faulty if the number of equations is not equal to the number of unknowns.[4] However, for the solution of the aggregation problem it is

[3] J. M. Keynes, *The General Theory of Unemployment, Interest and Money*, New York, 1936, Chapter 3.

[4] See for example Jan Tinbergen, "Econometric Business Cycle Research,"

convenient to consider systems with one or more degrees of freedom. We therefore leave undetermined the choice of an additional relation to complete the model (1)–(3) and consider it simply as a system of one degree of freedom in N, p, U, and Π/P.

3. COMPARISON WITH A TWO-INDUSTRY MODEL

Although we are primarily interested in justifying the model (1)–(3) in terms of a general equilibrium, we begin by comparing it with Evans' two-industry model in order to illustrate methods and results in a simple context. The two-industry model is given by the following equations:

$$U_1^{(1)} + U_1^{(3)} = \phi(U_1^{(1)}N_1), \tag{4.1}$$

$$U_3 = \theta(U_1^{(3)}N_3), \tag{4.2}$$

$$\phi_1 = 1, \tag{4.3}$$

$$\phi_2 = p_2/p_1, \tag{4.4}$$

$$\theta_1 = p_1, \tag{4.5}$$

$$\theta_2 = p_2, \tag{4.6}$$

$$N_1 + N_3 = N, \tag{4.7}$$

where U_3 is the output of consumption goods, $U_1^{(1)}$ and $U_1^{(3)}$ the amounts of capital goods used in the production of capital and consumption goods, N_1 and N_3 the amounts of labor employed in the two industries, and p_1 and p_2 real prices defined by $p_1 = P_1/P_3$ and $p_2 = w/P_3$, where P_1, P_3, and w are the money prices of capital goods, consumption goods, and labor. Equations (4.3)–(4.6) express conditions of maximum profits on the assumption of free competition, and of maximum output of consumption goods for a given N,[5] provided the following sufficient conditions are satisfied:

$$\begin{aligned} &\phi_{11}\phi_{22} - \phi_{12}^2 > 0, \qquad && \phi_{11} < 0,\ \phi_{22} < 0, \\ &\theta_{11}\theta_{22} - \theta_{12}^2 > 0, \qquad && \theta_{11} < 0,\ \theta_{22} < 0. \end{aligned} \tag{4.8}$$

The seven equations involve eight variables and hence define a system of one degree of freedom. Since we wish to compare it with the one-industry model, we consider N as the independent variable. The

Review of Economic Studies, Vol. 7, February, 1940, pp. 73–90: " . . . the Lausanne school has sometimes—and rightly, I think—been ridiculed for stopping after having found that there are as many equations as unknowns. Nevertheless, there is one thing that is worse, viz. finding that there are not the same number!" (p. 80).

[5] Griffith C. Evans, *loc. cit.* Evans' notation has been modified slightly. Subscripts indicate partial derivatives with respect to the first or second arguments.

system then determines each variable as a function of N. In economic terms, this means that to each level of employment there correspond equilibrium values of the other variables. We assume that the functions involved are such that these variables are continuous and differentiable functions of N. In particular we have

$$U_3 = \theta[U_1^{(3)}(N), N_3(N)] \tag{5}$$

and

$$p_2 = \theta_2[U_1^{(3)}(N), N_3(N)]. \tag{5.1}$$

In comparing the two models, it seems reasonable to identify U_3 with U, since all capital goods produced are used up in the same period in Evans' model and hence U_3 is the net output. It follows that P_3 is identified with P. Equations (5) and (5.1) therefore correspond with (2) and (3). If the one-industry model is to give the same relationships between p_2, N, and U as the two-industry model, we must have

$$\psi(N) \equiv \theta[U_1^{(3)}(N), N_3(N)] \tag{5.2}$$

and

$$\psi'(N) \equiv \theta_2[U_1^{(3)}(N), N_3(N)]. \tag{5.3}$$

The second equation is required in order that p_2 shall be given as the same function of N in both models. But

$$\psi'(N) \equiv \frac{d\theta}{dN} \equiv \theta_1 \frac{dU_1^{(3)}}{dN} + \theta_2 \frac{dN_3}{dN} \cdot \tag{5.31}$$

It follows that with $\psi(N)$ defined by (5.2) the necessary and sufficient condition that the models "agree" is the following identity in N:

$$\theta_2 \equiv \theta_1 \frac{dU_1^{(3)}}{dN} + \theta_2 \frac{dN_3}{dN} \cdot \tag{5.4}$$

At first sight, this appears a highly restrictive condition. However, from (4.1) we have after differentiating both sides by N:

$$\frac{dU_1^{(1)}}{dN} + \frac{dU_1^{(3)}}{dN} \equiv \phi_1 \frac{dU_1^{(1)}}{dN} + \phi_2 \frac{dN_1}{dN} \cdot \tag{5.5}$$

Also from (4.7) $dN_1/dN \equiv 1 - dN_3/dN$. When account is taken of this and of the relation (4.3), (5.5) reduces to

$$\frac{dU_1^{(3)}}{dN} \equiv \phi_2 - \phi_2 \frac{dN_3}{dN} \cdot \tag{5.51}$$

When this value of $dU_1^{(3)}/dN$ is substituted in the right-hand side of (5.4) it reduces to

$$\theta_1\phi_2 + (\theta_2 - \theta_1\phi_2)\frac{dN_3}{dN}\cdot$$

But from (4.3)–(4.6) it follows that

$$\theta_2 \equiv \theta_1\phi_2, \tag{5.52}$$

and hence the right-hand side of (5.4) is identically equal to θ_2.

It follows that if we define the production function of the one-industry model by equation (5.2), we may proceed without reference to the two-industry model and arrive at the same results with respect to the two variables involved. In this sense, the one-industry model is a uniquely determined and exact model of the two-industry system. No index problem is involved in this case, and the meaning of the function $\psi(N)$ and the legitimacy of the differentiation is established in terms of the more complete model.

4. DIFFERENTIATION OF CAPITAL GOODS

The general equilibrium involves any number of different factors and products. However, the aggregation problem is not symmetric with respect to these additional variables. Hence we introduce them one at a time in the general equilibrium. We first consider the following system involving n types of capital goods:[6]

$$\sum_{j=1}^{n+i} U_i^{(j)} = \varphi_{(i)}[U_1^{(i)}, \cdots, U_n^{(i)}, N_i], \qquad i = 1, \cdots, n, \tag{6.1}$$

$$Q = \theta[U_1^{(n+1)}, \cdots, U_n^{(n+1)}, N_{n+1}], \tag{6.2}$$

$$\varphi_{(i)j} = \frac{p_j}{p_i}, \quad \begin{matrix} i = 1, \cdots, n, \\ j = 1, \cdots, n, \end{matrix} \quad \varphi_{(i)j} = \frac{\partial\varphi_{(i)}}{\partial U_j^{(i)}}, \tag{6.3}$$

$$\varphi_{(i)n+1} = \frac{p}{p_i}, \qquad i = 1, \cdots, n, \tag{6.4}$$

$$\theta_j = p_j, \quad j = 1, \cdots, n, \qquad \theta_j = \frac{\partial\theta}{\partial U_j^{(n+1)}}, \tag{6.5}$$

$$\theta_{n+1} = p, \tag{6.6}$$

$$\sum_{i=1}^{n+1} N_i = N, \tag{6.7}$$

[6] Dresch, *loc. cit.* This is Dresch's general equilibrium for the special case of one type of labor and one type of consumption good. His notation is modified somewhat to facilitate distinction between the different factors.

where $U_j^{(i)}$ is the amount of U_j going into the production of U_i, Q is net output corresponding to U in the one-industry model, $p=w/P$, and the p_i are real prices given by P_i/P. A count of the equations and unknowns indicates one degree of freedom.

Again we consider the above system of equations as defining each variable as a function of N. As in (5.2), we take

$$\psi(N) \equiv \theta[U_1^{(n+1)}(N), \cdots, U_n^{(n+1)}(N), N_{n+1}(N)], \tag{6.8}$$

and wish to prove that p as given by (6.6) will be the same as that given by (3) and (6.8), i.e.,

$$\theta_{n+1} \equiv \frac{d\theta}{dN}\cdot \tag{6.9}$$

But the right-hand side of this equation is equal to

$$\sum_{i=1}^{n} \theta_i \frac{dU_i^{(n+1)}}{dN} + \theta_{n+1}\frac{dN_{n+1}}{dN},$$

and hence the result we wish to prove may be written in the differential form,

$$\theta_{n+1}dN \equiv \sum_{i=1}^{n} \theta_i dU_i^{(n+1)} + \theta_{n+1}dN_{n+1}. \tag{6.91}$$

Now the relations (6.1)–(6.7) become identities in N when the variables are replaced by their expressions in N. Hence from (6.1), by taking the total differential, we get

$$\sum_{j=1}^{n+1} dU_i^{(j)} - \sum_{j=1}^{n} \varphi_{(i)j}dU_j^{(i)} - \varphi_{(i)n+1}dN_i \equiv 0, \qquad i = 1, \cdots, n. \tag{7}$$

Multiplying by θ_i and summing over i from 1 to n, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n+1} \theta_i dU_i^{(j)} - \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_i\varphi_{(i)j}dU_j^{(i)} - \sum_{i=1}^{n} \theta_i\varphi_{(i)n+1}dN_i \equiv 0. \tag{7.01}$$

But from (6.3)–(6.6) it follows that

$$\theta_j = \theta_i\varphi_{(i)j} \tag{7.02}$$

for all i and j. Rewriting (7.01) in the following form

$$\begin{aligned}\sum_{i=1}^{n}\sum_{i=1}^{n} (\theta_j - \theta_i\varphi_{(i)j})dU_j^{(i)} &+ \sum_{1}^{n} \theta_i dU_i^{(n+1)} \\ &- \sum_{1}^{n} \theta_i\varphi_{(i)n+1}dN_i \equiv 0,\end{aligned} \tag{7.03}$$

we see that the first summation is zero and hence

$$(7.04) \qquad \sum_{i=1}^{n} \theta_i dU_i^{(n+1)} \equiv \sum_{i=1}^{n} \theta_i \varphi_{(i)n+1} dN_i.$$

But $\theta_i\varphi_{(i)n+1} \equiv \theta_{n+1}$ from (7.02) and $\sum_1^n dN_i \equiv dN - dN_{n+1}$ from (6.7). Hence (7.04) reduces as follows:

$$(7.05) \qquad \sum_{i=1}^{n} \theta_i dU_i^{(n+1)} \equiv \theta_{n+1} \sum_{i=1}^{n} dN_i \equiv \theta_{n+1}(dN - dN_{n+1}),$$

which is evidently equivalent to (6.91).

This result establishes the validity of the one-industry system as a simplified model of a system involving any number of capital goods. No index problems are involved since the simplified system is obtained merely by eliminating the capital goods that are used up in production and that are themselves functions of employment.

5. DIFFERENTIATION OF CONSUMPTION GOODS

We now proceed to a system involving m consumption goods Q_i, and n capital goods U_i. In order to connect this with the one-industry model it will be necessary to introduce an index of consumption goods and prices. We have the following general-equilibrium model with homogeneous labor:[7]

$$(7.1) \qquad \sum_{j=1}^{n+m} U_i^{(j)} = \varphi_{(i)}[U_1^{(i)}, \cdots, U_n^{(i)}, N_i], \qquad i = 1, \cdots, n,$$

$$(7.2) \qquad Q_i = \theta_{(i)}[U_1^{(n+i)}, \cdots, U_n^{(n+i)}, N_{n+i}] \qquad i = 1, \cdots, m,$$

$$(7.3) \qquad \varphi_{(i)j} = \frac{P_j}{P_i}, \qquad i = 1, \cdots, n, \; j = 1, \cdots, n,$$

$$(7.4) \qquad \varphi_{(i)n+1} = \frac{w}{P_i}, \qquad i = 1, \cdots, n,$$

$$(7.5) \qquad \theta_{(i)j} = \frac{P_j}{P_{n+i}}, \qquad i = 1, \cdots, m, \; j = 1, \cdots, n,$$

$$(7.6) \qquad \theta_{(i)n+1} = \frac{w}{P_{n+i}}, \qquad i = 1, \cdots, m,$$

$$(7.7) \qquad \sum_{i=1}^{n+m} N_i = N.$$

[7] Dresch, *loc. cit.* This is Dresch's general equilibrium for the special case of one type of labor.

As it stands, the system has $m+1$ degrees of freedom. We need additional relations to "complete" the system, which in our case means to reduce it to one degree of freedom. We could take, for example, budget functions that relate the amount spent on each type of consumption good and the total income. An additional degree of freedom could be eliminated by considering only the ratios of prices to one particular price. An equation of exchange may be added, but this will reduce the degrees of freedom only on the dubious assumption that the quantities of money and velocities of circulation are fixed. Many choices of additional equations are possible, but it turns out that the results are independent of how this question is decided. That is, we can always construct a one-industry model, although the form of ψ will of course depend on all functions in the general model including those not explicit in this discussion.

We suppose only that the system is reduced to one degree of freedom, so that all variables are functions of N. We will define net output by Divisia-type indices, where the parameter of integration is N. That is:

$$U = \exp\left[\int_{N_0}^{N} \frac{\sum_1^m P_{n+i}dQ_i}{\sum_1^m P_{n+i}Q_i}\right], \tag{7.8}$$

$$P = \exp\left[\int_{N_0}^{N} \frac{\sum_1^m Q_i dP_{n+i}}{\sum_1^m P_{n+i}Q_i}\right], \tag{7.9}$$

where the U and P are identified with the U and P of the one-industry model.[8] With the addition of these definitions, the system (7.1)–(7.9) still has one degree of freedom, and U and P, like the other variables, are functions of N. We now define the function ψ to be equal to the right-hand side of equation (7.8). With this definition we wish to derive (3) from the general system, or what amounts to the same thing, to prove that

$$PdU \equiv wdN. \tag{8}$$

Now from (7.8)

[8] F. Divisia, *Economique Rationelle*, Paris, 1928, p. 268. The Divisia indices, which were defined only with respect to time variations, were extended by Dresch (*loc. cit.*) to include virtual changes.

$$(7.81)\qquad \frac{dU}{U} = \frac{\sum_1^m P_{n+i}dQ_i}{\sum_1^m P_{n+i}Q_i}.$$

But for a convenient choice of units we have[9]

$$(7.91)\qquad PU = \sum_i^m P_{n+i}Q_i.$$

Hence the left-hand side of (8) equals $\sum_1^m P_{n+1}dQ_i$. If account is taken of the fact that $dN = \sum_{i=1}^{n+m} dN_i$ from (7.7), our desired result may be rewritten

$$(8.01)\qquad \sum_{i=1}^{m} P_{n+i}dQ_i - w\sum_{i=1}^{n+m} dN_i = 0,$$

and, on substitution of the value of dQ_i from (7.2), takes the form

$$(8.02)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} P_{n+i}\theta_{(i)j}dU_j^{(n+i)} + \sum_{i=1}^{m} P_{n+i}\theta_{(i)n+1}dN_{n+i} - w\sum_{i=1}^{n+m} dN_i \equiv 0,$$

which may be rearranged as

$$(8.03)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} P_{n+i}\theta_{(i)j}dU_j^{(n+i)} - w\sum_{i=1}^{n} dN_i + \sum_{i=1}^{m} (P_{n+i}\theta_{(i)n+1} - w)dN_{n+i} \equiv 0.$$

By the use of (7.5) and (7.6), this may be written

$$(8.04)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} P_j dU_j^{(n+i)} - w\sum_{i=1}^{n} dN_i \equiv 0,$$

an equation that is equivalent to (8). But from (7.1) we have

$$(8.05)\qquad \sum_{i=1}^{n+m} dU_j^{(i)} = \sum_{i=1}^{n} \varphi_{(j)i}dU_i^{(j)} + \varphi_{(j)n+1}dN_j, \qquad j = 1, \cdots, n,$$

[9] Griffith C. Evans, *Mathematical Introduction to Economics*, New York, 1930, pp. 101–103. The relation (7.91) follows directly from the general definition of the Divisia indices with

$$P(N_0) = 1 \text{ and } U(N_0) = 1.$$

or

$$(8.06) \qquad \sum_{i=1}^{m} dU_j^{(n+i)} = \sum_{i=1}^{n} \varphi_{(j)i} dU_i^{(j)} + \varphi_{(j)n+1} dN_j - \sum_{i=1}^{n} dU_j^{(i)}.$$

If this expression is substituted in (8.04), its left-hand member becomes

$$\sum_{j=1}^{n}\sum_{i=1}^{n} P_j\varphi_{(j)i} dU_i^{(j)} + \sum_{j=1}^{n} P_j\varphi_{(j)n+1} dN_j - \sum_{i=1}^{n}\sum_{j=1}^{n} P_j dU_j^{(i)} - w\sum_{i=1}^{n} dN_i,$$

which is the same as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (P_i\varphi_{(i)j} - P_j) dU_j^{(i)} - \sum_{i=1}^{n} (P_i\varphi_{(i)n+1} - w) dN_i,$$

an expression that is evidently identically equal to zero because of (7.3) and (7.4).

Hence in this case too the more general system defines a one-industry model such that macro- and microeconomic relations are analogous. Evans has shown[10] that the Divisia-Dresch indices may be approximated by the familiar

$$(8.07) \qquad P(N) = \frac{\sum_1^m P_{n+i}(N) Q_i(N_0)}{\sum_1^m P_{n+i}(N_0) Q_i(N_0)},$$

$$(8.08) \qquad U(N) = \frac{\sum_1^m P_{n+i}(N) Q_i(N)}{\sum_1^m P_{n+i}(N) Q_i(N_0)}.$$

Hence the aggregation involves no special practical difficulties. The result is similar to that of Dresch for the two-industry model, except that we have established ψ as a functional of the more general model and shown an exact rather than analogous relation between the two models. No conditions have to be placed on the functions involved other than those imposed by the general equilibrium itself.

6. INHOMOGENEOUS LABOR

So far we have assumed labor to be homogeneous. The removal of this simplification requires a definition of total employment and wages as indiees. However, for our purposes it is not convenient to define them as Divisia-Dresch indices. We desire to maintain N as the independent

[10] Evans, *Mathematical Introduction to Economics, loc. cit.*

variable and the parameter of integration in the indices P and U. At the same time it is convenient to have definitions of w and N that coincide with ordinary usage. For these reasons, and for formal reasons that will appear below, we take

$$N = \sum_{k=1}^{s} N_k, \tag{8.7}$$

$$w = \frac{1}{N} \sum_{k=1}^{s} w_k N_k. \tag{8.71}$$

These definitions are lacking in symmetry, but they correspond to the usual meanings of "employment" and "wages," the former being simply the unweighted sum of different types of employment and the latter, the weighted average of different wage rates.

We have a system of $m+s$ degrees of freedom determined by the following general equilibrium:

$$\sum_{j=1}^{n+m} U_i^{(j)} = \varphi_{(i)}[U_1^{(i)}, \cdots, U_n^{(i)}, N_1^{(i)}, \cdots, N_s^{(i)}), \qquad i = 1, \cdots, n, \tag{8.1}$$

$$Q_i = \theta_{(i)}[U_1^{(n+i)}, \cdots, U_n^{(n+i)}, N_1^{(n+i)}, \cdots, N_s^{(n+i)}], \qquad i = 1, \cdots, m, \tag{8.2}$$

$$\varphi_{(i)j} = \frac{P_j}{P_i}, \qquad \begin{matrix} i = 1, \cdots, n, \\ = 1, \cdots, n, \end{matrix} \tag{8.3}$$

$$\varphi_{(i)n+k} = \frac{w_k}{P_i}, \qquad \begin{matrix} i = 1, \cdots, n, \\ k = 1, \cdots, s, \end{matrix} \tag{8.4}$$

$$\theta_{(i)j} = \frac{P_j}{P_{n+i}}, \qquad \begin{matrix} i = 1, \cdots, m, \\ k = 1, \cdots, s, \end{matrix} \tag{8.5}$$

$$\theta_{(i)n+k} = \frac{w_k}{P_{n+i}}, \qquad \begin{matrix} i = 1, \cdots, m, \\ k = 1, \cdots, s, \end{matrix} \tag{8.6}$$

$$N = \sum_{k=1}^{s} N_k, \quad \text{where} \quad N_k = \sum_{i=1}^{n+m} N_k^{(i)}, \tag{8.7}$$

$$w = \frac{1}{N} \sum_{1}^{s} w_k N_k, \tag{8.71}$$

$$U = \exp\left[\int_{N_0}^{N} \frac{\sum_{1}^{m} P_{n+i} dQ_i}{\sum_{1}^{m} P_{n+i} Q_i} \right], \tag{8.72}$$

$$P = \exp\left[\int_{N_0}^{N} \frac{\sum_1^m Q_i dP_{n+i}}{\sum_1^m P_{n+i}Q_i}\right]. \tag{8.73}$$

We suppose that additional relations have been established so that the system has one degree of freedom. We wish to show that the aggregates defined by the last four equations satisfy equation (3) written in the form

$$P\frac{dU}{dN} = w \tag{9}$$

or from (8.71) in the equivalent form

$$P\frac{dU}{dN} = \frac{1}{N}\sum_1^s w_k N_k. \tag{9.01}$$

From (8.72) and (8.73) it follows, as we showed above in equations (7.81) and (7.91), that the left-hand side of this equation equals

$$\sum_{i=1}^m P_{n+i}\frac{dQ_i}{dN}\cdot$$

Substituting for dQ_i/dN as we did for dQ_i in the previous case, and similarly making use of the relations (8.3)–(8.6) and the results of differentiating the equations (8.01), we find that

$$P\frac{dU}{dN} = \sum_1^s w_k\frac{dN_k}{dN}\cdot \tag{9.02}$$

The right-hand side of this equality is not the same as that of (9.01), but it has a similar form which is emphasized if we write the desired relation in the form

$$P\frac{dU}{dN} = \sum_1^s w_k\frac{N_k}{N}\cdot \tag{9.01}$$

Evidently in the case $s=1$ the two expressions are identical, since $N_1/N = dN_1/dN = 1$ and $w_1 = w$. In fact this is the case of homogeneous labor. We could make (9) hold exactly by defining w as $\sum_1^s w_k(dN_k/dN)$, but it would be difficult to interpret this w or the corresponding index for N.

With our definition of w and N, the necessary and sufficient condition that (9) be satisfied is that the right-hand members of (9.01) and (9.02) be equal, i.e., that

$$\sum_1^s w_k \frac{dN_k}{dN} = \sum_1^s w_k \frac{N_k}{N}. \tag{9.03}$$

This equality will hold if

$$\frac{dN_k}{dN} = \frac{N_k}{N}, \tag{9.04}$$

which is equivalent to

$$N_k = C_k N, \qquad k = 1, \cdots, s, \tag{9.05}$$

in which the C_k are functions of k and $\sum_1^s C_k = 1$. In economic terms this means that as total employment changes, the amounts of different types of labor remain in proportion. Since different types of labor are distinguished here primarily on the basis of different wage rates, this is practically equivalent to the assumption that the proportionate distribution of wage rates remains the same with changes in total employment.

However, even if the relations (9.04) do not hold exactly, the equality (9.03) will be approximately valid. In fact its two members are weighted averages of the wage rates since $\sum_1^s N_k = N$ and $\sum_1^s dN_k = dN$ and it has been shown[11] that such averages are not greatly affected by variations in the weights. It is only necessary that the variations from proportionality be of a more or less random character.

Hence the desired result (9) holds exactly if the proportions of different types of labor remain constant and approximately otherwise. In any case, the significance of the function ψ, the legitimacy of differentiation, and the analogous behavior of the one-industry model have been derived from the general equilibrium.[12] The discussion was entirely in the context of free competition on the assumption that an equilibrium existed for each value of N. On this basis, the results were obtained independently of what relations were chosen to complete the models. Different assumptions, such as the introduction of limited competition, would require new treatment, but the above results suggest methods for treating other cases.

7. SECONDARY CONDITIONS

Throughout the discussion we have assumed that secondary conditions were satisfied so that the marginal relations corresponded to

[11] See for example, Arthur L. Bowley, *Elements of Statistics*, London, 1937, pp. 86–94.

[12] The relation $U = \psi(N)$ holds for all values of N, but evidently ψ depends on the equilibrium conditions. This suggests that *aggregate* production functions cannot have a purely technological character. For society as a whole output is a resultant of nontechnological as well as technological forces and no purely technical input-output relation exists.

maximum-profit positions. Can it be shown that the secondary condition (3.1) follows from the secondary conditions required for the more general models? In economic terms, is the position of maximum profit for each entrepreneur under conditions of free competition also a position of maximum total profit on the assumption that the entire economy behaves like a single enterprise? For simplicity, we will consider the question only in terms of a comparison of the one-industry and two-industry models.

Utilizing (5.3), we have

$$\psi''(N) = \theta_{21} \frac{dU_1^{(3)}}{dN} + \theta_{22} \frac{dN_3}{dN} \cdot \tag{9.1}$$

We may evaluate $dU_1^{(3)}/dN$ and dN_3/dN by implicit differentiation in the equations of the two-industry model. In fact

$$\frac{dU_1^{(3)}}{dN} = \frac{\phi_2\phi_{11}(\theta_{22} - \phi_2\theta_{12})}{D}, \tag{9.2}$$

$$\frac{dN_3}{dN} = \frac{\theta_1(\phi_{11}\phi_{22} - \phi_{12}^2) + \phi_2\phi_{11}(\phi_2\theta_{11} - \theta_{12})}{D}, \tag{9.3}$$

where

$$D = \phi_{11}(\phi_2^2\theta_{11} - 2\phi_2\theta_{12} + \theta_{22}) + \theta_1(\phi_{11}\phi_{12} - \phi_{12}^2),$$

from which we have by substitution in (9.1)

$$\psi''(N) = \frac{\theta_{22}\theta_1(\phi_{11}\phi_{22} - \phi_{12}^2) + \phi_{11}\phi_2^2(\theta_{11}\theta_{22} - \theta_{12}^2)}{D} \cdot \tag{9.4}$$

From the secondary conditions in the two-industry model (4.8), it is evident that the numerator is negative. To make the sign of D definite we need some further assumption. If we take $\theta_{12} \geqq 0$, which is a reasonable assumption since it is the change in marginal productivity of one factor with an increase in the other factor, the sign of the denominator will be positive. The secondary condition (3.1) is thus established.

Evidently, additional assumptions are necessary to establish the secondary conditions, but they are of an economically unexceptional character. There is no indication of new difficulties, except complications, in extending the results of this section to more general equilibrium systems.

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