THEORY OF THE FIRM AND OF INVESTMENT*

By Leonid Hurwicz

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INTRODUCTION

A. Formulation of the Problem

A-I. Some of the most fundamental controversies in the theory of
economic fluctuations are due to the lack of a well-integrated theory of
the entrepreneur’s investment policies. There exists a tendency, though
not without important exceptions, to separate the financial from the
technical and commercial aspects of the firm’s operations. Moreover,
there is no agreement on the fundamental maximizing principle to serve
as basis for a general theory of entrepreneurial behavior.

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The traditional principle of (money) profit maximization is subject to attack from several quarters. First, of course, it is the expected profits that are being maximized. Second, when risk and uncertainty are present (the case of "stochastic expectations"), not only the most probable value of profits is of relevance, but also the degree of uncertainty: a cautious entrepreneur will choose a course of action promising one bird in the hand rather than two in the bush.

Thus the entrepreneur's psychological make-up (somewhat belatedly) enters the picture, and, at least implicitly, profit maximization is replaced by utility maximization. Upon insertion of appropriate specializing assumptions, the utility maximization principle will yield most, but not all, existing theories of the firm and of investment behavior.

Among approaches inconsistent with this principle, a very important one is based on an analysis of the prevailing accounting practices and their implications. For instance, some of these practices seem to imply that businessmen follow a principle of profit-rate maximization.  

It is not a priori inconceivable that business is run more by routine than by rationality, and, if such were the case, theorizing based on the utility- (or profit-) maximization principle would be little more than idle pastime. Whether the "rational" or the "routine" approach is more realistic can only be decided by appeal to empirical evidence (interviewing, a study of firm histories, statistical analysis). But for such evidence to be conclusive, the implications of the "rational" approach must be analyzed more rigorously than has as yet been done. Many phenomena may appear irrational and "routine" from the viewpoint of a purely static theory devoid of the uncertainty elements, but turn out to be quite "rational" when elements of uncertainty and long-run effects are taken into account.

A-2. The present paper is an attempt to develop the theory of the entrepreneur's behavior, especially those aspects of it which are of importance for investment policies, starting with the postulate of utility maximization. Thus it is an exploration in the possibilities of "rationalizing" the entrepreneur's behavior. It is not the author's intention to belittle the importance of studying the routine practices prevalent in business, but rather to sharpen the tools of analysis so that fruitful empirical work can be carried out. It would be hardly surprising if reality turned out to be a mixture of "rationality" and "routine."

The theory, as here developed, is more general than previous contributions in some respects, but less so in others. Generality was espe-

1 Cf. Lutz [14], also [15], especially p. 817. [Numbers in square brackets refer to items in the list of References at the end of this article.] "Profit rate" is here defined as the ratio of net revenue to the cost of equipment.
cially sought with regard to those aspects of the firm's structure which are likely to affect its investment policies, credit operations, and liquidity. We shall try to construct a model explaining not only the entrepreneur's behavior with regard to purchases of equipment or utilization of labor and other factors of production, but also with regard to his financial operations (demand for loan funds, securities, cash) and inventories.

Simplifying assumptions are made when it is felt that this does not deprive the model of its essential features. But in many respects the level of generality is still too high; detailed study of special cases of interest cannot be undertaken here. Only occasionally will it be indicated how various theories in this field are related to our model. It should be made clear, however, that our interest is primarily centered on the logical relationships; the references to other theories, therefore, are highly incomplete and no attempt is made to establish priorities or to trace the origin of the ideas under consideration.

B. Some Limitations of the Treatment

B-1. While perfect competition in product, factor, or credit markets is not assumed, we do postulate subjective predictability, i.e., we assume that the entrepreneur is able to form some, possibly incorrect, expectations (with or without uncertainty), with regard to the behavior of those from whom he buys the factors of production, as well as the behavior of his customers and competitors. That this postulate breaks down completely in cases such as that of duopoly has been known to the economists for a considerable period of time. The duopolist $A$ cannot form the "reaction curve" of the duopolist $B$, since $B$'s response to $A$'s reaction depends on $B$'s expectations of $A$'s response to $B$'s reaction, etc. That it may not hold even in the more "ordinary" economic situations has only recently been emphasized. We shall nevertheless assume the postulate to hold, since the analytical techniques for dealing with the cases where it does not apply are not yet sufficiently developed. Thus the assumption of subjective predictability certainly makes our model inapplicable to oligopolistic or cartel situations and probably only approximately correct otherwise.

B-2. Another limitation is due to our identification of the interests of the firm with those of the entrepreneur as an individual. This may be realistic for a privately owned firm, but it certainly requires modification in the case of corporate enterprise with its conflicting interest groups (stockholders, directors, managers). A separate study is needed

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1 Cf. von Neumann and Morgenstern [21], Hurwicz [8], Marschak [18].

2 In this respect Lange's treatment is of greater generality: [13], especially pp. 39 ff., 69–70, 75 ff.
to evaluate the seriousness of error committed in applying the theory of investment by a privately owned firm to the case of corporate enterprise.

B-3. Our model is also deficient in that it ignores business and other taxes of great practical importance but it is hoped that this shortcoming will be remedied in a later paper.

B-4. A continuous model (i.e., with time as a continuous variable) would be more realistic, more elegant, and in many ways easier to handle mathematically than a discrete one. We have chosen the latter, however, because it facilitates the use of certain tools of stochastic theory that are still undeveloped for the continuous case. A discrete model is also easier to compare with traditional economic theory most of which is formulated in terms of discrete time processes.

The paper is divided into two parts. In Part I the entrepreneur’s expectations will be assumed to be free of risk and uncertainty elements; the latter will be introduced in Part II. This division appears justified since it makes possible some simplification in the exposition.

I. THE CASE OF NONSTOCHASTIC* EXPECTATIONS

C. The Principle of Utility Maximization

C-1. We picture the entrepreneur as making periodic* money withdrawals (usually positive, but sometimes zero or negative) from the firm’s stock of cash. These withdrawals will be denoted by $d_t$.

At the time $t_0$ (when decisions are being made) the entrepreneur considers the stream of prospective* withdrawals, i.e., the vector

$I.e.,$ free of subjective uncertainty or risk; stochastic expectations imply the presence of subjective uncertainty or risk or both.

* The frequency of withdrawals need not of course be equal to the frequency of decision-making; moreover, frequency of decision-making may be different for different aspects of the firm’s activities: say once a month for amount of labor used, but only once a year for major purchases of equipment; nevertheless, for the sake of simplicity of exposition we shall assume all these frequencies to be equal to the unit of time measurement and shall refer to this unit as a month; this in no way restricts the generality of the model. It is always possible during maximization to add a constant requiring that, say, every other month’s withdrawals should be zero and that investment decisions should be final for 12 months ahead but decisions with regard to inventories only one month ahead.

$^*$ $d$ is a “flow variable” (i.e., measured per unit of time); its subscript refers to the beginning of the time interval covered; $d_t$ are withdrawals during the time interval $(t, t+1)$; the same principle of notation is applied throughout the paper to all other flow variables.

$^*$ $\bar{d}$ is a symbol for the expected values of $d$; all expectations, as well as decisions, are made at the time $t_0$; thus $\bar{d}_{t+1}$ are withdrawals expected at $t_0$ for the time interval $(t_0+1, t_0+2)$; the same principle of notation is applied to all other expected variables.

We use as synonyms the terms “expected,” “prospective,” “imagined,” and
(1) \[ \bar{D}_t = (d_{t0}, d_{t0+1}, d_{t0+2}, \ldots). \]

\( \bar{D}_t \) is of (denumerably) infinite dimensionality, although its more remote components may have little effect on current decisions.

\( \bar{D}_t \) depends on three types of factors: (a) those known at the time of decision making, e.g., the firm’s assets; (b) some unknown ones that can only be expected, for instance the future market conditions; and (c) some that depend on the entrepreneur’s decisions, for instance, purchases of equipment, borrowing, inventory policies.

Given the knowledge of expectations of the factors beyond the firm’s control at the time of decision-making, the entrepreneur will decide on those matters where he does have a certain amount of choice.

These decisions will determine \( \bar{D}_t \). The entrepreneur is thus able, within limits, to choose the time pattern of the prospective withdrawals stream, as well as its size. What considerations will affect his choice?

In answering this question we shall start by making two highly restrictive assumptions: 1. that the entrepreneur’s utility depends only on the prospective real consumption stream, and 2. that the entrepreneur does not save within his household, i.e., that his money consumption within a given period equals the cash withdrawals from the firm for the same period. Then we have

(2) \[ u_t = u_t(\bar{C}_t), \quad \bar{C}_t = \{c_t, c_{t+1}, \ldots\}, \]

where \( c \) is real consumption, and also

(3) \[ \rho_t \bar{c}_t = \bar{d}_t, \]

where \( \rho_t \) represents the prices of consumer goods purchased by the entrepreneur. Substituting (3) into (2) we obtain

(4) \[ u_t = u_t\left(\frac{\bar{d}_t}{\bar{p}_t}, \frac{\bar{d}_{t+1}}{\bar{p}_{t+1}}, \ldots\right). \]

Of course, if private income tax exists it should be deducted from the withdrawals.

In this highly simplified model the entrepreneur’s decisions are determined by 1. his preferences as given by the utility function, and 2. his opportunities (as they exist in his mind) given by the limits within which he can vary \( \bar{D}_t \) and by the expected (consumer-goods) prices and tax rates. The entrepreneur’s decision in favor of earlier, as against later, withdrawals might be due to his “impatience” or to expected rises in consumer-goods prices or tax rates.

\(^*\) “estimated”; all these terms have been used in the literature of the subject. “Anticipated” has also been used, but it is apt to be interpreted in the sense of preventive speculative measures and has, therefore, been avoided.

\(^*\) Cf. Fisher [1].
A more general model can be obtained by relaxing both (2) and (3). First, we may introduce variables other than consumption influencing the entrepreneur's utility: the assets whose mere possession brings the entrepreneur to a higher utility level, effort connected with running the firm, amount of leisure,* or the level of the firm's activity, if this affects the entrepreneur's sense of self-importance and the pleasure he gets out of running his business. Second, we may allow for savings, as well as private sources of income, such as private lending by the entrepreneur. In this generalized model there are more decisions to be made; in order to see what determines decisions with regard to $\bar{D}_{t_0}$, we must consider the structure of the entrepreneur's preferences with regard to variables other than consumption, and also his opportunities as a private individual. A decision in favor of earlier withdrawals may be due to a number of causes in addition to those previously mentioned; for instance, it may be due to the fact that the entrepreneur does not want to assume the burden of running too big a business and that he expects the earlier withdrawals to arrest the growth of the firm; or he may have unusual opportunities of lending out privately and thus earning additional income, etc.

The decisions to be made within the entrepreneur's household and the dependence of utility on the various factors mentioned are thus of relevance for the theory of the firm. But they do not constitute a part of that theory. Their study is a proper branch of the theory of individual economic behavior and transcends the scope of this paper which is primarily concerned with the entrepreneur's activities as entrepreneur. In what follows, therefore, we shall write

$$u_{t_0} = \phi_{t_0}(\bar{D}_{t_0}),$$

which, formally, may be regarded as a special case of (4), viz., where stationary expectations with regard to prices of consumer goods prevail. However, factors not included in (5) must be kept in mind since they are of importance for theory, prediction, and public policy.

$\phi_{t_0}$ in (5) is usually assumed to be such that

$$\frac{\partial \phi_{t_0}}{\partial \bar{D}_{t'}} > \frac{\partial \phi_{t_0}}{\partial \bar{D}_{t''}} \quad \text{for} \quad t' < t'',$$

which implies "impatience" on the part of the entrepreneur. (This may be stated in terms of marginal substitution rates if we wish to avoid the use of marginal utilities.)

The problem now consists in expressing $\bar{D}_{t_0}$ in terms of variables controlled by the entrepreneur, and then performing the maximization of $u_{t_0}$ with regard to those variables.

* Cf. de Seitzovsky [23].
C-2. The approach of the preceding section is based on the postulate that the entrepreneur is simply an individual attempting to maximize his utility. This does not mean that the traits which distinguish him from other members of the community (e.g., workers, bankers) are ignored. He differs from them with regard to 1. initial assets (plant, market connections, patents, good will); 2. organizational and productive skill; 3. character of expectations as to market phenomena and technology; 4. structure of preferences (this is particularly important when risk and uncertainty are introduced).

These are the factors making it possible and desirable for him to become (or remain) an entrepreneur; these same factors also explain why firms behave differently under similar circumstances.

The fact that the emphasis is placed on the cash withdrawals $\overline{D}_t$ makes the model more realistic for, say, a modern manufacturing firm than for a small farmer.

C-3. Our model, because it is based on utility maximization, has greater generality than the more usual one which postulates that the entrepreneur maximizes the sum of discounted expected profits with a discount factor determined by the market rate of interest.

The need for explicit introduction of utility maximization as the appropriate criterion becomes especially apparent when risk and uncertainty are introduced into the picture, since the risk-preference pattern obviously determines the choice. No wonder, therefore, that those who have studied the cases of stochastic expectations were led to the utility-maximization principle. However, even in the absence of risk or uncertainty we are forced to start with utility considerations, once it becomes clear that the market rates of interest should not be used to determine the discount factors. For, as has been indicated by Mosak, such treatment is permissible only in evaluating the anticipated profits at the point of equilibrium (where in general the marginal rates of substitution equal the corresponding price ratios), but not in deriving the location of this point.

10 Cf., e.g., Hicks [6], Tintner [24].
11 Mosak [20], Tintner [24], Hart [3].
12 Marschak [17].
13 [20], p. 145.
14 The discounting procedure is valid when suitable type of credit market for private borrowing exists. In such a case the nature of $\phi$ in (5) will not affect decisions with regard to $\overline{D}_t$. Similarly, in the stochastic case the "risk preference pattern" $[\phi$ in (51) below] may also be without influence on decisions with regard to $\overline{D}_t$ if suitable market exchange possibilities exist. In such a case market rates of exchange between various types of risk, etc., could be used to form a discounted value.

In both stochastic and nonstochastic cases the discount factors must be
D. The Variables

D-1. Classification of the variables. As has been stated above, Section C-1, the stream of prospective withdrawals $D_t$ depends on three types of factors:

1. those known to the entrepreneur at the time of decision-making $t_0$; it is these factors that form the basis for anticipations and decisions; they will be called initial conditions and denoted by $\eta$;

2. those which are being expected; they will be called predictands and denoted by $\omega$;

3. those over which the entrepreneur has control, subject, of course, to certain restrictions (transformation equations, etc.); they will be called decision variables and denoted by $\zeta$.

The remainder of Section D is devoted to a description of the more important components of $\eta$, $\omega$, and $\zeta$.

D-2. The initial conditions. The initial conditions are important for two reasons. First, they impose certain restrictions on the future. Debts incurred in the past must be repaid in the future, past inputs may partly determine future outputs, etc. Second, the initial conditions affect the expectations. Thus expected prices may be the extrapolated values of the past prices, and so forth. This suggests a classification of the initial-conditions variables into two groups:

1. those which describe the state of the firm and whose values impose constraints on the entrepreneur's decisions, to be called assets and denoted by $\eta'$;

2. those whose values serve as basis for the formation of expectations, to be called predictors, and denoted by $\eta''$.

These two groups overlap somewhat, especially under conditions of imperfect competition; for instance, inventories will be obviously among assets, but their size may, in an imperfect market, affect the entrepreneur's prediction of future market conditions.

D-2a. Assets. There are three kinds of assets: i. "real" assets, including $\alpha$ inventories, $\beta$. plant, and $\gamma$. semi-finished products; ii. "financial" assets, including $\alpha$. cash, $\beta$. securities, and $\gamma$. debts; and iii. "intangible" assets.

i-α. Inventories. Stocks of the firm's $\mu$th product held at time $t$ will be denoted by $k_{t\mu}$; we write

\[ K_{t\mu} = \{ k_{t\mu} \} = \{ k_{t\mu}^1, \ldots, k_{t\mu}^s, \ldots \} \]

entered at their expected values and, in imperfect markets, may vary from one entrepreneur to another even if their "prediction equations" (cf. E below) are identical.

15 Cf. Hawtrey [5].
where \( \mu \) runs over all products. Factors of production other than equipment are treated as products with negative prices.

i-\( \beta \). Plant. The quantity of \( \sigma \)th type of equipment, installed during \( (t', t'+1) \) and still in use at \( t'' \) will be denoted by \( \sigma_{t', t''} \); we define investment \( v_{t', t''} \) as

\[
v_{t', t''} = z_{t', t''} - z_{t', t''} = \Delta z_{t', t''},
\]

and also write

\[
Z_{t_0} = \| z_{t, t_0} \|
\]

where \( \sigma \) runs over all types of equipment and \( t < t_0 \).

i-\( \gamma \). Semi-finished Products. These may be expressed in terms of past inputs and outputs; output of the \( \mu \)th product during \( (t, t+1) \) is denoted by \( x_{t, t'} \). We write

\[
X_{t_0} = ||x_{t, t'}|| = [ x_{t_0-1, t_0}, x_{t_0-2, t_0}, \ldots, x_{t_0-\sigma, t_0} ;
\]

\[
x_{t_0-1}, x_{t_0-2}, \ldots, x_{t_0-\sigma}, \ldots ]
\]

ii-\( \alpha \). Cash. Cash (including bank deposits) held at \( t \) is denoted by \( m_t \).

ii-\( \beta \). Securities. For simplicity's sake we consider only consols; other types may easily be introduced; \( s_{t', t'} \) denotes the market value, \( s_{t', t'}^{*} \) the par value of securities purchased during \( (t', t'+1) \) and still held at time \( t'' \); forward dealings are excluded; we write

\[
\left\{ \begin{array}{l}
s_{t', t''} - s_{t', t''}^{*} = \Delta s_{t', t''}^{*} \\
\sum_{t''} \Delta s_{t', t''}^{*} = \Delta s_{t', t'}^{*}
\end{array} \right. \ (t' \leq t'').
\]

The rate of interest payable during \( (t'', t''+1) \) on \( s_{t', t''+1}^{**} \) is denoted by \( r_{t', t''}^{**} \) and the prospective returns counted among the firm's assets.

We write

\[
S_{t_0} = \| s_{t_0-1, t_0}^{*} s_{t_0-2, t_0}^{*} \ldots \| s_{t_0-1, t_0}^{**} s_{t_0-2, t_0}^{**} \ldots \|
\]

to represent both the resale value and the interest aspects of the securities held at time \( t_0 \).

ii-\( \gamma \). Debts owed by the firm. The amount borrowed by the firm during \( (t', t'+1) \) to be repaid during \( (t'', t''+1) \) will be denoted by \( e_{t', t''}^{+} \).

The amount repaid during \( (t'', t''+1) \) on a loan incurred during \( (t', t'+1) \) will be denoted by \( e_{t', t''}^{-} \); hence (if the firm is assumed to pay its obligations on time) we have

\[
e_{t', t''}^{+} = e_{t', t''}^{-}.
\]
The symbol \( e_{t'', t'''} \) will be used for the proceeds (positive or negative) from borrowing (or repayment of debt) thus

\[
\begin{align*}
    e_{t', t''} &= e_{t', t''}^+ & \text{for } t' < t'', \\
    e_{t', t''} &= -e_{t', t''}^- & \text{for } t' > t''.
\end{align*}
\]

The total amount of debt outstanding at \( t \), no matter when incurred, and payable during \((t'', t'''+1)\) will be denoted by \( b_{t', t''} \); we have

\[
\begin{align*}
    b_{t', t''} &= \sum e_{t', t''}^+ & (t' < t < t''), \\
    b_t &= \sum b_{t', t''} & (t < t'''), \\
    \Delta b_t &= b_{t+1} - b_t.
\end{align*}
\]

The rate of interest payable during \((t''', t''''+1)\) on a loan \( c_{t', t''''} \) \((t' < t'''' < t''')\) will be denoted by \( r_{t', t''', t''''} \) and the prospective interest payments counted among the (negative) assets. We write

\[
B_{t_0} = \{b_{t_0, t'}\} \quad (t' > t_0)
\]

to represent the firm’s commitments with regard to both the principal and interest charges on debts outstanding at \( t \).

iii. "Intangible" assets. In addition to the above and other tangible assets, there exist some very important ones which cannot be represented in terms of quantities of goods or written documents: the firm’s connections and clientele, its reputation for honesty and fairness, its past price and labor policies, etc. Those intangible assets affect other people’s attitude toward the firm and hence influence its earning capacity.

When the entrepreneur’s behavior is such as seemingly to contradict the utility- (or profit-) maximization postulate, the economist will do well to examine the phenomenon from the viewpoint of the intangible assets; it may well be that the entrepreneur’s “irrationality” is actually due to his (highly rational) concern for the long-run standing of his firm; to this end he may be willing to sacrifice more immediate monetary advantages.\(^{17}\)

We may now define the array

\[
\eta_{t_0} = \{K_{t_0}, Z_{t_0}, X_{t_0}, m_{t_0}, S_{t_0}, B_{t_0}, I_{t_0}, \ldots\}
\]

where \( I_{t_0} \) represents the intangible assets at \( t_0 \). The listings (\( \ldots \)) in (10) indicate that not all conceivable types of assets have been listed explicitly.

D-2b. Predictors. These are any variables that affect the entrepre-

\(^{17}\) Thus, for instance, Lutz’s remarks [15], pp. 813–814, imply such considerations rather than lack of profit-maximization motive.
neur's expectations. There are two important groups among them: the past values of the predictand variables (e.g., past prices of firm's product, wages) and "barometer" type variables which presage developments for the firm's environment. How these predictors are used in the formation of expectations is indicated in Section E.

D-3. Predictands and decision variables. There is a certain amount of arbitrariness in deciding which variables are to be regarded as predictands and which ones as decision variables.¹⁸ Consider, for instance, the firm's expected revenue $py$ (where $p$ is the price and $y$ the quantity sold) from the sale of its (say single) product. It would not be correct to regard the expected revenue as a predictand since it depends on the firm's action; but it is not a pure decision variable either, for it partly depends on demand conditions that the entrepreneur can only expect but not control. Thus it is natural and customary to split $py$ into its two factors $p$ and $y$ and regard one of them as a predictand, the other as a decision variable. Under conditions of perfect competition we would be justified in regarding $p$ as the predictand since the individual firm has no control over it; $y$ would then be the decision variable.

But under imperfect competition the firm is free to fix either $p$ or $y$, though of course not both. The language often used (references to "pricing policies") implies that prices are commonly thought of as the decision variables while quantity sold becomes the predictand.

In this paper we have adopted the principle of regarding the expectations as applying to market variables (prices, wages, interest rates, etc.) while the quantities are regarded as decision variables.

In the case of perfect competition, therefore, prices are the predictands; in imperfect markets it is the parameters of the (imagined) demand and supply curves (e.g., their elasticities) that are the predictands. This approach makes it possible to treat simultaneously the case of perfect and imperfect competition; this could not be accomplished by making prices into decision variables. Our approach also has the advantage of yielding the solutions for quantity variables as the unknowns in terms of anticipated prices, wages, etc. Since the volume of investment is our unknown it is convenient to have it among decision variables.

However, it is clear that a more symmetrical method of treatment may in some cases be desirable. This can be accomplished by a transformation from the space of, say, $p$ and $y$ (in the above example) to another space, say, of $\alpha$ and $\sigma$ in the following manner.

¹⁸ This arbitrariness is due to the fact that, as will be seen later, the correct distinction applies to events or actions rather than variables; events or actions independent of the entrepreneur are the predictands, those subject to his control are decision "actions."
Let the expected ("imagined") demand function be written in the symmetrical form

\[ g(p, y; \alpha) = 0, \]

where \( g \) is a given function and \( \alpha \) its parameter entered at its expected value.

Let the decision of the entrepreneur with regard to his future selling policy be formulated in an equally symmetrical form as the "policy equation"

\[ h(p, y; \sigma) = 0, \]

where \( h \) is a known function and \( \sigma \) the parameter to be decided upon by the entrepreneur in such a way as to maximize (say) the revenue \( py \).

Then clearly the last two equations can be solved for \( p \) and \( y \) in terms of \( \alpha \) and \( \sigma \) so that \( py \) becomes a function of these two parameters; in order to maximize an expression involving \( py \) we shall now differentiate it with regard to \( \sigma \), keeping \( \alpha \) constant. Thus, effectively, \( \alpha \) has become the predictand and \( \sigma \) is the decision variable.\(^{19}\)

The artificiality of the above treatment consists in regarding the functions \( g \) and \( h \) as given and only their parameters as expected or being decided. More generally, let the expected demand function be written as

\[ \bar{g}(p, y) = 0 \]

and the "policy equation" as

\[ h(p, y) = 0, \]

where now the function \( h \) itself is regarded as the unknown to be decided upon, depending on the nature of \( \bar{g} \). If we now repeat the procedure followed in the parametric case, we shall have expressed \( py \) in terms of an unknown function \( h \) and a given function \( \bar{g} \), and can then carry out the maximization with regard to the unknown function \( h \), as in the calculus of variations.

\[ E. \text{ Formation of Expectations} \]

Just how expectations are actually formed is one of the most important questions and it can be answered by appeal to empirical evidence.

\(^{19}\) The approach adopted in this paper of regarding \( y \) (or any other quantity variables) as the decision variable is equivalent to assuming

\[ h(p, y; \sigma) = y - \sigma = 0, \]

which, of course, is a very special case.

Reference to "pricing policies" imply the opposite type of "policy equations" viz.,

\[ h(p, y; \sigma) = p - \sigma = 0. \]
But a model that would not indicate at least the general nature of the expectation process cannot be regarded as complete.20

In fact, most theoretical work is based on exceedingly simple theories of expectation formation. For instance, Hicks’s use of the elasticity of (say) price expectations,21

\[ \epsilon = \frac{p_{t+1}}{\bar{p}_t} \frac{d\bar{p}_t}{dp_{t+1}} \quad (t \geq t_0), \]

would seem to be based on a “forecast equation” of the type

\[ \bar{p}_t = \psi_t(p_{t+1}) \quad (t \geq t_0). \]

A special case of interest is obtained when (16) becomes

\[ \bar{p}_t = p_{t+1} \phi(t); \]

in this case the elasticity of expectations is constant in time. Specializing still further we get

\[ \bar{p}_t = p_{t+1} \gamma^{t-t_0+1}, \]

which implies

\[ \{ \begin{array}{l} \bar{p}_t = \gamma \bar{p}_{t-1} \quad (t = t_0 + 1, t_0 + 2, \ldots), \\ \bar{p}_{t_0} = \gamma p_{t_0-1}. \end{array} \]

The difference-equation form of the “forecast equation” is of interest in connection with the case of stochastic expectations where the exact difference equation (19) is replaced by a stochastic one.22

A “forecast equation” of type (16) has the great advantage of combining simplicity with relatively great wealth of possible types of solutions. But they can hardly be regarded as possessing an adequate degree of realism.

First, they treat the different components of the predictand as independent: thus the forecast of wage trends is separate from the forecast of price trends. Second, they are too restrictive with regard to the types of predictors used. Actually it seems that, apart from cases of direct knowledge, businessmen’s expectations of all the variables involved may often be expressed in terms of relatively few common “barometer” indices. For instance, certain types of governmental policies may be regarded as leading to a narrowing down of the ratio of prices to wages, while other policies may induce expectations of constancy of the average of these two variables. Then the structure of the expectations may be formalized as follows:

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21 Cf. Lange [13], p. 20, note 2.
22 Cf. (37) below.
\[ f_1(\tilde{p}_t, \tilde{w}_t; G_{t-1'}, G_{t-2''}) = 0, \]
\[ f_2(\tilde{p}_t, \tilde{w}_t; G_{t-1'}, G_{t-2''}) = 0, \]

where the \( G \)'s represent the two policy variables; in our example \( f_1 \) contains \( \tilde{p}/\tilde{w} \) and is independent of \( G'' \), while \( f_2 \) contains some weighted sum of \( \tilde{p} \) and \( \tilde{w} \) and is independent of \( G' \).

It should be noted that while (20) happens to have as many predictors as it has predictands, this need not in general be the case. On the one hand, the two policy variables might have been functionally related, so that effectively there would have been only one predictor; on the other hand, any number of additional policy variables might have been introduced in (20).

In (20) governmental policy variables have been chosen as the "barometer" type predictors. But there are many other such predictors. One of them deserves mention, since it seems to be implicit in a number of models: it is the \textit{past profit rate} for the market as a whole, possibly with an appropriately chosen distributed lag.\(^{23}\) (It would be of interest to see to what extent the implications of regarding the profit rate as an important "barometer" type predictor are similar to the consequences of the assumption that the entrepreneur is maximizing profit \textit{rates} rather than profits, or, more generally, utility.)

The above treatment is still inadequate in one important respect: the expectations are, in general, of a \textit{conditional} nature. An extremely simple example of this has already been given in Section D-3. When the market for the product is imperfect and the price is regarded as the predictand, we may write

\[ \hat{p}_t = \hat{p}_t(y_t), \]

where the parenthesis, in addition to the decision variable \( y_t \), may also include various predictors. Here, properly speaking, the entrepreneur is expecting not the price itself, but rather the shape of the demand function. Similarly the expected rate of interest may, \textit{inter alia}, depend on the degrees of commitment and also on the amount to be borrowed at any given time. Thus we have a case where the firm's assets assume the role of predictors. (Cf. Section D-2.)

This situation may be expressed by

\[ \hat{\omega}_t = \hat{\omega}_t(\xi_t, \eta_t'), \]

where the function \( \hat{\omega}_t \) is the true object of expectations. \( \xi \) in (22) must satisfy certain constraints (cf. F-2).

\textbf{F. Formation of Decisions}

\textit{F-1. "Budget equation" and "surplus."} We shall now relate the en-

\(^{23}\) Marschak, in an unpublished note.
entrepreneurial withdrawals $d_i$ to changes in the firm’s financial assets and its "surplus." This is accomplished through the budget (balance) equation

\[ \Delta m_i = \pi_i + \Delta b_i - \Delta s_i - d_i, \]

where $\Delta s_i$ is the current value of the securities purchased (or sold), and $\pi_i$ is the surplus, defined below in equation (25).

Equation (23) may now be rewritten

\[ d_i = \pi_i + \Delta b_i - \Delta s_i - \Delta m_i, \]

since it is the $d_i$’s that enter the expression that is being maximized. The surplus $\pi_i$ is given by

\[ \pi_i = \sum_s p_s y_s - \sum_s \sum_{t'} g_{s,t'} e_{s,t'} - \sum_{u} g_u k_{u+1} u - l_i^{(s)} + l_i^{(e)}, \]

where

\[ l_i^{(s)} = \sum_{t', t''} r_{t', t''} l_{t', t''} e_{t', t''} + \dot{\pi}_i e_{t', t''} \quad (t' < t < t''), \]

\[ l_i^{(e)} = \sum_{t'} r_{t', t''} s_{t', t''} e_{t', t''} - \dot{\pi}_i \Delta s_i \quad (t' < t). \]

Taxes, as noted before, are excluded from consideration. The market variables ($p, q, g, r', r''$) are assumed to be determined in imperfect markets. Thus $p$ may be a function, say $p(y)$, of the volume of sales. In the case of advertising we have the quantity of advertising $z^a$ among the negative inputs, so that an additional term $p^a x^a$ appears in $\pi$ and the demand function for the firm’s $n$th product becomes say $p(y', z^a)$.\textsuperscript{26}

Some of the imperfections may be of a frictional character, i.e., dependent on the rate of change of the relevant variables.\textsuperscript{27} For instance we may find that the (per unit) cost of storage $g$ depends not only on the quantity $k$ stored, but also on the rate of change $\Delta k$ of the stocks.

It is important to note that the values of the market variables that enter the entrepreneur’s calculations are the expected ones; similarly the demand curves [e.g., $p(y)$] are those imagined (i.e., expected) by the entrepreneur. They may or may not be near the true ones.

\textsuperscript{24} $y$ is quantity of product sold, $p$ its price; $g$ is the price of equipment, $g$ unit storage cost; the other symbols have already been defined.

\textsuperscript{25} $i$ are brokerage charges; they are introduced to justify holding of cash under nonstochastic expectations.

\textsuperscript{26} These imperfections may be of a discriminatory nature (objectively or subjectively) so that two otherwise identical firms may be facing entirely different markets.

\textsuperscript{27} In the economic literature the term "friction" is often applied to market imperfections or to the dynamic nature of economic phenomena, i.e., absence of instantaneous adjustments. In this paper the term will be used only as defined in the text.
When imperfections are present, they may be among the important factors in limiting the firm's scale of operations or its size. In particular, the imperfections of the credit market have attracted a good deal of attention. Thus

\[ r_{v',v''} = r_{v',v''}(\delta_r), \quad \frac{\partial r'}{\partial \delta} > 0, \]

where

\[ \delta = \frac{b}{m + s + \sum q_n} \]

is the "degree of commitment." [Equation (28) is merely an example of a possible definition of \( \delta \).]

The market for securities is thought of as perfect, but this assumption is in no way essential.

The expected withdrawals are given by a system of equations exactly equivalent to the above except for the expectation symbol - placed over all the predictands. Thus in (25) we write \( \bar{p}, \bar{w} \), instead of \( p, w \), etc.

**F-2. Constraints.** In choosing the optimal values of decision variables the entrepreneur is limited by several types of constraints.

(1) **Stock-flow identities.** The most obvious ones are those relating the stock variables to the flow variables. For instance the inventories are related to output and volume of sales by the identity

\[ k_{t+1} = x_t - y_t + k_t. \]

Thus either \( x \) or \( y \) could be completely eliminated, but the use of Lagrange multipliers makes this unnecessary and permits a more symmetrical treatment of all the variables involved.

(2) **Effects of financial operations.** It will be assumed that debts incurred necessitate repayment so that [cf. (7)]

\[ e_{v',v''} = e_{v',v''}. \]

Also the interest charges at any time [given by (26)] depend on the volume of past borrowing. Similarly purchase of securities implies revenue in the future, their sale a corresponding loss.

The above two types of constraints are of a somewhat trivial nature; although only the first one is implied by the definitions of the variables used, neither one involves any unknown relationships or parameters.

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* Kalecki [9].

* In the case of stochastic expectations fear of bankruptcy (i.e., the entrepreneur's utility function) will also discourage the firm from reaching too high a degree of commitment, but in the nonstochastic case "increasing risk" is a market phenomenon due to the lender's fears and his desire for protection in case of insolvency.
In the third and most important type of constraint, this is not the case.

(3) Transformation functions. A system of transformation functions may be written as

\[(31) \quad P^{(j)}(X_t, Z_t; E_t) = 0, \quad t = \ldots, t_0 - 1, t_0, t_0 + 1, \]
\[j = 1, 2, \ldots, J,\]

(The significance of \(E_t\) will be explained below.) \(X_t, N_t,\) and \(Z_t\) are matrices (or vectors) covering a specified span of time, thus implying a great variety of lag relationships among factors, among products, and between the two. The reason for there being \(J\) such relationships is that, for instance, there may be fixed technical relationships ("fixed coefficients," etc.) between, say, two factors.

It is important to see that even if \(J = 1\), we are still left with a system of constraints and not a single one. This is indicated by the domain of variability of the subscript \(t.\)

It is worth noting that \(Z_t\) does not become a scalar even if there is only one type of equipment, since equipment is differentiated according to the time of installation. This is of importance, because at the time of installation the entrepreneur has a considerable amount of freedom with regard to the most efficient arrangement of equipment within the plant; therefore, even apart from wear and tear, newly installed equipment may play a different role in the plant's output from that of equipment installed previously.

The variable \(E_t\) represents the various factors that influence technology: inventions, social and cultural phenomena. In the first approximation it may be represented as a trend function.

Since the transformation function that is of relevance is the one in the entrepreneur's mind (rather than the true one), its parameters are (like the parameters of the imagined demand functions) in the nature of expected variables. Ordinarily, we would assume that the entrepreneur knows his production function well enough so that there would be little justification for distinguishing between its subjective and objective forms.

A simple example may be helpful at this point. Let \(J = 1\) and the transformation function be

\[(31') \quad x_t^1 = f(x_{t-1}^1, x_{t-2}^1), \quad t = t_0, t_0 + 1, \ldots\]

This implies that (1) the entrepreneur is no longer free to choose \(x_t^1\) since \(x_t^1\) is predetermined by \(x_{t-1}^1\) and \(x_{t-2}^1\); (2) while \(x_{t+1}^1\) is not absolutely fixed, it is only partly free since it depends on \(x_{t-1}^1\); and, (3) if all the future \(x^1\)'s were chosen there would be no more choice with regard to the \(x^1\)'s.

A common error seems to be due to the use of only one of the values of \(t\) in systems like (31) or (31'). Hicks [5] for instance, has only a single Lagrange multiplier in his appendix, Section 23, instead of a whole set of them. Tinbergen may have had this in mind in regarding certain initial values as known in advance [24], p. 307.
jective version. But this is not always the case. With regard to untried innovations there is a great deal of difference between the two: it takes a certain amount of imagination to visualize possibilities of new productive processes, etc. The role played by these considerations in Schumpeter's model is well known.31

(4) Restrictions on the frequency and timing of withdrawals and decision-making. These are dictated partly by custom, partly by considerations of economy. As noted before frequency of decisions need not be the same for all variables.

(5) Inequalities. Assets are so defined that they cannot assume negative values. In order to take this into account we may proceed as follows:32

First find the optimal point disregarding the inequalities. In this manner the absolute maximum of utility is located. If this maximum is in a region where all assets have positive values, the optimum point is the correct one, since we can do no better than the absolute maximum. Suppose, however, that the optimal point is found in a region where some of the assets, say cash and securities, are negative. In that case the optimal level for these two assets will be zero; these zero values should be substituted in the function to be maximized and the maximization carried out with regard to other variables in the usual manner.33

F-3. Summary. The results so far obtained may now be summarized. The starting point is the requirement that utility, as a function of the prospective stream of withdrawals, be maximized subject to existing constraints:

(32) \[ u_{t_0} = \phi_{t_0}(\bar{D}_{t_0}) = \text{maximum}, \]

subject to

(33) \[
\begin{align*}
\{ \mathcal{F}_{t_0}^{(i)}(\eta_{t_0}, \xi_{t_0}) = 0, & \quad i = 1, 2, 3, 4, \\
\{ \mathcal{F}_{t_0}^{(i)}(\eta_{t_0}, \xi_{t_0}) \geq 0. & \quad i = 1, 2, 3, 4,
\end{align*}
\]

The relation (33) represents all the existing constraints and by its form indicates the variables entering the constraints. (The superscript of \( \mathcal{F} \) refers to the classification of constraints in F-2.)

The second stage consists in expressing the prospective stream of withdrawals in terms of assets, decision variables, and predictands, and,

31 Cf. Schumpeter, [22]; also Lange [13], pp. 71 ff. It is worth noting that similar type of vision may be required to estimate correctly the market conditions for various products and factors.


33 When secondary maxima exist, complications may arise; cf. footnote 35 below.
in turn, the predictands in terms of the decision variables and the predictors.

Thus we have

\[ D_t = D_t(\omega_t, \xi_t, \eta_t'), \]

and

\[ \omega_t = \omega_t(\xi_t, \eta_t''). \]

Finally, substituting (34) and (35) into (32) we have

\[ u_t = \psi_t(\xi_t, \eta_t) = \text{maximum subject to (33)}. \]

\( \psi \) in (5) depends on \( \phi \) as well as on \( \omega \). Thus the outcome of maximization, which is, of course, carried out with regard to \( \xi_t \) (with \( \eta_t \) as given), depends on the preference pattern, the structure of expectations, and the nature of the transformation functions.

**F.4. Maximization.** The process of maximization is straightforward; cf. Lange [12]. Its results may be written implicitly as

\[ \lambda_t(\hat{\xi}_t, \eta_t) = 0, \]

which is a system of equations each of which indicates maximization with regard to one particular decision variable.\(^9\) When the system is solved for the decision variables we obtain

\[ \hat{\xi}_t = \rho_t(\eta_t) \]

thus giving the optimal value \( \hat{\xi}_t \) of \( \xi_t \) as a function of the initial conditions \( \eta_t \). The nature of the system of functional relationships \( \rho \) depends, as implied by the remarks in the preceding section, on \( \phi \), \( \omega \), and \( \xi \).

Now \( \hat{\xi}_t \) consists of two types of values of decision variables. There are those referring to the period (more or less) immediately following \( t_0 \) which will no longer be revised; these form the basis for the entrepreneur’s final decisions and will be denoted by \( \hat{\xi}_t^{(0)} \). The remaining components of \( \hat{\xi}_t \), to be denoted by \( \hat{\xi}_t^{(1)} \), are in the nature of tentative decisions\(^8\) and are of a purely auxiliary nature. We thus obtain, as a subset of (38), the system

\[ \hat{\xi}_t^{(0)} = \rho_t^{(0)}(\eta_t), \]

\( \text{The Lagrange multipliers are not shown explicitly. They can always be eliminated with the help of the constraints.} \)
\( \text{Provided appropriate substitutions of the type } u = v \text{ are made for non-negative variables and absolute value terms, (37) is simply } \frac{\partial \psi_t}{\partial \xi_t} = 0. \) Cf. [16]. However, the existence of indistinguishabilities will modify the nature of solutions. They may, for instance, reduce to zero some variables (e.g., investment) that would otherwise be nonzero.
\( \text{Cf. Hart [4].} \)
which may be rewritten in the implicit form as

\[(40)\]

\[x_t^{(3)}(\hat{s}_t^{(0)}; \eta_t) = 0.\]

The latter is obtained as follows. Rewrite (37) as

\[(41.1)\]

\[x_t^{*}(\hat{s}_t^{(0)}, \hat{s}_t^{(1)}; \eta_t) = 0,\]

\[(41.2)\]

\[x_t^{**}(\hat{s}_t^{(0)}, \hat{s}_t^{(1)}; \eta_t) = 0,\]

where the equations in (41.1) perform maximization with regard to \(\hat{s}_t^{(0)}\) while (41.2) do it with regard to \(\hat{s}_t^{(0)}\). Then solve (41.2) for \(\hat{s}_t^{(0)}\) in terms of \(\hat{s}_t^{(0)}\) and \(\eta_t\) and obtain, say,

\[(42)\]

\[\hat{s}_t^{(1)} = \hat{s}_t^{(1)}(\hat{s}_t^{(0)}; \eta_t).\]

Then, substituting this result in (41.1), we get

\[(43)\]

\[x_t^{*}(\hat{s}_t^{(0)}, \hat{s}_t^{(1)}(\hat{s}_t^{(0)}; \eta_t); \eta_t) = 0,\]

which may be simplified into (40). The equations of which (40) consists formulate the laws of the entrepreneur's optimal behavior. One of these equations, for instance, carries out maximization with regard to, say, factor \(x_t^{*}\) and is what would usually be referred to as the demand function of the entrepreneur for \(x_t^{*}\) to be used at the time \(t\), provided decision at time \(t\) with regard to \(x_t^{*}\) is final. (In practice, these equations are written after the constraints have been used to eliminate the Lagrange multipliers. This procedure introduces a certain amount of arbitrariness as well as lack of symmetry.)

Thus the firm’s demand (or supply) functions for equipment, labor, other factors (e.g., raw materials, advertising space, etc.), inventories, products, cash, loans of various duration, and securities are obtained.

**G. Aggregation and Market Phenomena**

We shall now indicate, in a very formal manner and only for the sake of logical completeness, how the results of the preceding section are to be incorporated into macrodynamic economic models.

As a starting point, we shall take equation (39). We shall write it, however, in a somewhat modified form:

\[(44)\]

\[\hat{s}_t^{(0)}E_{t} = \rho_t^{(0)}E_{t} + \theta_t^{E_{t}}(\eta_tE_{t}).\]

Here the superscript \(E\) stands for the entrepreneurial group as a whole,\(^{37}\) while \(\epsilon\) denotes the individual entrepreneur. The demand (or supply)

\(^{37}\) \(E\) here is not related to the variable \(E_t\) in (26).
functions (1) of the individual entrepreneurs may now be aggregated\textsuperscript{38} for all entrepreneurs thus giving

\begin{equation}
\hat{x}_{t_0}^{(0)*E} = \sum_{s} \hat{\rho}_{t_0}^{(0)*E,s}(\eta_{t_0}^{s,E}) = \hat{\rho}_{t_0}^{(0)*E}(\eta_{t_0}^{E}),
\end{equation}

where

\begin{align}
\rho^{E} & = \{\rho^{s,s}\}, \\
\eta^{E} & = \{\eta^{s,s}\}.
\end{align}

Now suppose similar models have been developed for groups other than entrepreneurs, say workers, bankers, etc. We then obtain a system of equations like (45), one for each group:

\begin{equation}
\hat{x}_{t_0}^{(0)*G} = \hat{\rho}_{t_0}^{(0)*G}(\eta_{t_0}^{G}),
\end{equation}

where \( G \) runs over all groups. We then perform aggregation over groups thus obtaining

\begin{equation}
\hat{x}_{t_0}^{(0)} = \sum_{G} \hat{\rho}_{t_0}^{(0)*G}(\eta_{t_0}^{G}) = \hat{\rho}_{t_0}^{(0)}(\eta_{t_0}),
\end{equation}

where

\begin{align}
\rho & = \{\rho^{G}\}, \\
\eta & = \{\eta^{G}\}.
\end{align}

Now it would be desirable to throw (48) into the implicit form and also to scalarize\textsuperscript{39} the variables it contains. But the theory of such a process still remains to be worked out.\textsuperscript{40,41}

We shall therefore leave (48) as the final result. It is in a form that does not lend itself readily to immediate applications. But, in principle, it does enable us to predict the developments when the initial conditions \( \eta_{t_0} \) and the nature of the function \( \hat{\rho}_{t_0}^{(0)} \) are known.

H. Special Problems

H-1. Motivation of investment. It may be useful to give a brief explicit description of the functioning of the model from the viewpoint of investment decisions. It should be noted that this is still the case of nonstochastic anticipations and many interesting aspects of the problem are thus left out of consideration.

\textsuperscript{38} Under conditions approaching full employment decisions of individual entrepreneurs may be inconsistent and cannot always be aggregated. It may then be helpful to regard prices rather than quantities as decision variables.

\textsuperscript{39} I.e., to represent vectors by means of scalar functions of their components; e.g., total national income is such a function of all individual incomes; price index is such a function of prices of individual goods; etc.

\textsuperscript{40} Cf. Hicks [6], Marschak [17], Mosak [20].

\textsuperscript{41} It is possible to obtain the implicit form when the distribution of firms according to various criteria is known. This is related to the concept of a "representative firm."
Suppose the entrepreneur considers the desirability of purchasing during \((t_0, t_0+1)\) an additional piece of new equipment. Assume that the production plan is otherwise left unchanged. Then this implies a diminution of \(\pi_t\) by \(q_t\pi_t\). On the other hand, through the transformation function, it also implies increases in some future outputs or decreases in some inputs. This may become apparent during \((t_0, t_0+1)\), or it may only become apparent at some later time. Thus there will be a tendency for \(\pi_t\) to drop in the near future with a compensating rise at some later time; however, this tendency may be partly offset by other operations, say inventory variations.

Let us, for the sake of definiteness, say that \(\pi_t\) drops while \(\pi_{t+1}\) goes up. Then one thing the entrepreneur may do is to let \(d_t\) drop and \(d_{t+1}\) go up in parallel with the movements of the surplus; on the other hand, he may prefer to avoid such fluctuations in the level of his withdrawals by means of compensatory financial operations. This he can do by incurring debts (i.e., financing investment by borrowing), sale of securities, or diminishing his stock of cash (loss of liquidity). Higher debts will impose interest burdens on future surpluses, sale of securities previously held will also lower future surpluses by eliminating a part of the interest revenue. Loss of liquidity also implies certain additional future expenditures, but in the case of nonstochastic expectations this is of little consequence.

It is easy to see what types of price expectations would make the entrepreneur favor lower inventories, what interest-rate expectations would make him go into debt, sell securities, etc.

Clearly the entrepreneur’s “impatience,” his private financial condition, opportunities, etc. will also affect the decision.

Now it may turn out that even the optimal method of financing the investment (coupled with the best possible adjustments in the product plan) will leave the entrepreneur worse off in terms of the utility of prospective withdrawals than he would be without making the investment. Then, of course, the decision will be against purchase; in fact under extreme conditions it may be in favor of selling some equipment already in his possession or even dismantling the whole plant.

On the other hand, if the investment expenditure \(q_t\pi_t\) is found advantageous (again in utility terms) it will be carried out, and the advisability of additional purchases of equipment may then be considered. It will be noted that a special case of this situation is the problem of entering the industry. This is simply the question of desirability of a certain volume of investment where the initial values of certain assets, e.g., the plant, are zero. The reasoning that applies is the same as before, but various market imperfections (borrowing difficulties, patent restrictions, discrimination on the part of suppliers of factors, attach-
ment of consumers to known brands, etc.) may be of particularly great importance.

H-2. Size and rate of growth of the firm. The above considerations implicitly contain a (very general) theory of the size of the firm. (Since the size of the firm is usually regarded as determined by the size of its plant, a theory of investment is at the same time a theory of the firm's size. But even if other assets are included in the measure of the firm's size, our initial statement still holds.)

Among the factors limiting the firm's size we may list the more important ones:

1. initial assets,
2. market imperfections,
3. nature of the transformation functions,
4. the entrepreneur's lack of vision with regard to existing possibilities (markets, technology),
5. the entrepreneur's utility function.

The factors limiting the rate of growth of the firm are also of interest. The frictional imperfections of the markets are among the most important ones; the nature of transformation functions may also be slowing down the firm's growth.

With regard to both the firm's size and to the rate of growth, some of the most powerful limiting factors may be due to existence of uncertainty.

H-3. Inventories. In the absence of the stochastic element in expectations it is impossible to account for the "transaction" inventories. But the speculative inventories are present. Hawtrey's trader is a special case of an entrepreneur whose production function is such that the optimum level of output (but not sales) is zero, and the materials purchased are physically identical with those sold.

H-4. Acceleration principle. This principle can be written, in a generalized form, as

$$\hat{v}_t \propto \Delta a_{t-1},$$

where $a$ is a shift parameter in the demand function for the firm's product. The foregoing theory indicates how the importance of factors neglected in (50) can be estimated.

II. THE CASE OF STOCHASTIC EXPECTATIONS

I. Preference Pattern, Expectations, Decisions

The discussion of this subject, stimulated by Knight's Risk, Uncer-

* Cf. Hawtrey [5]; also von Haberler [2], p. 18.
tainty, and Profit,\textsuperscript{43} has led to considerable progress in economic thought.\textsuperscript{44}

First, it has served to give a more realistic theory of the demand for cash, inventories, and the like.

Secondly, it has led to a realization that the entrepreneur is facing a problem different from that of a gambler or insurance company.

The common element in the various possible ways of approaching the problem is that the object of maximization (i.e., utility) depends on the probability distributions of the economic variables. In the terminology of this paper the predictands become stochastic variables and the probability-distribution functions determine for a given value of the initial values and decision variables the joint-multivariate-probability distribution of the elements of the prospective-withdrawals vector. The optimum choice of decision variable is the one yielding a distribution of the prospective withdrawals that is more to the entrepreneur's liking than any alternative distribution. Clearly the psychological make-up of the entrepreneur plays a decisive role.

It may be helpful to summarize this, retaining the earlier notation.

We have the entrepreneur's utility as a functional on the probability distribution $G_{\tilde{D}_t}$ of the prospective vector withdrawals

$$u_t = \phi_t \{ G_{\tilde{D}_t} \},$$

subject to appropriate constraints. In turn

$$\tilde{D}_t = \tilde{D}_t(\tilde{\omega}_t, \tilde{x}_t, \eta_t'),$$

and $\tilde{\omega}_t$, itself is stochastic with a distribution

$$K_{\tilde{\omega}_t}(\tilde{\omega}_t| \tilde{x}_t, \eta_t'').$$

Hence when proper transformations are carried out we find

$$G_t(\tilde{D}_t| \tilde{x}_t, \eta_t).$$

Moreover, there are again restrictions on $\tilde{x}_t$ as before. Some of these are of a nonstochastic character and create no problem, but the important point is that the transformation functions do become stochastic.

It appears convenient to adopt here a method that is less symmetrical than that used in the nonstochastic case: with the help of constraints (stochastic and nonstochastic) we eliminate some of the decision variables and also modify the distribution function in (54). Let the remaining free decision variables be denoted by $\xi_t$.

Then we have that $\tilde{D}_t$ is distributed according to

$$K_{\tilde{\omega}_t}(\tilde{D}_t| \xi_t, \eta_t).$$

\textsuperscript{43} [10].

\textsuperscript{44} Marschak [17]; Tintner [24], [25]; Hart [3], [4].
and hence we have from (46)

\[ u_t = \psi_t(x_t, \eta_t) , \]

which is a function, not a functional and, formally, the problem is no different from that obtained in the case of nonstochastic expectations.

**J. Risk, Uncertainty, Rational Expectations**

At this point reference must be made to the two aspects of the stochastic nature of the expectations, customarily associated in the economic literature with the terms risk and uncertainty.

Consider, as an example, the (true) demand function for the firm's product\(^{45}\)

\[ p_t = \alpha y_t + \alpha_2 p_{t-1} + \alpha_0 + u_t , \]

where \( u_t \) is a stochastic variable with a mean zero and variance \( \sigma^2 \). Here the variance of \( u \) is a measure of variability in the buyers' behavior. Even if the entrepreneur knew the true values of the \( \alpha \)'s and of \( \sigma^2 \), he still would not be able to predict the exact future price \( p_t \) corresponding to some given \( y_t \) and \( p_{t-1} \), but he could determine the probability distribution of \( p_t \). The dispersion of the distribution of \( D_{\alpha} \) (due to the dispersion of \( p_t \)) may be called the entrepreneur's risk (this definition is somewhat narrow, but sufficient for our purpose).

Actually, however, the entrepreneur realizes that he does not know the true values of the \( \alpha \)'s and of \( \sigma^2 \). This lack of knowledge is referred to as uncertainty and it can be treated in two ways.

One approach, that of Tintner, is to postulate that in the entrepreneur's mind there exist some a priori known distributions of the \( \alpha \)'s and of \( \sigma^2 \); these parameters are now thought of as stochastic variables.

Another approach, that of Marschak, is to assume that the entrepreneur will estimate the \( \alpha \)'s and \( \sigma^2 \) by some (optimal) statistical methods; this, of course, does not imply the neglect of any a priori information the entrepreneur may possess, but it does not require that such information should be available. (Actually, Marschak treats estimation as an integral part of the decision-making rather than separate the two.)

Since in practice we are not always justified in assuming a priori distributions, it is obvious that procedures of statistical inference should be introduced whenever possible. But the question arises whether they will always be sufficient.

This problem is equivalent to that investigated by Wald,\(^{46}\) viz., of the conditions under which optimal estimates exist ("optimal" in the

\(^{45}\) Cf. equations (19) and (21).

\(^{46}\) Wald [26].
sense of minimizing the maximum average loss⁴⁷). Since our model has not been shown to fulfill all of Wald’s assumptions, his theorem (stating that in certain circumstances optimal estimates can be found) need not follow. This question deserves more study, but we shall confine ourselves here to a general description of certain important cases.

Case A. The entrepreneur expects the structure of the economy to remain what it had been during the period prior to the time of decision-making. This implies, inter alia, that the demand curve for his product remains unchanged. It can be shown⁴⁸ that in this case ordinary least-squares procedures will give the entrepreneur the conditional distribution of \( p_t \) given \( y_t \) and \( p_{t-1} \) which is all that is needed; it is then not necessary to estimate the “structural coefficients” \( \alpha \) and \( \sigma \).

Case B. The entrepreneur does expect some structural changes, say a change in \( \alpha_1 \). The new value of \( \alpha_1 \) (say \( \alpha_1' \)) is a function of the old one. This means that either \( \alpha_1' \) is given directly or, at least it can be found if the old value \( \alpha_1 \) is known. Now it is here that the ordinary regression methods will usually fail and there is need for the “simultaneous equation” approach. It is then, in general, necessary to know the structural parameters⁴⁹. Unfortunately, however, the structural parameters cannot always be found. We may now distinguish two cases.

Case B1. Parameters to be estimated are identifiable, i.e., it is possible to estimate them. Then it is still possible for the entrepreneur to find a basis for his decisions, provided he uses the correct estimation procedures.⁵⁰

Case B2. Parameters to be estimated are not (or, at least, not completely) identifiable. This means that no matter how large a sample of past observations the entrepreneur has, he will not be able completely to determine the probability distribution of future prices.

It might then so happen that any decision on the part of the entrepreneur would imply the possibility of an infinite average loss. When this is the case, maximum average loss cannot be minimized and there is no rational basis for decision.

But this lack of identifiability of the parameters of the demand function will not always have such extreme consequences. The minimization of maximum average loss may lead to a policy of following certain routine procedures, or else the entrepreneur may not realize how deep his ignorance is. This may lead him to wrong decisions, but at least it does not paralyze him, as the knowledge of his ignorance would. Other-

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⁴⁷ Wald uses the term “risk” instead of “average loss”; the use of the latter term was suggested by Professor Marschak to avoid confusion with “risk” in the sense defined above.
⁴⁸ Hurwicz [7].
⁴⁹ Marschak [10].
wise, we must again resort to Tintner's approach if we believe that the entrepreneurs always do make some decisions.

K. Special Problems

There is need for explicit construction of preference functionals (or functions) and of the stochastic expectation equations. With these it would be possible to show how, for instance, "transaction demand" for cash or inventories is implied by the utility-maximization postulate. But the detailed study of these phenomena, especially of their impact on investment decisions is beyond the scope of this paper.⁸¹

It should be mentioned that in practice maximization is not performed accurately; that is, the values of decision variables chosen are not exactly those which would maximize \( u_t \); also, the entrepreneur's preferences (i.e., \( \phi \)) and ways of forming expectations (i.e., the function \( \phi \)) fluctuate in time. If these fluctuations are given functions of time they are covered by the foregoing treatment (since all the relevant functions have a time subscript). But when these fluctuations are of a stochastic nature, our equations acquire an additional component, the "disturbance." It is worth noting that the presence or absence of the disturbance is unrelated to the presence of a stochastic element in the expectations.

Cowles Commission for Research in Economics
The University of Chicago

REFERENCES


