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MACROECONOMICS AND THE THEORY OF RATIONAL BEHAVIOR[1]

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I. THE PROBLEM

MANY of the newly constructed mathematical models of economic systems, especially the business-cycle theories, are very loosely related to the behavior of individual households or firms which must form the basis of all theories of economic behavior. In these mathematical models, the demand equations for factors of production in the economy as a whole are derived from the assumption that entrepreneurs collectively attempt to maximize some aggregate profit; whereas the usually accepted assumption is that the individual firm attempts to maximize its own profit. For example Evans,[2] Keynes,[3] Hicks,[4] and Pigou[5] all have in their systems marginal-productivity (i.e., profit-maximizing) equations for the total economy or for some very large subsections such as the consumer-goods or producer-goods industries. These marginal-productivity equations are written, without justification, for the economy as a whole, in exactly the same form as the marginal-productivity equations for a single firm producing a single commodity. These aggregative theories have often been criticized on the grounds that they mislead us by taking attention away from basic individual behavior. The problem of bridging the gap between the traditional theories based on individual behavior and the theories based on community or class behavior is, to a large extent, a problem of proper measurement. This paper attempts to make a very modest contribution towards the formulation and solution of the problem.

We have a body of theory which develops the economic behavior of

[1] Cowles Commission Papers, New Series, No. 14. Part of the work on this paper was done under a fellowship of the Social Science Research Council. The author is indebted to other members of the Cowles Commission staff for constructive criticism.

[2] "Maximum Production Studied in a Simplified Economic System," ECONOMETRICA, Vol. 2, January, 1934, pp. 37–50.

[3] *The General Theory of Employment, Interest and Money*, New York, Harcourt Brace, 1936.

[4] "Mr. Keynes and the 'Classics': A Suggested Interpretation," ECONOMETRICA, Vol. 5, April, 1937, pp. 147–159.

[5] *Employment and Equilibrium*, London, Macmillan, 1941.

individual households and firms. We also have many index numbers compiled according to definite formulas from individual observations. If we consider the index numbers as transformations of the variables that appear in the behavior equations of microeconomics, there possibly exists a definite set of relations among the index numbers which we may call our model of macroeconomics. But for most of the common index numbers, it is very difficult to determine whether a well-defined macrosystem follows from our theories of microeconomics. Consequently we may be forced to attempt to solve our problem in another way. Instead of assuming the theory of microeconomics and the index numbers, let us assume the theory of micro- and of macroeconomics, and then construct aggregates (usually in the form of index numbers) which are consistent with the two theories.

All too often, index-number theorists have devised arbitrary and even mutually inconsistent criteria which are imposed upon the construction of index numbers. We can well begin by setting down objective criteria of properly constructed economic aggregates which are consistent with the practices and aims of business-cycle theory. The general economic system is composed of equations relating to the behavior of households, firms, and interactions in the market between households and firms. We shall give detailed consideration in this paper only to those equations relating to the behavior of firms. Many of the propositions can be easily carried over to the equations of household behavior.

II. TWO CRITERIA FOR AGGREGATES

Our first criterion that an aggregate must satisfy is that *if there exist functional relations that connect output and input (production functions) for the individual firm, there should also exist functional relations that connect aggregate output and aggregate input for the economy as a whole or an appropriate subsection.* For example, we have for the firm, in microeconomics,

$$(1) \qquad F_\alpha(x_{1\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}) = 0, \qquad \alpha = 1, 2, \cdots, A.$$

This relation states that the αth firm produces the m commodities $\{x_{i\alpha}\}$ through the input of the services of r kinds of labor $\{n_{i\alpha}\}$ and of s kinds of capital $\{z_{i\alpha}\}$. We demand now that there exist a function, in macroeconomics,

$$(2) \qquad F(X, N, Z) = 0$$

which states that the entire community of firms produces the aggregate output X through the input of the services of labor N and of capital Z.

A second criterion that we shall impose upon our aggregates is the following: *If profits are maximized by the individual firms so that the marginal-productivity equations,*

$$
(3) \qquad \begin{aligned}
\frac{\partial x_{i\alpha}}{\partial n_{j\alpha}} &= \frac{w_j}{p_i}, &\quad & i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A, \\
\frac{\partial x_{i\alpha}}{\partial z_{j\alpha}} &= \frac{q_j}{p_i}, &\quad & i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A,
\end{aligned}
$$

hold under perfect competition, then the aggregative marginal-productivity equations,

$$
(4) \qquad \begin{aligned}
\frac{\partial X}{\partial N} &= \frac{W}{P}, \\
\frac{\partial X}{\partial Z} &= \frac{Q}{P},
\end{aligned}
$$

must also hold, where w_j = the wage of the jth type of labor, p_i = the price of the ith commodity, q_j = the price of the jth type of capital service, W = the wage aggregate, P = the output-price aggregate and Q = the capital-service-price aggregate.

Obviously the second criterion cannot be satisfied without the first.

These criteria imply that we derive our macrosystem of N commodities and M factors as though we were writing down the equations for a hypothetical microsystem of N commodities and M factors. Particular interest is attached to the case where $N = 1$ and $M = 2$.

A theory based on the second criterion alone has been studied extensively by Dresch[6] and has also been treated by Hicks[7] and Lange.[8] Hicks has shown that the "fundamental equation of value theory" (Slutsky equation) remains formally invariant if we lump together (treat as one good) any group of goods whose prices change all in the same proportion. This is clearly a sufficient condition for the solution of the aggregation problem, but it may not be the most satisfactory condition to impose because most prices do not change in the same proportion.

[6] "Index Numbers and the General Economic Equilibrium," *Bulletin of the American Mathematical Society*, Vol. 44, February, 1938, pp. 134–141.

[7] *Value and Capital*, Oxford, Clarendon Press, 1939, p. 312.

[8] *Price Flexibility and Employment*, Cowles Commission Monograph No. 8, Bloomington, Indiana, Principia Press, 1944, pp. 103–106.

III. THE ATTEMPT OF FRANCIS DRESCH

Dresch,[9] in a suggestive article, has attempted to show that all the necessary conditions for maximum profits in the case of firm behavior hold in analogy for the economy as a whole if the macrovariables are properly defined in terms of the microvariables. Dresch's properly defined variables are Divisia[10] index numbers in every case. We shall show below that Dresch's aggregates do not satisfy our criteria.

We can best discuss the Dresch theory in a simple case of competitive firms making one product each. Let the production function for the αth good produced by the αth firm be

$$x_\alpha = f_\alpha(n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}), \qquad \alpha = 1, 2, \cdots, A. \tag{5}$$

Profit maximization under perfect competition leads to the necessary conditions

$$\frac{\partial x_\alpha}{\partial n_{i\alpha}} = \frac{w_i}{p_\alpha}, \qquad \begin{array}{l} i = 1, 2, \cdots, r, \\ \alpha = 1, 2, \cdots, A, \end{array} \tag{6}$$

$$\frac{\partial x_\alpha}{\partial z_{i\alpha}} = \frac{q_i}{p_\alpha}, \qquad \begin{array}{l} i = 1, 2, \cdots, s, \\ \alpha = 1, 2, \cdots, A, \end{array} \tag{7}$$

where w_i is the wage rate paid to the ith type of labor, p_α is the price of the αth good and q_i is the cost of the services of the ith type of capital.

The Divisia index of total output, X, is defined by the differential equation

$$dX = \frac{X}{V_x} \sum_{\alpha=1}^{A} p_\alpha dx_\alpha; \qquad V_x \equiv \sum_{\alpha=1}^{A} p_\alpha x_\alpha. \tag{8}$$

But from the production function (5) we obtain

$$dx_\alpha = \sum_{i=1}^{r} \frac{\partial f_\alpha}{\partial n_{i\alpha}} dn_{i\alpha}, \tag{9}$$

if all $dz_{i\alpha} = 0$, i.e., if we consider variations in output when labor alone varies and capital services of all types are held constant. Hence on substitution of (9) into (8) we get

$$(dX)_N = \frac{X}{V_x} \sum_{\alpha=1}^{A} \sum_{i=1}^{r} p_\alpha \frac{\partial f_\alpha}{\partial n_{i\alpha}} dn_{i\alpha}, \tag{10}$$

where $(dX)_N$ is defined as the change in total output when labor alone varies. Similarly the Divisia definition of the labor index is obtained from the differential equation

[9] F. W. Dresch, *op. cit.*

[10] F. Divisia, *Economique Rationnelle*, Paris, Doin, 1928, pp. 265–280.

$$(11) \qquad dN = \frac{N}{V_N} \sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i dn_{i\alpha}; \qquad V_N \equiv \sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}.$$

The definition of marginal productivity is now taken to be the ratio $(dX)_N/dN$ or

$$(12) \qquad \frac{(dX)_N}{dN} = \frac{\dfrac{X}{V_X} \displaystyle\sum_{\alpha=1}^{A} \sum_{i=1}^{r} p_\alpha \frac{\partial f_\alpha}{\partial n_{i\alpha}} dn_{i\alpha}}{\dfrac{N}{V_N} \displaystyle\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i dn_{i\alpha}}.$$

But if we substitute the equilibrium conditions for profit maximization (6) we obtain

$$(13) \qquad \frac{(dX)_N}{dN} = \frac{\dfrac{X}{V_X} \displaystyle\sum_{\alpha=1}^{A} \sum_{i=1}^{r} p_\alpha \frac{w_i}{p_\alpha} dn_{i\alpha}}{\dfrac{N}{V_N} \displaystyle\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i dn_{i\alpha}} = \frac{\dfrac{V_N}{N}}{\dfrac{V_X}{X}}.$$

The ratio V_N/N represents the wage bill deflated by an employment index and can be called the average wage rate, an aggregate. Also V_X/X represents the value of output deflated by an output index, and can be called the price aggregate. Thus the proposition that the marginal productivity of labor equals the real wage rate in equilibrium holds in analogy for the macrosystem if the corresponding proposition holds for the microsystem. By a parallel procedure it follows that

$$(14) \qquad \frac{(dX)_Z}{dZ} = \frac{\dfrac{V_Z}{Z}}{\dfrac{V_X}{X}}.$$

It is also true that this technique can be extended to the theory of consumer behavior except for the fact that the aggregations can only be taken over groups of commodities and not over individuals because of the difficulties of interpersonal comparisons of utility.

What is the meaning of the ratio $(dX)_N/dN$? Can this ratio properly be defined as marginal productivity, $\partial X/\partial N$? If such a partial derivative is to have meaning, then there must exist a differentiable aggregate production function, from which we can derive the marginal productivity for the economy as a whole or for some subsection of the economy.

This means that our first criterion must be satisfied. Formally, if there exists a set of production functions referring to the individual firms,

$$x_\alpha = f_\alpha(n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}) \qquad \alpha = 1, 2, \cdots, A, \tag{5}$$

with well-defined partial derivatives

$$\frac{\partial f_\alpha}{\partial n_{i\alpha}}, \qquad \begin{matrix} i = 1, 2, \cdots, r, \\ \alpha = 1, 2, \cdots, A, \end{matrix}$$

$$\frac{\partial f_\alpha}{\partial z_{i\alpha}}, \qquad \begin{matrix} i = 1, 2, \cdots, s, \\ \alpha = 1, 2, \cdots, A, \end{matrix}$$

then the criterion requires that there must also exist a function

$$X = f^*(N, Z) \tag{15}$$

with well-defined partial derivatives

$$\frac{\partial f^*}{\partial N}, \qquad \frac{\partial f^*}{\partial Z}.$$

It is by no means evident that an acceptable production function measured in terms of Divisia indexes exists; furthermore it is not evident that, if such a production function does exist, it has a partial derivative equal to $(dX)_N/dN$ as calculated above.

A precise statement of the conditions under which an aggregate production function exists can be made with the help of some propositions from the theory of functional dependence.[11] Let us write individual production functions, for the most general case, as

$$F_\alpha(x_{1\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}) = 0, \qquad \alpha = 1, 2, \cdots, A. \tag{1}$$

If these production functions are sufficiently well-behaved, as is generally assumed, we can rewrite them as

$$x_{1\alpha} = f_\alpha(x_{2\alpha}, \cdots, x_{m\alpha}; n_{1\alpha}, \cdots, n_{r\alpha}; z_{1\alpha}, \cdots, z_{s\alpha}), \qquad \alpha = 1, 2, \cdots, A. \tag{16}$$

We shall now define our aggregates as

$$X = G(x_{11}, \cdots, x_{m1}, \cdots, x_{1A}, \cdots, x_{mA}), \tag{17}$$

$$N = H(n_{11}, \cdots, n_{r1}, \cdots, n_{1A}, \cdots, n_{rA}), \tag{18}$$

$$Z = I(z_{11}, \cdots, z_{s1}, \cdots, z_{1A}, \cdots, z_{sA}). \tag{19}$$

The definitions of the output, labor, and capital aggregates define three transformation functions sending the variables $\{x_{i\alpha}\}$, $\{n_{i\alpha}\}$, and $\{z_{i\alpha}\}$ into X, N, Z, subject to the restraints of the production functions.

[11] Leonid Hurwicz was very helpful in formulating the proposition to follow.

It is well known that the transformed variables are functionally related, uniquely, by a relation

$$\Phi(X, N, Z) = 0, \tag{20}$$

if the following rectangular matrix is of rank 2:

$$\left\| \begin{matrix} \left[\dfrac{\partial G}{\partial x_{1\alpha}} \dfrac{\partial x_{1\alpha}}{\partial x_{i\alpha}} + \dfrac{\partial G}{\partial x_{i\alpha}}\right] & 0 & 0 \\ \left[\dfrac{\partial G}{\partial x_{1\alpha}} \dfrac{\partial x_{1\alpha}}{\partial n_{i\alpha}}\right] & \left[\dfrac{\partial H}{\partial n_{i\alpha}}\right] & 0 \\ \left[\dfrac{\partial G}{\partial x_{1\alpha}} \dfrac{\partial x_{1\alpha}}{\partial z_{i\alpha}}\right] & 0 & \left[\dfrac{\partial I}{\partial z_{i\alpha}}\right] \end{matrix} \right\|.$$

Each of the elements of this matrix are column vectors, the vectors of the first row having $(m-1)A$ elements $(i=2, 3, \cdots, m; \alpha=1, 2, \cdots, A)$, the vectors of the second row having rA elements $(i=1, 2, \cdots, r; \alpha=1, 2, \cdots, A)$, and the vectors of the third row having sA elements $(i=1, 2, \cdots, s; \alpha=1, 2, \cdots, A)$.

The conditions that all third-order determinants vanish, *identically*, where $\partial H/\partial n_{i\alpha}$ and $\partial I/\partial z_{i\alpha}$ are not all zero, are

$$\frac{\partial x_{1\alpha}}{\partial x_{i\alpha}} \equiv -\frac{\dfrac{\partial G}{\partial x_{i\alpha}}}{\dfrac{\partial G}{\partial x_{1\alpha}}}, \qquad \begin{aligned} i &= 2, \cdots, m, \\ \alpha &= 1, 2, \cdots, A, \end{aligned} \tag{21}$$

$$\frac{\dfrac{\partial x_{1\alpha}}{\partial n_{i\alpha}}}{\dfrac{\partial x_{1\beta}}{\partial n_{j\beta}}} \equiv \frac{\dfrac{\partial H}{\partial n_{i\alpha}}}{\dfrac{\partial H}{\partial n_{j\beta}}} \cdot \frac{\dfrac{\partial G}{\partial x_{1\beta}}}{\dfrac{\partial G}{\partial x_{1\alpha}}}, \qquad \begin{aligned} i &= 1, 2, \cdots, r, \\ j &= 1, 2, \cdots, r, \\ \alpha &= 1, 2, \cdots, A, \\ \beta &= 1, 2, \cdots, A, \end{aligned} \tag{22}$$

$$\frac{\dfrac{\partial x_{1\alpha}}{\partial z_{i\alpha}}}{\dfrac{\partial x_{1\beta}}{\partial z_{j\beta}}} \equiv \frac{\dfrac{\partial I}{\partial z_{i\alpha}}}{\dfrac{\partial I}{\partial z_{j\beta}}} \cdot \frac{\dfrac{\partial G}{\partial x_{1\beta}}}{\dfrac{\partial G}{\partial x_{1\alpha}}}, \qquad \begin{aligned} i &= 1, 2, \cdots, s, \\ j &= 1, 2, \cdots, s, \\ \alpha &= 1, 2, \cdots, A, \\ \beta &= 1, 2, \cdots, A. \end{aligned} \tag{23}$$

The choice of the aggregative functions, G, H, and I must be such as to satisfy these identical relationships. The relationships state in a loose sense that marginal rates of substitution among variables of the aggregative functions must be the same as the marginal rates of substitution among the variables of the production function. It seems clear from these conditions that there must be some similarities in form between the basic production functions and the aggregative functions. It will be necessary to have some specifications, in any case, on the individual functions in order to know how to construct the aggregates so as to satisfy the theorem on functional dependence.

The conditions (21), (22), (23) give us an exact judgment as to the desirability of any particular type of aggregation. For example, there may be considered the special case in which the different types of output and of factors are of the same dimensionality. Then we may be led to believe that simple summation is the natural type of aggregation. We would have

$$X = \sum_{\alpha=1}^{A} \sum_{i=1}^{m} x_{i\alpha}, \tag{17a}$$

$$N = \sum_{\alpha=1}^{A} \sum_{i=1}^{r} n_{i\alpha}, \tag{18a}$$

$$Z = \sum_{\alpha=1}^{A} \sum_{i=1}^{s} z_{i\alpha}, \tag{19a}$$

and (21), (22), (23) would become

$$\frac{\partial x_{1\alpha}}{\partial x_{i\alpha}} \equiv -1, \qquad i = 1, 2, \cdots, m, \quad \alpha = 1, 2, \cdots, A,$$

$$\frac{\partial x_{1\alpha}}{\partial n_{i\alpha}} \equiv \frac{\partial x_{1\beta}}{\partial n_{j\beta}}, \qquad i = 1, 2, \cdots, r, \quad j = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A, \quad \beta = 1, 2, \cdots, A,$$

$$\frac{\partial x_{1\alpha}}{\partial z_{i\alpha}} \equiv \frac{\partial x_{1\beta}}{\partial z_{j\beta}}, \qquad i = 1, 2, \cdots, s, \quad j = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A, \quad \beta = 1, 2, \cdots, A.$$

This seemingly obvious type of aggregation would thus be suitable only if the marginal productivity of any type of labor (capital) in any firm were *identically* the same as the marginal productivity of any other type of labor (capital) in any firm. The restriction can be somewhat reshaped if the sums in (17a), (18a), and (19a) are changed to linear combinations. Then the marginal productivities need not be equal, but merely proportional.

It should be remarked that the functions G, H, I were made to depend only upon the physical quanties $\{x_{i\alpha}\}$, $\{n_{i\alpha}\}$, $\{z_{i\alpha}\}$. Most index numbers are constructed so that quantity indexes depend upon prices as weights, as well as upon quantities. We might construct our transformations as follows:

$$X = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{m} p_i x_{i\alpha}}{\dfrac{\sum_{\alpha=1}^{A} \sum_{i=1}^{m} p_i x_{i\alpha}^0}{\sum_{\alpha=1}^{A} \sum_{i=1}^{m} p_i^0 x_{i\alpha}^0}}, \tag{17b}$$

$$N = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}}{\dfrac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}^0}{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i^0 n_{i\alpha}^0}}, \tag{18b}$$

$$Z = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}}{\dfrac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}^0}{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i^0 z_{i\alpha}^0}}. \tag{19b}$$

The aggregates (17b), (18b), (19b) are all value aggregates deflated by fixed-base price indexes.

By differentiating (17b), we find

$$-\frac{\dfrac{\partial X}{\partial x_{i\alpha}}}{\dfrac{\partial X}{\partial x_{1\alpha}}} \equiv -p_i/p_1. \tag{24}$$

This relation holds identically because of the definition of the aggregative function. It is also true that

$$\frac{\partial x_{1\alpha}}{\partial x_{i\alpha}} = -\frac{p_i}{p_1}, \tag{25}$$

but this relation does *not* hold identically; it holds only for the equilibrium conditions under profit maximization. It is not a relation that depends solely upon technological possibilities of substitution via the production function. Hence condition (21) is not *identically* satisfied for a very common type of index number. The same is true of (22) and (23).

It needs to be further pointed out that the inclusion of prices and wages as variables in the aggregation functions, G, H, and I complicates the functional matrix by the addition of more rows provided it is desired to find a relation

$$\Phi(X, N, Z) = 0 \tag{20}$$

that does not depend explicitly on the individual prices and wages. The simple addition of more rows, however, will have no influence on the previously stated conditions (21), (22), (23) that the matrix be of rank 2. These conditions become necessary but not sufficient in this case.

It can be seen from this discussion that the use of some very common types of index numbers is not justified on the basis of the criteria which have been stated at the outset.

The Divisia-type indexes which Dresch has employed are not covered by the functions G, H, and I above because these functions are ordinary point functions, while it is well known that the Divisia indexes are line integrals, i.e., functionals They depend upon the entire paths of prices and quantities rather than merely upon point values An investigation of the conditions under which a functional relation exists among X, N, Z when they are defined by functionals as opposed to point functions is more complicated. But it happens in that case also, that the appropriate determinants do not vanish identically. Dresch's theory has intuitive significance, but fails to satisfy both of the criteria put forth at the beginning of this paper.

IV. A SUGGESTION

An alternative approach that retains the same goals can now be shown in an example. This approach is not general or unique but holds for a class of production functions that are very significant. By specifying, more closely, the shape of the production functions, we can derive a satisfactory explanation of the meaning of an aggregative production function.

Let

$$x_\alpha = B_\alpha f_\alpha(n_{1\alpha}, \cdots, n_{r\alpha}) g_\alpha(z_{1\alpha}, \cdots, z_{s\alpha}), \quad \alpha = 1, 2, \cdots, A, \tag{26}$$

be the production function for the αth firm. A special case of this function is the logical extension of the Cobb-Douglas type function

(26a) $$x_\alpha = C_\alpha \prod_{i=1}^{r} n_{i\alpha}^{a_i} \cdot \prod_{i=1}^{s} z_{i\alpha}^{b_i}.$$

Our requirement is that the production function partition into a product of a labor function and a capital function. We also attribute a single output variable to each firm, but this is done for simplicity; it is not essential.

The transformations[12] will be defined according to

(27) $$X = \left[\prod_{\alpha=1}^{A} x_\alpha \right]^{1/A},$$

(28) $$N^a = \left[\prod_{\alpha=1}^{A} f_\alpha(n_{1\alpha}, \cdots, n_{r\alpha}) \right]^{1/A},$$

(29) $$Z^b = \left[\prod_{\alpha=1}^{A} g_\alpha(z_{1\alpha}, \cdots, z_{s\alpha}) \right]^{1/A},$$

(30) $$X = DN^aZ^b.$$

The first criterion is satisfied because the aggregate production (30) does exist in explicit form. In order to apply the second criterion, we distinguish between two cases.

Case I: *a and b, the elasticities of output, are constants.* If p_α is the price of the αth good, w_i is the wage rate paid to the ith type of labor, and q_i is the cost of the ith type of capital services, then we define

(31) $$P = \frac{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}{A \cdot X},$$

(32) $$W = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}}{A \cdot N},$$

(33) $$Q = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}}{A \cdot Z},$$

as the corresponding aggregates for average price of output, average

[12] In this discussion, the macrovariables are averages, but the entire analysis also follows if the averages are changed to aggregates. We use averages in order that the macrovariable be made less sensitive to variations in the output or input of a single firm.

It should also be pointed out that firms with zero output are excluded; otherwise the entire aggregate would vanish if a single term vanished.

wage, and average price of capital. These definitions lead by simple division to

$$(34)\qquad \frac{W}{P} = \frac{X}{N}\,\frac{\sum_{\alpha=1}^{A}\sum_{i=1}^{r} w_i n_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha},$$

$$(35)\qquad \frac{Q}{P} = \frac{X}{Z}\,\frac{\sum_{\alpha=1}^{A}\sum_{i=1}^{s} q_i z_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}.$$

Also by differentiation of the aggregate production functions, we get

$$(36)\qquad \frac{\partial X}{\partial N} = a\,\frac{X}{N},$$

$$(37)\qquad \frac{\partial X}{\partial Z} = b\,\frac{X}{Z}.$$

Combining (34), (35), (36), (37), we have

$$(38)\qquad \frac{\partial X}{\partial N} = \frac{W}{P}\left\{a\,\frac{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A}\sum_{i=1}^{r} w_i n_{i\alpha}}\right\},$$

$$(39)\qquad \frac{\partial X}{\partial Z} = \frac{Q}{P}\left\{b\,\frac{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A}\sum_{i=1}^{s} q_i z_{i\alpha}}\right\}.$$

The aggregative marginal productivities are not in general equal to W/P or Q/P, but they will be when

$$(40)\qquad a = \frac{\sum_{\alpha=1}^{A}\sum_{i=1}^{r} w_i n_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha},$$

$$b = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha} . \tag{41}$$

Equations (40) and (41) are to be considered as equilibrium conditions for the macrosystem. The constant elasticities, a and b, are to be chosen as the average values over the time path of the *observed* ratios

$$\frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}$$

and

$$\frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}$$

respectively. The observed values of labor's share and capital's share will fluctuate about the average or equilibrium values and, therefore, cause $\partial X/\partial N$ and $\partial X/\partial Z$ in (38) and (39) to deviate from their equilibrium values W/P and Q/P. The macroequations for the firm will assume their equilibrium forms only when labor's share and capital's are at their equilibrium values.

Our equilibrium system, in abbreviated form, is then

$$X = DN^a Z^b, \tag{30}$$

$$\frac{\partial X}{\partial N} = \frac{W}{P}, \tag{42}$$

$$\frac{\partial X}{\partial Z} = \frac{Q}{P} . \tag{43}$$

This is a complete analogue of the equilibrium system of microeconomics.

Case II: *a and b, the output elasticities, are not constant.*

Define

$$a_{i\alpha} = \frac{n_{i\alpha}}{x_\alpha} \frac{\partial x_\alpha}{\partial n_{i\alpha}}, \qquad i = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A,$$

$$b_{i\alpha} = \frac{z_{i\alpha}}{x_\alpha} \frac{\partial x_\alpha}{\partial z_{i\alpha}}, \qquad i = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A.$$

In addition to the transformation equations (27), (28), (29), we also have

$$a = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} a_{i\alpha} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}, \tag{44}$$

$$b = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} b_{i\alpha} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}. \tag{45}$$

According to (44) and (45), the elasticities of output for the aggregative system are weighted averages of the elasticities of the individual firms. We retain the same definitions of P, W, Q given in (31), (32), (33); consequently (38) and (39) still hold. We now propose to show that the equilibrium conditions (40) and (41) are true profit-maximizing conditions which hold whenever profits are at a maximum for each individual firm. In the microsystem, we have for equilibrium,

$$\frac{\partial x_\alpha}{\partial n_{i\alpha}} = \frac{w_i}{p_\alpha}, \qquad i = 1, 2, \cdots, r, \quad \alpha = 1, 2, \cdots, A, \tag{6}$$

$$\frac{\partial x_\alpha}{\partial z_{i\alpha}} = \frac{q_i}{p_\alpha}, \qquad i = 1, 2, \cdots, s, \quad \alpha = 1, 2, \cdots, A. \tag{7}$$

Then, on substituting the definitions of $a_{i\alpha}$ and $b_{i\alpha}$ into (6) and (7), we get

$$a_{i\alpha} = \frac{w_i n_{i\alpha}}{p_\alpha x_\alpha}, \tag{46}$$

$$b_{i\alpha} = \frac{q_i z_{i\alpha}}{p_\alpha x_\alpha}. \tag{47}$$

Summing over the i subscript in each case and then over the α subscript, we get

$$\sum_{\alpha=1}^{A} \sum_{i=1}^{r} a_{i\alpha} p_\alpha x_\alpha = \sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}, \tag{48}$$

$$\sum_{\alpha=1}^{A} \sum_{i=1}^{s} b_{i\alpha} p_\alpha x_\alpha = \sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}. \tag{49}$$

Divide both sides of (48) and (49) by $\sum_{\alpha=1}^{A} p_\alpha x_\alpha$ to get our previously stated equilibrium conditions,

$$a = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} a_{i\alpha} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha} = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} w_i n_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}, \tag{40}$$

$$b = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{r} b_{i\alpha} p_\alpha x_\alpha}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha} = \frac{\sum_{\alpha=1}^{A} \sum_{i=1}^{s} q_i z_{i\alpha}}{\sum_{\alpha=1}^{A} p_\alpha x_\alpha}. \tag{41}$$

The abbreviated equilibrium system, (30), (42), (43), holds as before in Case I.

In the formulations above, a and b are like elasticities in that they are invariant under a change of units. But the quantity aggregates, X, N, Z, like any physical variable of economics, depend upon the choice of units.

If the functions f_α and g_α are known explicitly, then it is possible to show the precise manner in which the aggregates should be calculated. For example if

$$N^a = \left(\prod_{\alpha=1}^{A} f_\alpha \right)^{1/A} = \left(\prod_{\alpha=1}^{A} \prod_{i=1}^{r} n_{i\alpha}{}^{a_i} \right)^{1/A}, \tag{28a}$$

$$Z^b = \left(\prod_{\alpha=1}^{A} g_\alpha \right)^{1/A} = \left(\prod_{\alpha=1}^{A} \prod_{i=1}^{s} z_{i\alpha}{}^{b_i} \right)^{1/A}, \tag{29a}$$

then the logarithm of N is a linear combination of the logarithms of the various types of labor employed by the various firms, and similarly for capital.

As a practical method of procedure, we should calculate functions of the type (26a) for a large sample of cases. From the sample, calculate weighted geometric means of output, labor, and capital and weighted arithmetic means of the elasticities of output of labor and of capital. Knowing these averages and the numbers of firms, products, and factors, we can get good approximations of the proper aggregates. The problem of calculating the aggregates is mainly one of sampling.

The above demonstration has to be somewhat modified for the case of imperfect competition, but in any event the idea is clear for an important case. If we want to simplify mathematical models of general equilibrium into a small number of equations, it is useful to know that operationally significant concepts exist which justify such simplifications. It is only in models of macroeconomics that we can see through all the complex interrelationships of the economy in order to form intelligent judgments about such important magnitudes as aggregate employment, output, consumption, investment.

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