

THE THEORY OF ECONOMIC BEHAVIOR¹

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Had it merely called to our attention the existence and exact nature of certain fundamental gaps in economic theory, the *Theory of Games and Economic Behavior* by von Neumann and Morgenstern would have been a book of outstanding importance. But it does more than that. It is essentially constructive: where existing theory is considered to be inadequate, the authors put in its place a highly novel analytical apparatus designed to cope with the problem.

It would be doing the authors an injustice to say that theirs is a contribution to economics only. The scope of the book is much broader. The techniques applied by the authors in tackling economic problems are of sufficient generality to be valid in political science, sociology, or even military strategy. The applicability to games proper (chess and poker) is obvious from the title. Moreover, the book is of considerable interest from a purely mathematical point of view. This review, however, is in the main confined to the purely economic aspects of the *Theory of Games and Economic Behavior*.

To a considerable extent this review is of an expository² nature. This seems justified by the importance of the book, its use of new and unfamiliar concepts and its very length which some may find a serious obstacle.

The existence of the gap which the book attempts to fill has been known to the economic theorists at least since Cournot's work on duopoly, although even now many do not seem to realize its seriousness. There is no adequate solution of the problem of defining "rational economic behavior" on the part of an individual when the very rationality of his actions depends on the probable behavior of other individuals: in the case of oligopoly, other sellers. Cournot and many after him have attempted to sidetrack the difficulty by assuming that every individual has a definite idea as to what others will do under given conditions. Depending on the nature of this expected behavior of other individuals, we have the special, well-known solutions of Bertrand and Cournot, as well as the more general Bowley concept of the

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¹ *Theory of Games and Economic Behavior*. By John von Neumann and Oskar Morgenstern. (Princeton: Princeton Univ. Press. 1944. Pp. xviii, 625. \$10.)

² The exposition is mostly carried out by means of comparatively simple numerical examples. This involves loss of generality and rigor, but it may be hoped that it will make the presentation more accessible.

"conjectural variation."³ Thus, the individual's "rational behavior" is determinate *if* the pattern of behavior of "others" can be assumed *a priori* known. But the behavior of "others" cannot be known *a priori* if the "others," too, are to behave rationally! Thus a logical *impasse* is reached.

The way, or at least *a way*,⁴ out of this difficulty had been pointed out by one of the authors⁵ over a decade ago. It lies in the rejection of a narrowly interpreted maximization principle as synonymous with rational behavior. Not that maximization (of utility⁶ or profits) would not be desirable if it were feasible, but there can be no true maximization when only one of the several factors which decide the outcome (of, say, oligopolistic competition) is controlled by the given individual.

Consider, for instance, a duopolistic situation⁷ where each one of the duopolists A and B is *trying* to maximize his profits. A's profits will depend not only on his behavior ("strategy") but on B's strategy as well. Thus, *if* A could control (directly or indirectly) the strategy to be adopted by B, he would select a strategy for himself and one for B so as to maximize his own profits. But he cannot select B's strategy. Therefore, he can in no way make sure that by a proper choice of his own strategy his profits will actually be unconditionally maximized.

It might seem that in such a situation there is no possibility of defining rational behavior on the part of the two duopolists. But it is here that the novel solution proposed by the authors comes in. An example will illustrate this.

Suppose each of the duopolists has three possible strategies at his disposal.⁸ Denote the strategies open to duopolist A by A_1 , A_2 , and A_3 , and those open to duopolist B by B_1 , B_2 , and B_3 . The profit made by A, to be denoted by a , obviously is determined by the choices of strategy made by the two duopolists. This dependence will be indicated by subscripts attached to a , with the first subscript referring to A's strategy and the second subscript to that of B; thus, *e.g.*, a_{13} is the profit which will be made by A if he chooses strategy A_1 while B chooses the strategy B_3 . Similarly, b_{13} would denote the profits

³ More recent investigations have led to the idea of a kinked demand curve. This, however, is a special—though very interesting—case of the conjectural variation.

⁴ Cf. reference to von Stackelberg in footnote 17 and some of the work quoted by von Stackelberg, *op. cit.*

⁵ J. von Neumann, "Zur Theorie der Gesellschaftsspiele," *Math. Annalen* (1928).

⁶ A side-issue of considerable interest discussed in the *Theory of Games* is that of measurability of the utility function. The authors need measurability in order to be able to set up tables of the type to be presented later in the case where utility rather than profit is being maximized. The proof of measurability is not given; however, an article giving the proof is promised for the near future and it seems advisable to postpone comment until the proof appears. But it should be emphasized that the validity of the core of the *Theory of Games* is by no means dependent on measurability or transferability of the utilities and those who feel strongly on the subject would perhaps do best to substitute "profits" for "utility" in most of the book in order to avoid judging the achievements of the *Theory of Games* from the point of view of an unessential assumption.

⁷ It is assumed that the buyers' behavior may be regarded as known.

⁸ Actually the number of strategies could be very high, perhaps infinite.

A's Profits

B's choice of strategies A's choice of strategies	B ₁	B ₂	B ₃
A ₁	a ₁₁	a ₁₂	a ₁₃
A ₂	a ₂₁	a ₂₂	a ₂₃
A ₃	a ₃₁	a ₃₂	a ₃₃

Table 1a

B's Profits

B's choice of strategies A's choice of strategies	B ₁	B ₂	B ₃
A ₁	b ₁₁	b ₁₂	b ₁₃
A ₂	b ₂₁	b ₂₂	b ₂₃
A ₃	b ₃₁	b ₃₂	b ₃₃

Table 1b

by B under the same circumstances. The possible outcomes of the "duopolistic competition" may be represented in the following two tables:

Table 1a shows the profits A will make depending on his own and B's choice of strategies. The first row corresponds to the choice of A₁, etc.; columns correspond to B's strategies. Table 1b gives analogous information regarding B's profits.

In order to show how A and B will make decisions concerning strategies we shall avail ourselves of a numerical example given in Tables 2a and 2b.

A's Profits

B's choice of strategies A's choice of strategies	B ₁	B ₂	B ₃
A ₁	2	8	1
A ₂	4	3	9
A ₃	5	6	7

Table 2a

B's Profits

B's choice of strategies A's choice of strategies	B ₁	B ₂	B ₃
A ₁	11	2	20
A ₂	9	15	3
A ₃	8	7	6

Table 2b

Now let us watch A's thinking processes as he considers his choice of strategy. First of all, he will notice that by choosing strategy A₃ he will be sure that his profits cannot go down below 5, while either of the remaining alternatives would expose him to the danger of going down to 3 or even to 1.

But there is another reason for his choosing A_3 . Suppose there is a danger of a "leak": B might learn what A's decision is before he makes his own. Had A chosen, say, A_1 , B—if he knew about this—would obviously choose B_3 so as to maximize his own profits; this would leave A with a profit of only 1. Had A chosen A_2 , B would respond by selecting B_2 , which again would leave A with a profit below 5 which he could be sure of getting if he chose A_3 .

One might perhaps argue whether A's choice of A_3 under such circumstances is the only way of defining rational behavior, but it certainly is a way of accomplishing this and, as will be seen later, a very fruitful one. The reader will verify without difficulty that similar reasoning on B's part will make him choose B_1 as the optimal strategy. Thus, the outcome of the duopolistic com-

B's choice of strategies A's choice of strategies		A's Profits		
		B_1	B_2	B_3
A_1	2	8	1	
A_2	4	3	9	
A_3	5	6	7	

Table 3a

B's choice of strategies A's choice of strategies		B's Profits		
		B_1	B_2	B_3
A_1	8	2	9	
A_2	6	7	1	
A_3	5	4	3	

Table 3b

petition is determinate and can be described as follows: A will choose A_3 , B will choose B_1 , A's profit will be 5, B's 8.

An interesting property of this solution is that neither duopolist would be inclined to alter his decision, even if he were able to do so, after he found out what the other man's strategy was.

To see this, suppose B has found out that A's decision was in favor of strategy A_3 . Looking at the third row of Table 2b, he will immediately see that in no case could he do better than by choosing B_1 , which gives him the highest profit consistent with A's choice of A_3 . The solution arrived at is of a very stable nature, independent of finding out the other man's strategy.

But the above example is artificial in several important respects. For one thing, it ignores the possibility of a "collusion" or, to use a more neutral term, coalition between A and B. In our solution, yielding the strategy combination (A_3, B_1), the joint profits of the two duopolists amount to 13; they could do better than that by acting together. By agreeing to choose the strategies A_1 and B_3 respectively, they would bring their joint profits up to 21; this sum could then be so divided that both would be better off than under the previous solution.

A major achievement of the *Theory of Games* is the analysis of the conditions and nature of coalition formation. How that is done will be shown below. But, for the moment, let us eliminate the problem of coalitions by considering a case which is somewhat special but nevertheless of great theoretical interest: the case of *constant sum* profits. An example of such a case is given in Tables 3a and 3b.

Table 3a is identical with Table 2a. But figures in Table 3b have been selected in such a manner that the joint profits of the two duopolists always amount to the same (10), no matter what strategies have been chosen. In such a case, A's gain is B's loss and *vice versa*. Hence, it is intuitively obvious (although the authors take great pains to show it rigorously) that no coalition will be formed.

The solution can again be obtained by reasoning used in the previous case and it will again turn out to be (A_3, B_1) with the respective profits 5 and 5 adding up to 10. What was said above about stability of solution and absence of advantage in finding the opponent⁹ out still applies.

There is, however, an element of artificiality in the example chosen that is responsible for the determinateness of the solution. To see this it will suffice to interchange 5 and 6 in Table 3a. The changed situation is portrayed in Table 4 which gives A's profits for different choices of strategies.¹⁰

There is no solution now which would possess the kind of stability found in the earlier example. For suppose A again chooses A_3 ; then if B should find that out, he would obviously "play" B_2 which gives him the highest possible profit consistent with A_3 . But then A_3 would no longer be A's optimum strategy: he could do much better by choosing A_1 ; but if he does so, B's optimum strategy is B_3 , not B_2 , etc. There is no solution which would not give at least one of the opponents an incentive to change his decision if he found the other man out! There is no stability.¹¹

		A's Profits		
		B_1	B_2	B_3
A's choice of strategies	A_1	2	8	1
	A_2	4	3	9
	A_3	6	5	7

Table 4

What is it in the construction of the table that insured determinateness in

⁹ In this case the interests of the two duopolists are diametrically opposed and the term "opponents" is fully justified; in the previous example it would not have been.

¹⁰ The table for B's profits is omitted because of the constant sum assumption. Clearly, in the constant sum case, B may be regarded as minimizing A's profits since this implies maximization of his own.

¹¹ There is, however, a certain amount of determinateness, at least in the negative sense, since certain strategy combinations are excluded: e.g. (A_2, B_1) ; A would never choose A_2 if he knew B had chosen B_1 , and *vice versa*.

the case of Table 3 and made it impossible in Table 4? The answer is that Table 3 has a *saddle point* ("minimax") while Table 4 does not.

The saddle point has the following two properties: it is the highest of all the row minima and at the same time it is lowest of the column maxima. Thus, in Table 3a the row minima are respectively 1, 3, and 5, the last one being highest among them (*Maximum Minimorum*); on the other hand, the column maxima are respectively 5, 8, and 9 with 5 as the lowest (*Minimum Maximorum*). Hence the combination (A_3, B_1) yields both the highest row minimum and the lowest column maximum, and, therefore, constitutes a saddle point. It is easy to see that Table 4 does *not* possess a saddle point. Here 5 is still the *Maximum Minimorum*, but the *Minimum Maximorum* is given by 6; the two do not coincide, and it is the absence of the saddle point that makes for indeterminateness in Table 4.

Why is the existence of a unique saddle point necessary (as well as sufficient) to insure the determinateness of the solution? The answer is inherent in the reasoning used in connection with the earlier examples: if A chooses his strategy so as to be protected in case of any leakage of information concerning his decision, he will choose the strategy whose row in the table has the highest minimum value, *i.e.*, the row corresponding to the *Maximum Minimorum*— A_3 in case of Table 4—for then he is sure he will not get less than 5, even if B should learn of this decision. B, following the same principle, will choose the column (*i.e.*, strategy) corresponding to the *Minimum Maximorum*— B_1 in Table 4—thus making sure he will get at least 4, even if the information does leak out.

In this fashion both duopolists are sure of a certain minimum of profit—5 and 4, respectively. But this adds up to only 9. The residual—1—is still to be allocated and this allocation depends on outguessing the opponent. It is this residual that provides an explanation, as well as a measure, of the extent of indeterminacy. Its presence will not surprise economists familiar with this type of phenomenon from the theory of bilateral monopoly. But there are cases when this residual does equal zero, that is, when the *Minimum Maximorum* equals the *Maximum Minimorum*, which (by definition) implies the existence of the saddle point and complete determinacy.

At this stage the authors of the *Theory of Games* had to make a choice. They could have accepted the fact that saddle points do not always exist so that a certain amount of indeterminacy would, in general, be present. They preferred, however, to get rid of the indeterminacy by a highly ingenious modification of the process which leads to the choice of appropriate strategy.

So far our picture of the duopolist making a decision on strategy was that of a man reasoning out which of the several possible courses of action is most favorable ("*pure strategy*"). We now change this picture and put in his hands a set of dice which he will throw to determine the strategy to be chosen. Thus, an element of chance is introduced into decision making ("*mixed strategy*").¹² But not everything is left to chance. The duopolist A

¹²The authors' justification for introducing "mixed strategies" is that leaving one's decision to chance is an effective way of preventing "leakage" of information since the individual making the decision does not himself know which strategy he will choose.

A's Profits

B's choice of strategies A's choice of strategies	B ₁	B ₂
A ₁	5	3
A ₂	1	5

ROW MINIMA
3
1

} MAXIMUM MINIMORUM

COLUMN MAXIMA	5	5
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} MINIMUM MAXIMORUM

Table 5

must in advance formulate a rule as to what results of the throw—assume that just one die is thrown—would make him choose a given strategy. In order to illustrate this we shall use a table that is somewhat simpler, even if less interesting than those used previously. In this new table (Table 5)¹³ each duopolist has only two strategies at his disposal.

An example of a rule A might adopt would be:

- If the result of the throw is 1 or 2, choose A₁;
- if the result of the throw is 3, 4, 5, or 6, choose A₂.

If this rule were followed, the probability that A will choose A₁ is 1/3, that of his choosing A₂ is 2/3. If a different rule had been decided upon (say, one of choosing A₁ whenever the result of the throw is 1, 2, or 3), the probability of choosing A₁ would have been 1/2. Let us call the fraction giving the

¹³In Table 5 there is no saddle point.

probability of choosing A_1 , A's *chance coefficient*; in the two examples, A's chance coefficients were $1/3$ and $1/2$ respectively.¹⁴

As a special case the value of the chance coefficient might be zero (meaning, that is, definitely choosing strategy A_2) or one (meaning that A is definitely choosing strategy A_1); thus in a sense "pure strategies" may be regarded as

Mathematical Expectations of A's Profits

B's chance coefficients A's chance coefficients	0	$\frac{1}{3}$	$\frac{2}{3}$	1	ROW MINIMA
0	5	$3\frac{2}{3}$	$2\frac{1}{3}$	1	1
$\frac{1}{3}$	$4\frac{1}{3}$	$3\frac{2}{3}$	3	$2\frac{1}{3}$	$2\frac{1}{3}$
$\frac{2}{3}$	$3\frac{2}{3}$	$3\frac{2}{3}$	$3\frac{2}{3}$	$3\frac{2}{3}$	$3\frac{2}{3}$ } MAXIMUM MINIMORUM
1	3	$3\frac{2}{3}$	$4\frac{1}{3}$	5	3
COLUMN MAXIMA	5	$3\frac{2}{3}$	$4\frac{1}{3}$	5	
		MINIMUM MAXIMORUM			

Table 6

a special case of mixed strategies. However, this last statement is subject to rather important qualifications which are of a complex nature and will not be given here.

Now instead of choosing one of the available strategies the duopolist A must choose the optimal (in a sense not yet defined) chance coefficient. How

¹⁴ Since the probability of choosing A_2 is always equal to one minus that of choosing A_1 , specification of the probability of choosing A_1 is sufficient to describe a given rule. However, when the number of available strategies exceeds two, there are several such chance coefficients to be specified.

is the choice of the chance coefficient made? The answer lies in constructing a table which differs in two important respects from those used earlier. Table 6 provides an example. Each row in the table now corresponds to a possible value of A's chance coefficient; similarly, columns correspond to possible values of B's chance coefficient. Since the chance coefficient may assume any value between zero and one (including the latter two values), the table is to be regarded merely as a "sample." This is indicated by spaces between rows and between columns.

The numbers entered in the table are the average values (mathematical expectations) corresponding to the choice of chance coefficients indicated by the row and column.¹⁵ (One should mention that Table 6 is only an expository device: the actual procedures used in the book are algebraic and much simpler computationally.)

If we now assume with the authors that each duopolist is trying to maximize the mathematical expectation of his profits (Table 6) rather than the profits themselves (Table 5), it might seem that the original source of difficulty remains if a saddle point does not happen to exist. But the mixed strategies were not introduced in vain! It is shown (the theorem was originally proved by von Neumann in 1928) that in the table of mathematical expectations (like Table 6) a saddle point *must* exist; the problem is always determinate.¹⁶

The reader who may have viewed the introduction of dice into the decision-making process with a certain amount of suspicion will probably agree that

¹⁵ To see this we shall show how, e.g., we have obtained the value in the second row and third column of Table 5 (*viz.*, 3).

We construct an auxiliary table (valid only for this particular combination of chance coefficients (A's 1/3, B's 2/3).

This table differs from Table 5 only by the omission of row maxima and column minima and by the insertion of the probabilities of choosing the available strategies corresponding to the second row third column of Table 6. The computation of the mathematical expectation is indicated in Table 6.

COMPUTATION OF THE MATHEMATICAL EXPECTATION FOR THE 2ND ROW, 3RD COLUMN IN TABLE 6

		B's choice of strategies	
		B ₁	B ₂
A's choice of strategies	B's chance coefficients	$\frac{2}{3}$	$\frac{1}{3}$
	A's chance coefficients	$\frac{1}{3}$	$\frac{2}{3}$
A ₁		5	3
A ₂		1	5

$$\begin{aligned} & \frac{1}{3} \times \frac{2}{3} \times 5 + \frac{1}{3} \times \frac{1}{3} \times 3 \\ & + \frac{2}{3} \times \frac{2}{3} \times 1 + \frac{2}{3} \times \frac{1}{3} \times 5 \\ & = 2\frac{7}{9} = 3 \end{aligned}$$

¹⁶ In Table 6 the saddle point is in the third row second column; it is to be stressed that Table 5 has no saddle point.

this is a rather spectacular result. Contrary to the initial impression, it is possible to render the problem determinate. But there is a price to be paid: acceptance of mixed strategies, assumption that only the mathematical expectation of profit (not its variance, for instance) matters, seem to be necessary. Many an economist will consider the price too high. Moreover, one might question the need for introducing determinateness into a problem of this nature. Perhaps we should consider as the "solution" the interval of indeterminacy given by the two critical points: the *Minimum Maximorum* and *Maximum Minimorum*.

As indicated earlier in this review, one should not ignore, in general, the possibility of a collusion. This is especially evident when more complex economic situations are considered.

We might, for instance, have a situation where there are two sellers facing two buyers. Here a "coalition" of buyers, as well as one of sellers, may be formed. But it is also conceivable that a buyer would bribe a seller into some sort of coöperation against the other two participants. Several other combinations of this type can easily be found.

When only *two* persons enter the picture, as in the case of duopoly (where the rôle of buyers was ignored), it was seen that a coalition would not be formed if the sum of the two persons' profits remained constant. But when the number of participants is *three* or more, subcoalitions can profitably be formed even if the sum of all participants' profits is constant; in the above four-person example it might pay the sellers to combine against the buyers even if (or, perhaps, especially if) the profits of all four always add to the same amount.

Hence, the formation of coalitions may be adequately treated without abandoning the highly convenient constant-sum assumption. In fact, when the sum is known to be non-constant, it is possible to introduce (conceptually) an additional fictitious participant who, by definition, loses what all the real participants gain and *vice versa*. In this fashion a non-constant sum situation involving, say, three persons may be considered as a special case of a constant-sum four-person situation. This is an additional justification for confining most of the discussion (both in the book and in the review) to the constant-sum case despite the fact that economic problems are as a rule of the non-constant sum variety.

We shall now proceed to study the simplest constant-sum case which admits coalition formation, that involving three participants. The technique of analysis presented earlier in the two-person case is no longer adequate. The number of possibilities increases rapidly. Each of the participants may be acting independently; or else, one of the three possible two-person coalitions (A and B *vs.* C, A and C *vs.* B, B and C *vs.* A) may be formed. Were it not for the constant-sum restriction, there would be the additional possibility of the coalition comprising all three participants.

Here again we realize the novel character of the authors' approach to the problem. In most¹⁷ of traditional economic theory the formation—or absence—

¹⁷ In his *Grundlagen einer reinen Kostentheorie* (Vienna, 1932) H. von Stackelberg does point out (p. 89) that "the competitors [duopolists] must somehow unite; they must . . .

of specific coalitions is *postulated*. Thus, for instance, we discuss the economics of a cartel without rigorously investigating the necessary and sufficient conditions for its formation. Moreover, we tend to exclude *a priori* such phenomena as collusion between buyers and sellers even if these phenomena are known to occur in practice. The *Theory of Games*, though seemingly more abstract than economic theory known to us, approaches reality much more closely on points of this nature. A complete solution to the problems of economic theory requires an answer to the question of coalition formation, bribery, collusion, etc. This answer is now provided, even though it is of a somewhat formal nature in the more complex cases; and even though it does not always give sufficient insight into the actual workings of the market.

Let us now return to the case of three participants. Suppose two of them are sellers, one a buyer. Traditional theory would tell us the quantity sold by each seller and the price. But we know that in the process of bargaining one of the sellers might bribe the other one into staying out of the competition. Hence the seller who refrained from market operations would make a profit; on the other hand, the nominal profit made by the man who did make the sale would exceed (by the amount of bribe) the actual gain made.

It is convenient, therefore, to introduce the concept of *gain*: the bribed man's gain is the amount of the bribe, the seller's gain is the profit made on a sale minus the bribe, etc. A given distribution of gains among the participants is called an *imputation*. The imputation is not a number: it is a set of numbers. For instance, if the gains of the participants in a given situation were g_A, g_B, g_C , it is the set of these three g 's that is called the imputation. The imputation summarizes the outcome of the economic process. In any given situation there are a great many possible imputations. Therefore, one of the chief objectives of economic theory is that of finding those among all the possible imputations which will actually be observed under rational behavior.

In a situation such as that described (three participants, constant-sum) each man will start by asking himself how much he could get acting independently, even if the worst should happen and the other two formed a coalition against him. He can determine this by treating the situation as a two-person case (the opposing coalition regarded as one person) and finding the relevant *Maximum Minimax*, or the saddle point, if that point does exist; the saddle point would, of course, exist if "mixed strategies" are used. Next, the participant will consider the possibility of forming a coalition with one of the other two men. Now comes the crucial question: under what conditions might such a coalition be formed?

Before discussing this in detail, let us summarize, in Table 8, all the relevant information.

supplement the economic mechanics, which in this case is inadequate, by economic politics." But no rigorous theory is developed for such situations (although an outline of possible developments is given). This is where the *Theory of Games* has made real progress.

TABLE 8

I. If A acts alone, he can get	5
If B acts alone, he can get	7
If C acts alone, he can get	10.
II. If A and B form a coalition, they can get	15
If A and C form a coalition, they can get	18
If B and C form a coalition, they can get	20.
III. If A, B, and C act together, they can get	25.

Among the many possible imputations, let us now consider the three given in Table 9.

TABLE 9

	A	B	C
#1	6.5	8.3	10.2
#2	5.0	9.5	10.5
#3	4.0	10.0	11.0

It will be noted that under imputation #1, B and C are each better off than if they had been acting individually: they get respectively 8.3 and 10.2 instead of 7 and 10. Hence, there is an incentive for B and C to form a coalition since without such a coalition imputation #1 would not be possible. But once the coalition is formed, they can do better than under #1; *viz.*, under #2, where each gets more (9.5 and 10.5 instead of 8.3 and 10.2, respectively). In such a case we say that imputation #2 *dominates* imputation #1. It might seem that #3, in turn, dominates #2 since it promises still more to both B and C. But it promises too much: the sum of B's and C's gains under #3 is 21, which is more than their coalition could get (*cf.* Table 8)! Thus #3 is ruled out as unrealistic and cannot be said to dominate any other imputation.

Domination is an exceptionally interesting type of relation. For one thing, it is not transitive: we may have an imputation i_1 dominating the imputation i_2 and i_2 dominating i_3 , without thereby implying that i_1 dominates i_3 ; in fact, i_1 might be dominated by i_3 .¹⁸ Moreover, it is easy to construct examples of, say, two imputations, neither of which dominates the other one.¹⁹

To get a geometric picture of this somewhat unusual situation one may turn

¹⁸ *I.e.*, domination may be a *cyclic* relation. For instance, consider the following three imputations in the above problem: #1 and #2 as in Table 9, and #4, where

	A	B	C
#4	6.0	7.0	12.0.

Here #2 (as shown before) dominates #1 (for the coalition B, C), #4 dominates #2 (for coalition A, C), but at the same time #1 dominates #4 (for the coalition A, B): the cycle is completed.

¹⁹ For instance, #2 and #3 in Table 9.

to Figure 1, where points on the circle represent different possible imputations. (The reader must be cautioned that this is merely a geometrical analogy, though a helpful one.) Let us now say that point #1 dominates point #2 if #2 is less than 90° (clockwise) from #1. It is easy to see in Figure 1 that #1 dominates #2 and #2 dominates #3, but in spite of that, #1 does not dominate #3.

This geometrical picture will help define the very fundamental concept of a *solution*.

Consider the points (imputations) #1, 3, 5, and 7 in Figure 1. None of them dominates any other since any two are either *exactly* or more than 90° apart. But any other point on the circle is dominated by at least (in this case: exactly) one of them: all points between #1 and #3 are dominated by #1, etc. There is no point on the circle which is not dominated by one of the above four points. Now we *define* a solution as a set of points (imputations) with two properties: (1) no element of the set dominates any other element of the set, and (2) any point outside the set must be dominated by at least one element within the set.

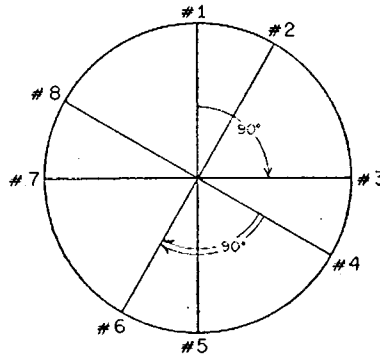


FIGURE 1

We have seen that the points #1, 3, 5, 7 do have both of these properties; hence, the four points together form a solution. It is important to see that none of the individual points by itself can be regarded as a solution. In fact, if we tried to leave out any one of the four points of the set, the remaining three would no longer form a solution; for instance, if #1 were left out, the points between #1 and #3 are not dominated by any of the points #3, 5, 7. This violates the second property required of a solution and the three points by themselves are not a solution. On the other hand, if a fifth point were added to #1, 3, 5, 7, the resulting five element set would not form a solution either; suppose #2 is the fifth point chosen; we note that #2 is dominated by #1 and it also dominates #3. Thus, the first property of a solution is absent.

Contrary to what would be one's intuitive guess, an element of the solution may be dominated by points outside the solution: #1 is dominated by #8, etc.

There can easily be more than one solution. The reader should have no trouble verifying the fact that #2, 4, 6, 8 also form a solution, and it is clear that infinitely many other solutions exist.

Does there always exist at least one solution? So far this question remains unanswered. Among the cases examined by the authors none has been found without at least one solution. But it has not yet been proved that there must always be a solution. To see the theoretical possibility of a case without a

solution we shall redefine slightly our concept of domination (*cf.* Figure 2): #1 dominates #2 if the angle between them (measured clockwise) does not exceed 180° .

Hence, in Figure 2 point #1 dominates #3, but not #4, etc. It can now be shown that in this case *no* solution exists. For suppose there is one; then we may, without loss of generality, choose #1 as one of its points. Clearly, #1 by itself does not constitute a solution, for there are points on the circle (*e.g.*, #4) not dominated by #1; thus the solution must have at least two points. But any other point on the circle either is dominated by #1 (*e.g.*, #2), or it dominates #1 (*e.g.*, #4), or both (#3), which contradicts the first requirement for the elements of a solution. Hence there is no solution consisting of two points either. *A fortiori*, there are no solutions containing more than two points. Hence we have been able to construct an example without a

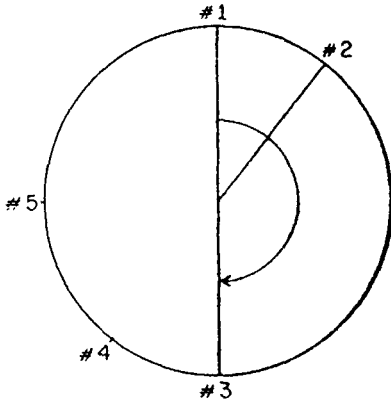


FIGURE 2

solution. But whether this type of situation could arise in economics (or in games, for that matter) is still an open question.

Now for the economic interpretation of the concept of solution. Within the solution there is no reason for switching from one imputation to another since they do not dominate each other. Moreover, there is never a good reason for going outside a given solution: any imputation outside the solution can be "discredited" by an imputation within the solution which dominates the one outside. But, as we have seen, the reverse is also usually true: imputations within the solution may be dominated by those outside. If we are to assume that the latter consideration is ignored, the given solution acquires an institutional, if not accidental, character. According to the authors, a solution may be equivalent to what one would call the "standards of behavior" which are accepted by a given community.

The multiplicity of solutions can then be considered as corresponding to alternative institutional setups; for a given institutional framework only one solution would be relevant. But even then a large number of possibilities remains since, in general, a solution contains more than one imputation. More indeterminacy yet would be present if we had refrained from introducing mixed strategies.

It would be surprising, therefore, if in their applications von Neumann and Morganstern should get no more than the classical results without discovering imputations hitherto neglected or ignored. And there are some rather interesting "unorthodox" results pointed out, especially in the last chapter of the book.

In one case, at least, the authors' claim to generality exceeding that of

economic theory is not altogether justified in view of the more recent literature. That is the case of what essentially corresponds to bilateral monopoly (p. 564, proposition 61:C). The authors obtain (by using their newly developed methods) a certain interval of indeterminacy for the price; this interval is wider than that indicated by Böhm-Bawerk, because (as the authors themselves point out) of the dropping of Böhm-Bawerk's assumption of a unique price. But this assumption has been abandoned, to give only one example, in the theories of consumer's surplus, with analogous extension of the price interval.

It will stand repeating, however, that the *Theory of Games* does offer a greater generality of approach than could be attained otherwise. The existence of "discriminatory" solutions, discovered by purely analytical methods, is an instance of this. Also, the possibility of accounting for various types of deals and collusions mentioned earlier in connection with the three-person and four-person cases go far beyond results usually obtained by customarily used methods and techniques of economic theory.

The potentialities of von Neumann's and Morgenstern's new approach seem tremendous and may, one hopes, lead to revamping, and enriching in realism, a good deal of economic theory. But to a large extent they are only potentialities: results are still largely a matter of future developments.

The difficulties encountered in handling, even by the more powerful mathematical methods, the situations involving more than three persons are quite formidable. Even the problems of monopoly and monopsony are beyond reach at the present stage of investigation. The same is true of perfect competition, though it may turn out that the latter is not a "legitimate" solution since it excludes the formation of coalitions which may dominate the competitive imputations. A good deal of light has been thrown on the problem of oligopoly, but there again the results are far from the degree of concreteness desired by the economic theorist.

The reviewer therefore regards as somewhat regrettable some of the statements made in the initial chapter of the book attacking (rather indiscriminately) the analytical techniques at present used by the economic theorists. True enough, the deficiencies of economic theory pointed out in the *Theory of Games* are very real; nothing would be more welcome than a model giving the general properties of a system with, say, m sellers and n buyers, so that monopoly, duopoly, or perfect competition could simply be treated as special cases of the general analysis. Unfortunately, however, such a model is not yet in sight. In its absence less satisfactory, but still highly useful, models have been and no doubt will continue to be used by economic theorists. One can hardly afford to ignore the social need for the results of economic theory even if the best is rather crude. The fact that the theory of economic fluctuations has been studied as much as it has is not a proof of "how much the attendant difficulties have been underestimated" (p. 5). Rather it shows that economics cannot afford the luxury of developing in the theoretically most "logical" manner when the need for the results is as strong as it happens to be in the case of the ups and downs of the employment level!

Nor is it quite certain, though of course conceivable, that, when a rigorous

theory developed along the lines suggested by von Neumann and Morgenstern is available, the results obtained in the important problems will be sufficiently remote from those obtained with the help of the current (admittedly imperfect) tools to justify some of the harsher accusations to be found in the opening chapter of the book. It must not be forgotten, for instance, that, while theoretical derivation of coalitions to be formed is of great value, we do have empirical knowledge which can be used as a substitute (again imperfect) for theory. For example, cartel formation may be so clearly "in the cards" in a given situation that the economic theorist will simply include it as one of his assumptions while von Neumann and Morgenstern would (at least in principle) be able to *prove* the formation of the cartel without making it an additional (and logically unnecessary) assumption.

The authors criticize applications of the mathematical methods to economics in a way which might almost, in spite of protests to the contrary, mislead some readers into thinking that von Neumann and Morgenstern are not aware of the amount of recent progress in many fields of economic theory due largely to the use of mathematical tools. They also seem to ignore the fact that economics developed in literary form is, implicitly, based on the mathematical techniques which the authors criticize. (Thus it is not the methods of mathematical economics they are really questioning, but rather those elements of economic theory which literary and mathematical economics have in common.) While it is true that even mathematical treatment is not always sufficiently rigorous, it is as a rule more so than the corresponding literary form, even though the latter is not infrequently more realistic in important respects.

There is little doubt in the reviewer's mind that nothing could have been further from the authors' intentions than to give aid and comfort to the opponents of rigorous thinking in economics or to increase their complacency. Yet such may be the effect of some of the vague criticisms contained in the first chapter; they hardly seem worthy of the constructive achievements of the rest of the book.

Economists will probably be surprised to find so few references to more recent economic writings. One might almost form the impression that economics is synonymous with Böhm-Bawerk plus Pareto. Neither the nineteenth century pioneers (such as Cournot) nor the writers of the last few decades (Chamberlin, Joan Robinson, Frisch, Stackelberg) are even alluded to. But, perhaps, the authors are entitled to claim exemption from the task of relating their work to that of their predecessors by virtue of the tremendous amount of constructive effort they put into their opus. One cannot but admire the audacity of vision, the perseverance in details, and the depth of thought displayed on almost every page of the book.

The exposition is remarkably lucid and fascinating, no matter how involved the argument happens to be. The authors made an effort to avoid the assumption that the reader is familiar with any but the more elementary parts of mathematics; more refined tools are forged "on the spot" whenever needed.

One should also mention, though this transcends the scope of the review, that in the realm of strategic games proper (chess, poker) the results obtained are more specific than some of the economic applications. Those interested in the nature of determinacy of chess, in the theory of "bluffing" in poker, or in the proper strategy for Sherlock Holmes in his famous encounter with Professor Moriarty, will enjoy reading the sections of the book which have no direct bearing on economics. The reader's views on optimum military or diplomatic strategies are also likely to be affected.

Thus, the reading of the book is a treat as well as a stage in one's intellectual development. The great majority of economists should be able to go through the book even if the going is slow at times; it is well worth the effort. The appearance of a book of the caliber of the *Theory of Games* is indeed a rare event.