TABLE 2 (*Continued*)

Reference[1]	Place	Type of Study[2]	Date	a_1	a_2	a_1+a_2	$\bar{w}^1$
[14]	U.S.A.	*C*	1914	0.612 ±0.027	0.359 ±0.025	0.971	0.586
[15]	U.S.A.	*C*	1904	0.647 ±0.023	0.311 ±0.022	0.958	0.635
	[8]	*C*	1904	0.654 ±0.022	0.312 ±0.021	0.966	0.635
[16]	Canada [9]	*C*	1923	0.48 ±0.038	0.48 ±0.035	0.96	0.50
	[9]	*C*	1927	0.46 ±0.039	0.52 ±0.036	0.98	0.48
	[9]	*C*	1935	0.50 ±0.044	0.52 ±0.040	1.02	0.40
	[9]	*C*	1937	0.43 ±0.041	0.58 ±0.036	1.01	0.52
			average	0.468	0.525	0.992	0.475
[4]	New Zealand	*T*	1915–1935	0.483	. . .	[10]	. . .

[1] With the exception of [4] (Clark) the references are to papers by Douglas and his collaborators [5] and [8]–[16] listed at the end of the present article and arranged according to the date of the publication.
[2] T = intertemporal study (based on time series of national totals); C = interindustrial ("cross-section") study.
[3] $a_1 = 0.67$ fits better for 1899–1916.
[4] Cf. also Table 3.
[5] Output includes depreciation, hence low $\bar{w}_1$.
[6] Averages per plant (but moments of logarithms unweighted).
[7] Industry aggregates (not divided by number of firms).
[8] Includes proprietors and firm members under "labor."
[9] Depreciation apparently not deducted.
[10] $a_1 + a_2$ bound at 1.

to permit the application of either method; and though the technological differences between the 80 or so industries can be regarded as random, yet the ensuing, presumably multimodal, distribution (strong clustering of the values of a_0 around certain magnitudes characteristic of each industry) may produce great inaccuracies[23]—though perhaps not much greater than those due to the imperfection of our knowledge of the boundary constants, such as c_{00}, c_{01}, c_{02}. If a firm's belonging to a given industry is regarded as a "systematic" rather than a "random" cause, and is, accordingly, studied by properly isolating it—i.e., by studying each industry separately, perhaps after a preliminary analysis of variance as a guide to such separation—more accurate results are

[23] We have no proof for this assertion, however.

obtained. For the analogous reason, the "intertemporal" studies cannot render reliable estimates unless one isolates the factor time by introducing a proper trend. This is an alternative to isolating each year (as is done in the "cross-section studies"); but in either case the systematic variations from industry to industry will spoil the accuracy even if the systematic variations from year to year have been removed.

The computations and diagrams of Chapter II were based on figures of the U.S.A. Census of Manufactures for 1909 [45] using the assumption that all firms belonging to one industry were identical. Because this assumption is dubious and because of systematic variations as between industries, the results are probably very inaccurate. They mainly serve the purpose of showing the kind of uncertainty that would prevail in these studies even if the data by individual firms were available, and the assumption of "identical firms" unnecessary. In addition, Figures 1 and 2 help to illustrate the relationship between the least-squares "single-equation" estimates and the true values of output elasticities.

§28

Some of the "single-equation" least-squares estimates of output elasticities, computed by Douglas and others from census data, are listed below.[24] One has to keep in mind that:

(1) in some earlier studies—e.g., [5], [8], [9]—the systematic effect of a time trend was not eliminated;

(2) in some earlier studies, the coefficients α_1 and α_2 were bound by the a priori condition of "constant returns to scale," $\alpha_1+\alpha_2=1$; this does not seem justified (see Appendix 1, §43);

(3) the dividing of aggregate output, labor, and capital of a year or an industry by the number of firms was not performed except in later studies;

(4) when such dividing was done, the computed moments were not weighted though they should be (see §25).

Table 3 takes up in somewhat more detail one of the results listed in Table 2, viz., U.S.A. 1909 [10]. The effect is shown (1) of dividing or not dividing each industrial aggregate by the number of firms in the industry; (2) of computing weighted or unweighted moments of the logarithms. (As explained in §25, the correct procedure is to divide the industrial aggregates and to weight the moments of logarithms.) We have applied one further adjustment. Some of the industrial groupings used by Bronfenbrenner and Douglas [10] were rather heterogeneous

[24] In our notation, a_1 and a_2 are the least-squares estimates of elasticities of net output with respect to labor and capital, respectively. In Douglas' notation, $a_1=k$, $a_2=j$. Further, our $\bar{w}_1$ equals, in Douglas' notation, W/P, where $W=$national pay roll and $P=$national net output of manufactures.

TABLE 3

EFFECT OF WEIGHTING AND OF OTHER PROCEDURES ON SOME SINGLE-EQUATION ESTIMATES OF OUTPUT ELASTICITIES (a_1, a_2) AND THEIR SUMS
(Estimates are followed by their standard errors)

U. S. 1909[1] (cf. Table 2, Ref. [10])	a_1	a_2	a_1+a_2
Bronfenbrenner-Douglas: 90 industries			
Total-capital figures			
Industry aggregates			
Moments unweighted	0.742 ±0.045	0.319 ±0.043	1.061 ±0.027
Fixed-capital figures			
Industry aggregates			
Moments unweighted	0.857 ±0.045	0.083 ±0.035	0.940 ±0.031
Andrews: 85 industries reclassified			
Total-capital figures			
Industry aggregates			
Moments unweighted	0.707 ±0.039	0.261 ±0.034	0.968 ±0.023
Industry averages per firm			
Moments unweighted	0.677 ±0.043	0.281 ±0.034	0.958 ±0.020
Moments weighted	0.656 ±0.042	0.266 ±0.037	0.922 ±0.021

[1] Weighting of moments (of logarithms) discussed in §25; "industrial totals" (vs. "industrial averages") in §28, sub. (3). For the reclassification of industries and the use of capital figures, see footnote 25 below.

composites of Census items. To make the assumption of "identical firms in each industry" more plausible, the list had to be revised.[25] It

[25] Of the 90 "industries" used by Douglas and Bronfenbrenner, 22 had to be regarded as mere composites. We have listed, instead, 86 industries each representing a single Census item, [45] p. 186, and each employing 10,000 or more wage-earners. One industry (locomotives not made by railroad companies) was omitted since it did not appear in one of the Census tables from which data were used. Of the remaining 85 industries, 63 duplicate exactly those of Bronfenbrenner-Douglas, while 22 form part of their "composite" items. Those items included in addition 50 other (small) industries.

To compute net output, Bronfenbrenner and Douglas used the Census figure of "value added" after adjusting for various items the most important of which was depreciation. These adjustments were taken over for convenience and comparability. Where one of the 85 industries was part of the Bronfenbrenner-Douglas "composites," the same adjustment was applied as had been applied by them to the whole group.

Bronfenbrenner and Douglas used two alternative series for capital: the total-capital series, taken directly from the Census for 1909, [45], pp. 507 ff; and a series for fixed capital only. The latter was obtained by multiplying total capital by the ratio, fixed capital ÷ total capital, from the 1904 Census (since breakdown of capital was not given in 1909). We used the total-capital series.

is from these revised data that the weighted moments (§21, Table 1) were computed by formula (3.3), and used for the construction of Figures 1 and 2.

§29

In this and the following sections, the two main findings of Douglas and his collaborators will be analyzed. It is claimed by those authors that: (1) the sum a_1+a_2—the sum of the least-squares estimates—tends to be close to 1; (2) a_1 tends to be close to the national ratio of pay rolls to net output, $\bar{w}_1$.

From these findings, conclusions were drawn regarding the prevalence of perfect competition in the industry. This may mean (1) absence of monopoly in the markets of products (§6); (2) rapid free entry into or exit out of industry, of firms having nonzero profits (Appendix 1, §§41 seq.); (3) absence of monopsony in the labor market (§6); (4) absense of monopsony in the capital market (§6); (5) absence of monopolistic arrangements among customers, among workers, or among lenders (§7).

We shall first analyze whether, or under what conditions, any of these economic implications do follow from the above two findings, granted the statistical reliability of those findings. We shall then (§34) come back to the statistical question.

It should be kept in mind that whether the conclusions as to the prevalence of perfect competition be justified or not, the estimation of output elasticities (α_1, α_2) and of the other parameters involved [β_j; σ_{jk} (j, $k=0$, 1, 2)] has theoretical interest and particular importance as a tool of policy of firms and of governments. Hence the importance of the statistical question implied.

§30

To begin with (see §17), a_1 and a_2 are estimates not of the output elasticities α_1 and α_2 but of the "output elasticities corrected for monopoly," viz., of $\alpha_1'=\alpha_1\beta_0$ and $\alpha_2'=\alpha_2\beta_0$. This follows from equations (1.31) which must be applied when the data used are: output (or net output) in dollars, labor in men, capital in dollars.[26] To obtain, instead, the estimates of α_1 and α_2 one would have to use equations (1.29): but this would imply that output is measured in physical, not in money units (hence independent of prices, i.e., of degree of monopoly)—a kind of data possible for interfirm studies in certain one-commodity industries, but not for a study embracing different industries. Since β_0 is equal to or smaller than one, the presence of imperfect competition in the mar-

[26] Reder seems to have in mind the same criticism: [38], p. 261 under 1.

ket of the products of at least some of the industries studied makes α_1' and α_2' lower than the true output elasticities.

§31

The first of the two findings listed in §29 must, therefore (granted the statistical reliability of the estimates) be written in the form

$$(\alpha_1 + \alpha_2)\beta_0 = 1. \tag{3.4}$$

If it were known that $\beta_0 = 1$, equation (3.4) would indeed mean that the production function assumed common to all firms is homogeneous of first degree: "constant returns to scale" at any combination of resources. This would not in itself imply the absence of monopoly in the markets of products; nor the absence of monopsony in the markets of resources; nor the rapidity of "free entry and exit." But if those monopolies and monopsonies were absent, the homogeneity, of first degree, of the production function of each firm would indeed imply necessarily, though it is not implied by (see Appendix 1, §§42–43), the absence of profits and losses; and this absence, in turn, is implied by (though does not necessarily imply) instantaneous "free entry and exit." The question thus arises whether the data yield, in addition to (3.4), also $\beta_0 = \beta_1 = \beta = 1$, i.e., the absence of firms' monopolies and monopsonies in the three markets.

§32

The absence of monopsony in at least one of the markets concerned, viz., the absence of monopsony for labor, has, in fact, been inferred from the second finding, viz., from a_1 being close to the national ratio of pay rolls to net output. Still granting that a_1 is a reliable approximation of $\alpha_1' = \alpha_1\beta_0$, let us formulate the finding more precisely:

$$\alpha_1\beta_0 = \frac{\sum_{f=1}^{n} y_{1f}}{\sum_{f=1}^{n} y_{0f}} \equiv \frac{\sum_{f=1}^{n} w_{1f}y_{0f}}{\sum_{f=1}^{n} y_{0f}} \equiv \bar{w}_1 \quad \text{(say)}, \tag{3.5}$$

a weighted average of the ratios (w_{1f}) of each firm's pay roll to each firm's net output value. On the other hand, for each such individual ratio we know, rewriting (1.13) for the firm f, that, if it has its profit at a maximum,

$$\alpha_1\beta_0 = \beta_1 w_{1f}. \tag{3.6}$$

The finding (3.5) together with the assumption of maximized profits would lead thus to

$$\beta_1 = \frac{\bar{w}_1}{w_{1f}} \cdot \tag{3.7}$$

Thus, if all profits were maximized, and if all the individual pay-roll shares w_{1f} were equal (and hence equal to their weighted average $\bar{w}_1$), it would have been proved that

$$\beta_1 = 1,$$

i.e., absence of monopsony for labor. Unfortunately, to assume that all w_{1f} are equal is not realistic. A look at the Census figures shows great fluctuations even between industrial averages of w_{1f}. Even after all industries are grouped into large complexes, there remain strong fluctuations of the average labor shares (w_1), as seen, for example, from the following calculations of Bronfenbrenner and Douglas for U.S.A., 1909 [10]: Clothing and textiles, $w_1 = 0.69$; Foods and beverages, $w_1 = 0.56$; Metals and machinery, $w_1 = 0.73$. Fluctuations among single firms are, of course, larger: it is likely (§19) that, within an industry, the more efficient firms have lower values of w_{1f} then the technically less efficient ones. Thus (3.7) would imply that β_1, too, fluctuates from firm to firm, stronger monopsony being accompanied by lower labor share and correlated with higher technical efficiency. Alternatively, one may doubt the exact validity of (3.7)—even while still granting statistical reliability to the finding (3.5)—because (3.7) is based on the assumption of exactly maximized profits, which cannot be granted and need not be made in realistic studies (§§12, 13).

§33

To sum up: granted the statistical reliability of the claimed two findings, i.e., granted that

$$(\alpha_1 + \alpha_2)\beta_0 = 1 \quad \text{(approximately)}, \tag{3.4}$$

and

$$\alpha_1\beta_0 = \bar{w}_1 \text{ (approximately)}, \tag{3.5}$$

the following inferences can be made, provided there is no monopoly in product markets ($\beta_0 = 1$): (1) all production functions are homogeneous of first degree; (2) if there is no monopsony in the resources markets ($\beta_0 = \beta_1 = \beta_2 = 1$), there is "free entry and exit"; (3) either the degree of labor monopsony varies strongly from industry to industry (and firm to firm), fluctuating around unity; or the profits are not maximized. These inferences have, however, to compete with another one, viz., the statistical reliability of the findings may be doubted.

§34

First of all, the degree of approximation in (3.4) and (3.5) is not very satisfactory. This is seen from Tables 2 and 3. The sum a_1+a_2 does deviate from 1, and a_1 does deviate from $\bar{w}_1$ considerably if account is taken of the standard error.

These deviations might become either larger or smaller if the necessary revisions (enumerated in §28) were made in the calculations.

§35

Moreover, we should remember that the assumption that α_1, α_2 (though not A_0) are the same for all firms is due only to the limitation of our statistical tools (§15). What is actually computed are certain "average" values of these parameters. Therefore, even if it should be true that estimates of α_1 and α_2 added up to 1, this would imply "constant returns to scale" only in the sense that if some firms have increasing returns, these are, on the whole, offset by others having decreasing returns. The proportion of firms to have "almost" constant returns to scale will be the smaller, the more strongly the values of α_1 and α_2 fluctuate from firm to firm—thus making hypotheses such as (1.23) [the one used in the least-squares estimates and also used as part of our own hypothesis (1.31)] with A_0 as the only random parameter, more and more erroneous.

On the other hand, another limitation, not of statistical tools, but (a more accidental one) of available published data, has made it necessary, in the studies of Douglas as well as in our own illustration, to assume that all firms within an industry are identical (§25).

§36

The two particular assumptions mentioned in §35 are needed for our estimates not less than for those of Douglas and his collaborators. But we have removed certain further assumptions that are tacitly made when the "single-equation" method is applied, and that need not be made. This is seen in our Figures 1 and 2. For example, the "single-equation" least-squares estimate becomes worse and worse the stronger the variation of technical efficiency from firm to firm (or from industry to industry), as measured by the sample variance s_{00} which can be used as an approximation (being a consistent estimate) of the population variance σ_{00} (§§21–22).

Furthermore, the sample correlations r_{01} and r_{02} are both nonnegative at the point A whose co-ordinates are the least-squares estimates a_1, a_2. This fact is not due to the particular data on which Figure 1 is based; it can be derived from (2.3), substituting for α_1', α_2' their "single-equation" least-squares estimates a_1, a_2.[27] As the sample covariances can be used as approximations (being consistent estimates) of the population covariances, the population correlation coefficients ρ_{01}, ρ_{02} are likely to be nonnegative if the least-squares estimates a_1, a_2 happen to coincide with the true values of α_1', α_2'. Economic considerations (§19) make it probable, however, that ρ_{01} is positive but ρ_{02} *negative.* This would throw doubt on any estimate of α_1', α_2' represented by a point lying far above the line $\rho_{02}=0$. To be sure, the hypothesis $\rho_{02}<0$ is only approximate: it is the less exact, the stronger the differences between the firms' supply curves (which are all horizontal under perfect competition).

§37

Consider now Figure 1. It is based on the assumption that labor and capital markets are perfect ($\beta_1=\beta_2=1$). Using first the considerations of §18: if the "best" firm as there defined is technically at the most 4 times more efficient than the worst, and if a Gibrat distribution of technical efficiency is assumed, then $\sigma_{00}=0.010$, and the consistent estimate of α_1', α_2' will lie on or within the corresponding ellipse. As our "firms" are actually industrial averages (§§25–26), the ratio 4:1 is perhaps a reasonable upper limit. The next larger ellipse on Figure 1, $\sigma_{00}=0.020$, would correspond to a ratio $10^{6\sqrt{\sigma_{00}}}:1=7:1$ which some estimators may prefer as a more cautious upper limit.

Using now the considerations of §19: if $\rho_{01}>0$ and $\rho_{02}<0$, the consistent estimates of α_1', α_2' must lie within the triangle formed by the lines $\rho_{01}=0$, $\rho_{02}=0$, and the horizontal axis (since $\alpha_1'>0$). Together with the elliptic boundary ($\sigma_{00}=0.010$) just discussed, this will reduce the region to the southeastern part of the ellipse. The other boundary derived in §19, viz., equation (1.41), does not prove effective as it practically coincides, in the relevant part of the figure, with one of the larger ellipse. ($\bar{w}_1=0.68$, assumed profit $\bar{w}=0.10$, so that $\bar{w}_2=0.22$.)

Finally, the condition of §20 is, under perfect competition, that of nonincreasing returns (and maximized profits). A straight line (denoted

[27] Write $|m|$ = determinant $|m_{00}m_{11}m_{22}|$; $\|m^{ij}\|=\|m_{ij}\|^{-1}$. Then $a_i=-m^{0i}/m^{00}$ ($i=1, 2$). Inserting into the second equation in (2.3), we have

$$s_{01}m^{00}=\sum_{j=0}^{2}m_{0j}m^{0j}-\beta_1\sum_{j=0}^{2}m_{1j}m^{0j}=1;$$

hence $s_{01}=s_{02}=1/m^{00}$ which is nonnegative, as $|m|$ is a moment determinant. Therefore both r_{01} and r_{02} are nonnegative.

by $|\pi| = 0$), sloping at $-45°$ to both axes, chops another bit off the ellipse. The resulting region is shaded.

On Figure 2, four alternative assumptions (the same as on Figure 3) have been made as to the size of β_1 and β_2; and, in addition to the assumptions of the preceding paragraphs, it was assumed $\rho_{01} \geqq 0.7$. The resulting regions are:

If	$\beta_1 = \beta_2 = 1,$		*CDEFGC*,
If	$\beta_1 = 1,$	$\beta_2 = 1.5$	*CDEHIGC*,
If	$\beta_1 = 1.5,$	$\beta_2 = 1$	*JKLMJ*,
If	$\beta_1 = \beta_2 = 1.5$		*JNPMJ*.

It appears that the region of values of α_1', α_2' compatible with the data and with certain economic considerations shrinks as β_1 increases. One might thus be able to form a judgment as to the upper limits of β_1; in the case of a single industry these would be indicative of the degrees of monopsony in the labor market. However, the economic considerations would need further treatment before such judgment is made. For example (as follows from the reasoning of §19), the bounding lines $\rho_{01} = 0$, $\rho_{02} = 0$ must themselves be modified as the markets become more imperfect and the supply curves to single firms less similar.

Naturally, the region of consistent estimates of the set of parameters α_1', α_2', β_1, β_2, compatible with the data as well as with other information we have is the narrower, the more precise that information: that is, the narrower the inequalities such as thus suggested in §§18–20.[28] With information vague, the region is wide, and the conclusions, on the whole, of a negative type. For actual applications a still wider region would have to be used: a confidence interval would have to be constructed around every estimation point. This could not be done here, because, when using industrial averages instead of single firms, we were ignorant about what to consider the size of the sample (§26).

If data on the output and used resources of individual firms of a single industry become available, the method outlined in Chapter II and resulting so far in Figures 1 and 2 will become more promising for three reasons: (1) the assumption "all firms in an industry are identical" will be dropped; (2) it will become possible, at least in the case of one-commodity industries, to apply hypothesis (1.29) and estimate

[28] The use of such boundary conditions has exactly the same logical meaning as the use, very common in recent economics, of series of plausible alternative assumptions or combinations of assumptions. See, for example, Pigou [37], Machlup [30], Slichter [41].

α_1, α_2 instead of $\alpha_1\beta_0$, $\alpha_2\beta_0$; (3) for a given single industry, the knowledge of the variability of technical efficiency, and of its correlation with labor, capital, and profit share can be more precise than for all industries, so that the region of consistent estimates can be narrowed down; (4) confidence intervals can be constructed, the size of sample being equal to the number of firms.

With the type of data used here (and also used by Douglas and his collaborators) the discussion was necessarily critical and illustrative. Douglas' estimates listed in Table 2 do exhibit a stability from one year to another which presumably would not be much affected by applying corrections as in Table 3. Since the least-squares estimates are functions of the parameters of the joint distribution of output, labor, and capital—i.e., functions of the α's, as well as of the variances and covariances of the random components (such as technical and economic efficiency)—the stability of Douglas' results suggest that those parameters change only slowly. This promises significant results as soon as better data are available for the estimation of those parameters.

IV. APPENDIX 1. DECREASING, INCREASING, AND CONSTANT RETURNS TO SCALE

§38

It is often believed that, for "a priori reasons," the production function must belong to a certain class. Such restrictions are derived from the assumption of maximized profits and perfect competition in the markets of products as well as resources; to these assumptions sometimes is added the assumption of instantaneous wiping out of profits and losses by free entry of firms into profit-making, free exit of firms out of loss-making industries. In the empirical work, outlined in the preceding chapters, no such assumptions need be made though some of them may be tested. However, the questions have played a role in the theory and statistical study of production. They deserve clarification if only to establish how these various assumptions are related.

In Chapter I, only the first-order conditions for a maximum profit of a firm were stated:

(1.11), (1.12) $$\frac{\partial \pi}{\partial x_1} = 0, \qquad \frac{\partial \pi}{\partial x_2} = 0.$$

These conditions are necessary for the profit to be at a maximum, i.e., to exceed the profits in all points of the neighborhood. However, these conditions are also compatible with profits being at a minimum, or being at an "improper maximum,"[29] and with still other situations. In

[29] See footnote 35 for a geometrical example.

other words, the vanishing of the first-order derivatives of the profit is necessary but not sufficient for a proper maximum.

Denoting now the second derivatives $\partial^2\pi/\partial x_h \partial x_i$ by π_{hi} ($i=1, 2$), there exist, for a proper maximum of π, the following *necessary* conditions

$$(4.1), (4.2), (4.3) \qquad \pi_{11} \leqq 0, \qquad \pi_{22} \leqq 0, \qquad |\pi| \geqq 0$$

where $|\pi|$ is the determinant $\pi_{11}\pi_{22} - \pi_{12}^2$. There exist the following *sufficient* conditions (to be taken together):

$$(4.4), (4.5) \qquad \pi_{11} < 0, \qquad |\pi| > 0$$

(from the latter two follows also $\pi_{22} < 0$).

(4.1)–(4.5) will be seen in the next sections to imply, in the case of a firm, that, if the output of and the quantities of resources used by the firm are determined by its making profits a maximum, and if there is perfect competition in the markets of its product as well as of its resources, the firm's production function must not show *increasing* returns to scale; if it shows *decreasing* returns to scale, while all the necessary first-order conditions are satisfied, the proper maximum of profit is secured. This leaves open (until §42) the case of *constant* returns to scale. The three cases were defined in §5.

§39

Let us start with the special case (1.19)–(1.22), used in the present article as an empirical approximation: the case of constant output elasticities (α_1, α_2) and constant elasticities of revenue and of outlays (β_0, β_1, β_2). Then the profit

$$\pi \equiv b_0 \cdot (a_0 x_1^{\alpha_1} x_2^{\alpha_2})^{\beta_0} - b_1 x_1^{\beta_1} - b_2 x_2^{\beta_2}.$$

If the necessary first-order conditions, $\partial\pi/\partial x_1 = \partial\pi/\partial x_2 = 0$, are satisfied, the necessary second-order conditions (4.1)–(4.3) become[30]

$$(4.6) \qquad (\beta_0 \alpha_i - \beta_i) \leqq 0 \qquad (i = 1, 2),$$

$$\beta_1\beta_2 - \beta_0 \cdot (\alpha_1\beta_2 + \alpha_2\beta_1) \geqq 0.$$

From §6 and from these inequalities we have (since α_1, α_2, β_1, β_2 are all positive)

$$(4.7) \qquad \alpha_1 \frac{\beta_0}{\beta_1} + \alpha_2 \frac{\beta_0}{\beta_2} \leqq 1.$$

[30] In differentiating, use the fact that $\partial x_0/\partial x_1 = \alpha_1 x_0/x_1$, where $x_0 = a_0 x_1^{\alpha_1} x_2^{\alpha_2}$.

If $\beta_0 = \beta_1 = \beta_2 = 1$, i.e., if there exists perfect competition in all three markets, the sufficient condition (4.7) becomes

$$\alpha_1 + \alpha_2 \leqq 1, \tag{4.8}$$

i.e., increasing returns to scale are excluded.

If, instead of combining the necessary first-order conditions with the necessary second-order conditions (4.1)–(4.3), we combine the necessary first-order conditions with the sufficient second-order conditions (4.4), (4.5), we obtain, as sufficient conditions for proper maximum of profits, the same propositions as (4.6), (4.7), (4.8), but with the equality signs dropped:

$$\alpha_1 \frac{\beta_0}{\beta_1} + \alpha_2 \frac{\beta_0}{\beta_2} < 1; \tag{4.7'}$$

and, under perfect competition in all three markets,

$$\alpha_1 + \alpha_2 < 1; \tag{4.8'}$$

i.e., under perfect competition, the vanishing of the first-order derivatives of profit of a firm with decreasing returns to scale ensures proper maximum profits.

Figure 3 illustrates these results. The axes are $\alpha_1' = \alpha_1\beta_0$ and $\alpha_2' = \alpha_2\beta_0$. Those points in the (α_1', α_2')-plane which lie neither in the area bounded by the axes and by the straight line with intercepts β_1, β_2, nor on that straight line itself, are not compatible with profit maximization. The equation of the straight line is

$$\frac{\alpha_1'}{\beta_1} + \frac{\alpha_2'}{\beta_2} = 1. \tag{1.43}$$

In the case of perfect competition in all three markets the excluded area is that bounded by line (a) which has equation $\alpha_1 + \alpha_2 = 1$. That is, the returns to scale must not be increasing. In a somewhat more general case $\beta_1 = \beta_2$ $(= \beta$, say$) \geqq \beta_0$, we have $\alpha_1 + \alpha_2 \leqq \beta/\beta_0$, and since $\beta/\beta_0 \geqq 1$, increasing returns to scale are possible (i.e., compatible with maximum profits). The line (b) illustrates at the same time the case $\beta_0 = 1$, $\beta_1 = \beta_2 = \frac{3}{2}$ (perfect competition in the sale of the product); or the case $\beta_0 = \frac{2}{3}$, $\beta_1 = \beta_2 = 1$ (perfect competition in the purchase of the factors); or a case like $\beta_0 = \frac{8}{9}$, $\beta_1 = \beta_2 = \frac{4}{3}$ (imperfect competition in all three markets, with the two outlay elasticities equal). In all these cases the sum $\alpha_1 + \alpha_2 \leqq \frac{3}{2}$. Still more general is a case (c) when all three elasticities are unequal: the sum $\alpha_1 + \alpha_2$ is $\leqq \beta_1/\beta_0$, or $\leqq \beta_2/\beta_0$, whichever is larger. Other cases, (d) and (e), are also shown on Figure 3 as well as on Figure 2.

This exclusion of a certain area represents a necessary condition (4.7) for a proper maximum. We further have the sufficient condition (4.7′):

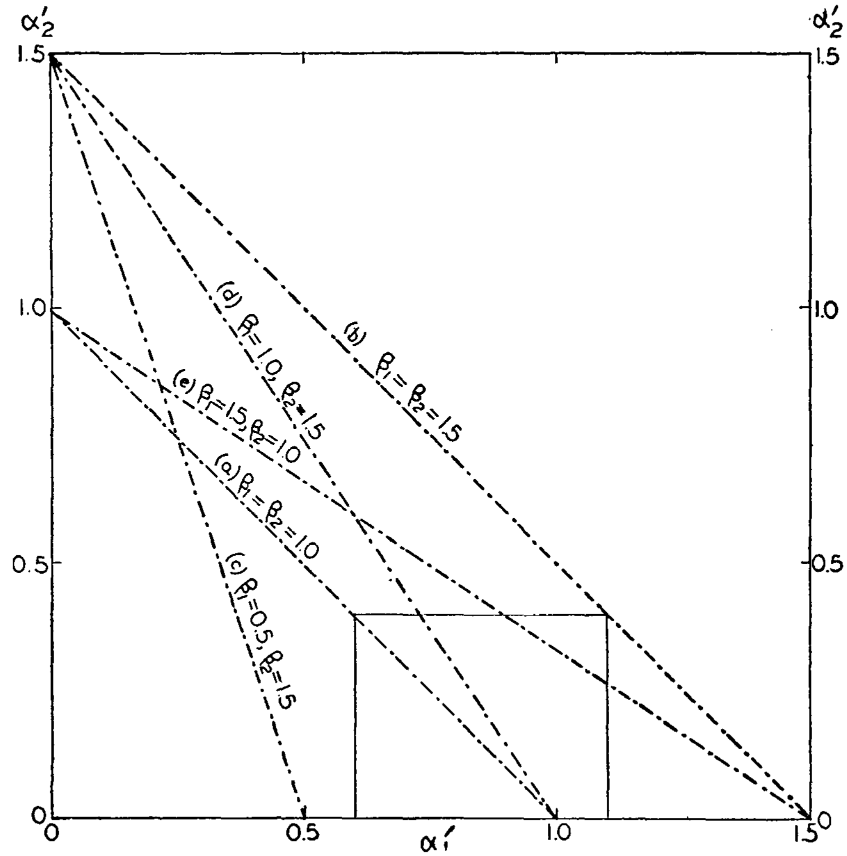

FIGURE 3.—Boundary lines for profit maxima, $|\pi| = 0$, under varying degrees of labor and capital monopsony.

$$\frac{\alpha_1'}{\beta_1} + \frac{\alpha_2'}{\beta_2} = 1, \quad (\alpha_1' = \alpha_1\beta_0,\ \alpha_2' = \alpha_2\beta_0).$$

(Small rectangle represents field of Figure 2.)

if the point lies neither in the excluded area *nor on the boundary line*, and the necessary conditions (vanishing of first derivative) are satisfied, proper maximum profits are assured. The boundary line represents, under perfect competition, the case of *constant returns*: they leave the sufficient conditions unfulfilled.

§40

Dropping now our special case (1.19)–(1.22), we can conveniently treat the general case (1.2)–(1.5) by introducing the "elasticities of elasticities,"

$$\alpha_{hi} = \frac{\partial \alpha_h}{\partial x_i} \cdot \frac{x_i}{\alpha_h} \qquad (h,\ i = 1,\ 2),$$

and writing further

$$\beta_j^* = \beta_j + \frac{d\beta_j}{dx_j} \cdot \frac{x_j}{\beta_j} \qquad (j = 0,\ 1,\ 2).$$

From definitions we have

$$\frac{\alpha_{12}}{\alpha_2} = \frac{\alpha_{21}}{\alpha_1} \equiv \alpha \quad \text{(say)}. \tag{4.9}$$

If profit is maximized, we obtain by straightforward differentiation the following restrictions upon the functions involved (necessary conditions; or, if equality signs are dropped, sufficient conditions)

$$\begin{aligned} \beta_0 \cdot (\alpha_i \beta_0^* - \beta_i^* + \alpha_{ii}) &\leqq 0 \qquad (i = 1,\ 2), \\ (\alpha_1\beta_0^* - \beta_1^* + \alpha_{11})(\alpha_2\beta_0^* - \beta_2^* + \alpha_{22}) - \alpha_1\alpha_2(\beta_0^* + \alpha)^2 &\geqq 0. \end{aligned} \tag{4.10}$$

These inequalities have to take the place of (4.7), (4.8) which had already shown, for the special case then considered, that under imperfect competition increasing returns to scale are compatible with maximum profits. Those special inequalities can be derived from the general ones (4.10), by making the α's and β's constant, so that $\alpha_{hi} = \alpha = 0$, $\beta_j^* = \beta_j\ (h,\ i = 1,\ 2;\ j = 0,\ 1,\ 2)$.

Another important case (which includes the special case just mentioned) is that of a production function homogeneous of degree $\alpha_1 + \alpha_2 = r = \text{const.}$ [see equation (1.9)], the α's being, in general, not indepent of the x's. Differentiating (1.9) with respect to x_1 or x_2, we have, by (4.9),

$$\frac{\alpha_{11}}{\alpha_2} = \frac{\alpha_{22}}{\alpha_1} = -\alpha,$$

and, by (4.10), if $\beta_0 = \beta_1 = \beta_2 = 1$ (perfect competition),

$$\begin{aligned} \alpha_1 - 1 - \alpha_2\alpha &\leqq 0, \\ \alpha_2 - 1 - \alpha_1\alpha &\leqq 0, \\ (1 - r)(\alpha r + 1) &\geqq 0. \end{aligned}$$

Adding the first two inequalities gives $(1-r) \geqq -(\alpha r + 1)$, so that in the third inequality the factors cannot both be negative; hence

$$1 - r \geqq 0,$$
$$\alpha_1 + \alpha_2 \leqq 1.$$

Again [as in the special case (4.8)], this restriction upon the production function (exclusion of increasing returns to scale) would not be obtained if at least one of the three markets was imperfect.

§41

If it is assumed that firms enter or leave the industry instantaneously in response to positive or negative profits, an additional equation is provided:

$$\pi_f = 0 \qquad (f = 1, \cdots, n), \tag{4.11}$$

making thus for each firm the number of equations[31] larger by one than the number of variables. The system thus becomes inconsistent unless certain further restrictions are imposed upon the functions involved. As the process of the entry and exit of firms involves interrelationships among firms, it is necessary to set up the system of equations for the economy as a whole. No generality is lost, for the purpose at hand, if all (n) firms are supposed to produce the same commodity (identified by subscript 0). Before the condition of zero profits is introduced, we have the following determinate system ($f=1, \cdots, n$; $j=0, 1, 2$; $i=1, 2$), where the last column refers to the single-firm system of equations in Chapter I:

(4.12)	$3n$ identities	$y_{jf} = p_{jf} x_{jf}$	(1.1),
(4.13)	n identities	$\pi_f = y_{0f} - y_{1f} - y_{2f}$	(1.10),
(4.14)	$3n$ revenue and outlay equations	$y_{jf} = y_{jf}(x_{jf})$	(1.3)–(1.5),
(4.15)	n production equations	$x_{0f} = \phi_f(x_{1f}, x_{2f})$	(1.2),
(4.16)	$2n$ equilibrium equations	$\dfrac{\partial \pi_f}{\partial x_{if}} = 0$	(1.11)–(1.12).

This system of $10n$ equations and unknowns reproduces exactly that given in Chapter I for one firm. If perfect competition is assumed, in the sense that firms receive (and pay) the same prices, and no single firm can affect them, the number of unknown prices is reduced by $3(n-1)$; and the $3n$ revenue and outlay equations have to be replaced by 3 equations of aggregate demand (and supply): say[32]

[31] These were given in §§4–8 with the firm subscript omitted.

[32] To express the fact that demand and supply functions F_j depend on the aggregate incomes of workers and of lenders, these quantities can be included in the

$$p_j = F_j\left(\sum_{f=1}^{n} x_{jf}\right) \qquad (j = 0, 1, 2). \tag{4.17}$$

We can then present the system more compactly, eliminating the y's (revenues and outlays) and performing the differentiation in (4.16):

$$\begin{aligned} \pi_f &= p_0 x_{0f} - p_1 x_{1f} - p_2 x_{2f}, \\ x_{0f} &= \phi_f(x_{1f}, x_{2f}) \qquad (f = 1, \cdots, n), \\ \frac{p_i}{p_0} &= \frac{\partial \phi_f}{\partial x_i} \qquad (j = 0, 1, 2; i = 1, 2), \\ p_j &= F_j\left(\sum_{f=1}^{n} x_{jf}\right). \end{aligned} \tag{4.18}$$

§42

If, now, the condition of "free entry" is added,

$$\pi_f = 0 \qquad (f = 1, \cdots, n), \tag{4.11}$$

the system has n equations more than unknowns. It becomes inconsistent unless such restrictions are imposed upon the production functions and other functions involved, as would make n equations derivable from the rest of the system.

Economists have looked for such restrictions, discussing in particular the case of perfect competition, i.e., the case when (4.14) is replaced by (4.17). It was suggested that the conditions of perfect competition and maximum profits can be reconciled with the condition of zero profits if all production functions ϕ_f $(f = 1, \cdots, n)$ are homogeneous of first degree, i.e., the returns to scale are constant. However, other restrictions on the functions involved might be imposed instead.[33]

---

parentheses in (4.17) without changing the number of equations. On the other hand, the system breaks down if bilaterally imperfect competition is introduced.

[33] As an illustration consider production functions in one variable (x_1) only (say, labor, all firms having the same capital; or labor and capital in fixed proportions, etc.): output $x_{0f} = \phi_f(x_{1f})$ $(f = 1, \cdots, n)$. Further, under perfect competition, $p_1 = F(\sum x_{1f})$, $p_0 = F_0(\sum x_{0f})$, as in (4.17). Then, if profits are both zero and maximum, we have

$$\frac{p_1}{p_0} = \frac{\partial \phi_f}{\partial x_{1f}} = \frac{x_{0f}}{x_{1f}} \qquad (f = 1, \cdots, n).$$

Assume each ϕ_f to pass through the origin and have a point of inflexion in the positive quadrant: such S-shaped production curves, corresponding to a U-shaped average-cost curve, are often set up—Knight [29], Bronfenbrenner [3]—and can be approximated—Allen [2]—by a cubic,

$$x_{0f} = a_f x_{1f}^3 + b_f x_{1f}^2 + c_f x_{1f}.$$

§43

If all n production functions ϕ_f are homogeneous of first degree, and equations (4.18) are satisfied, profits π_f $(f=1, \cdots, n)$ vanish identically by Euler's theorem (1.8a). In this case the n equations (4.11) do not make the system inconsistent. But the production functions being homogeneous of first degree make the system indeterminate. This is seen by rewriting the production function for each firm (dropping the subscript f for brevity) in the form $x_0 = x_2 \cdot g(x_1/x_2)$ and defining $u = x_1/x_2$, $v = x_0/x_2$. The first three lines in (4.18) become

$$\text{(4.19)} \qquad \frac{\pi}{p_0 x_2} = v - \frac{p_1}{p_0} u - \frac{p_2}{p_0},$$
$$v = g(u),$$
$$\frac{p_1}{p_0} = g'(u), \qquad \frac{p_2}{p_0} = g(u) - ug'(u).$$

The last two equations in (4.19) impose a restriction on prices; if this is satisfied, one of these two equations becomes redundant, and (4.19) determines the ratios u and v, and the profit $(=0)$, but not the x's themselves. These remain still indeterminate when the n individual firms' systems (4.19) are supplemented by the 3 equations (4.17) of aggregate demand and supply.

The amount of resources and product being indeterminate, the question whether the profit is at its maximum loses much of its meaning. To be sure, the same formal property of the homogeneous linear function ϕ that makes the system indeterminate renders also the sufficient conditions for profit maximum unsatisfied. The determinant $|\phi| = |\partial^2\phi/\partial x_h \partial x_i|$ is the Jacobian of the system of equations $p_0\phi_i - p_i = 0$, and its vanishing shows the system indeterminate. On the other

This gives x_{1f} the equilibrium value $-b_f/2a_f$ and imposes n conditions upon the parameters of the functions involved, viz.,

$$\frac{d_1}{4a_1} = \cdots = \frac{d_n}{4a_n} = \frac{F(-\sum b_f/2a_f)}{-F_0(\sum b_f d_f/8a_f^2)},$$

where $d_f = b_f^2 - 4a_f c_f$ $(f=1, \cdots, n)$. Further if $d^2\pi_f/dx_{1f}^2 < 0$, then $b_f > 0$ is a sufficient condition for maximum profits.

Bronfenbrenner's Figure 1—[3], p. 36—well illustrates the necessity for some such restrictions: not every set of S-shaped production curves will have the property that all of them will touch the same straight line; and not every price ratio p_1/p_0 represented by the slope of that straight line will clear the market. In presenting the system of equations, however, Bronfenbrenner, while counting the n profits as unknowns, omits to count the n profit-defining identities—(4.13) above—as equations; thus failing to make clear that, in the absence of special restrictions, the "free entry" postulate makes the system inconsistent.

hand the determinant $|\pi| = |\partial^2\pi/\partial x_h \partial x_i| = p_0|\phi|$; and its vanishing indicates that the profit need not be at the maximum.[34]

Now consider profit as a function of the ratio $u = x_1/x_2$, denoting by $\hat{u}$ the "equilibrium" value of $\hat{u}$, i.e. the value that satisfies the last two equations in (4.19). Will this ratio be the most profitable one? We have

$$\frac{\pi}{p_0 x_2} = g(u) - ug'(\hat{u}) - g(\hat{u}) + \hat{u}g'(\hat{u})$$

(so that, as would be expected, $\pi = 0$ when $u = \hat{u}$). If, now, u varies while x_2 remains constant,

$$\frac{1}{p_0 x_2} \cdot \frac{d\pi}{du} = g'(u) - g'(\hat{u})$$

(so that, as would be expected, $d\pi/du = 0$ when $u = \hat{u}$); and, still varying u while x_2 remains constant, we have

$$\frac{1}{p_0 x_2} \cdot \frac{d^2\pi}{du^2} = g''(u);$$

thus, if u varies while x_2 remains constant, $d^2\pi/du^2$ has (when $u = \hat{u}$) the same sign as $g''(u)$. The same can be shown, at the point $u = \hat{u}$, if u varies while x_1 remains constant in the two successive differentiations; or when u varies while first x_1 and then x_2 (or vice versa) are kept constant. Hence, at the point $u = \hat{u}$, $\pi(u)$ has a maximum (or minimum, or neither) if $g(u)$ has a maximum (or minimum, or neither). For example, if in our special case (1.19) the production function is made homogeneous of first degree,

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1, \\ x_0 &= a_0 x_1^{\alpha_1} x_2^{\alpha_2} = a_0 x_2 u^{\alpha_1}, \\ g(u) &= a_0 u^{\alpha_1}, \\ g''(u) &= \alpha_1(\alpha_1 - 1) = -\alpha_1\alpha_2 < 0 \quad \text{(by 1.6)}; \end{aligned}$$

hence π is larger when $u = \hat{u}$ than at other neighboring values of u.[35]

[34] For the second result see Mosak [35], Durand [17]. Hicks [26] ("Dans le cas d'une fonction de production homogène et du premier degré, l'ensemble de notre analyse s'effondre") points out that the property $|\phi| = 0$ makes it necessary to change the definition of the elasticity of substitution. However, the "effondrement" is even more radical.

It is easily seen that the Jacobian of the market system (4.18) also vanishes if $|\phi| = 0$.

[35] To illustrate geometrically, think of undersea reefs: let profit be represented by the altitude above sea level, and let the amounts of the factors used be represented by longitude and latitude respectively. If the production function is homo-

With some other production functions, also homogeneous of first degree, $g''(u)$ might have been positive, making profit a minimum at the "equilibrium" ratio of resources used; or zero, and requiring further investigation.

To sum up: making all production functions homogeneous of the first degree does in general render perfect competition and profit maximization consistent with zero profits inferred from "free entry," but makes the system indeterminate: only the ratios between the amounts of resources used by each firm and not those amounts themselves can be determined. And it depends on further properties of the production functions assumed, whether those ratios are indeed the most (or perhaps, e.g., the least) profitable ones.

It was also shown that other restrictions, having nothing to do with "constant returns," may be imposed upon the functions involved and render the system consistent.

Finally and probably most important: the whole problem of reconciling perfect competition, profit maximization, and instantaneous wiping out of profits hardly arises in reality. A workable empirical hypothesis must not be based on any of these three. It is especially hard to believe that the functions involved would "adjust themselves" to the appropriate shapes with such rapidity, or that the profits and losses would vanish so quickly that the statistician's survey would record "long-run" functions and the absence of profits and losses.[36]

The "constant returns to scale" is a tricky boundary case which, as

---

geneous of the first degree and $g''(u)<0$, then the highest attainable profits (zero) lie on a horizontal ridge just touching at all its points the sea surface. The ridge is a straight line which goes through the point where latitude, longitude, and altitude are all zero. All points of the ridge are equally high, viz., zero above sea level. Hence, the fact that the highest altitude (profit) is chosen by the entrepreneur determines the ratio of longitude to latitude (direction of the ridge on a map: ratio of labor used to capital used) but not their absolute amounts. But if $g''(u)>0$ we shall have not a reef but a boat touching with its keel the surface of a (quicksilver) sea: zero profits will be lowest, not highest profits.

Under imperfect competition, the condition of constant returns is generalized into $|\pi|=0$ in (4.3) or into the second equation in (4.10) (dropping the inequality sign). If the production function is linear in the logarithms of the x's, this condition becomes (4.7′) and is the same as the condition $|\gamma|=0$, cf. (1.42). See footnotes 19 and 38.

[36] The nonnecessity of assuming constant returns to scale was pointed out by Wicksell [48], as reported by the historians of the problem, Hicks [24] and Stigler [42]. These two authors also hint only briefly in their more recent work, [25], [26], [43] at the awkward implications of that assumption. A definitive critical history of the subject will require thorough counting and interpretation of the equations and inequalities involved: even Pareto and Walras do not seem to have settled that count to mutual satisfaction: cf. Walras [46].

so often in economics, is too much used in classrooms because of its mistaken theoretical simplicity. The real interest of nondecreasing returns to scale lies in this: in theory they would preclude maximum profits under perfect competition; nevertheless, they may occur or even prevail in actual fact. Therefore the deviations of profits from their maxima and the deviations of prices from competitive ones (and the causes of such deviations) must be studied empirically. In the present study, the existence of such deviations was, in fact, not excluded.

V. APPENDIX 2: CRITICAL NOTE

§44

The gist of the recent discussion on "interfirm production functions" would have lent itself to easier understanding if the authors had explicitly formulated the simultaneous random equations constituting the economist's hypothesis on firms' behavior. Mr. Reder's argument in [38] seems to indicate correctly that the hypothesis must consist of equations listed in our Introduction as (1), (2), (3)—preferably in a generalized form admitting of imperfect competition. He correctly states (in his last footnote) that the equations involved will vary from firm to firm. This seems contradicted, however, by his assumption (in introducing the article) that the production function "is known with complete accuracy, i.e., the values which its parameters take are not affected by random fluctuations. This assumption is made in order to simplify our discussion which is primarily theoretical." If this means (in contradiction to his footnote just mentioned) that all firms have the same production function, then, as illustrated in our Introduction in the case of demand and supply functions, the production function will exactly coincide with the "empirical" (or "interfirm") function. On the other hand, if all functions involved vary from firm to firm, the quantities of output, labor, and capital will not trace out a single surface (or, at constant capital, a single curve, whose derivative with respect to L is curve ww' on Reder's diagram), unless some additional constraints on the form of their variation are imposed. Presumably, the surface (or curve, or its derivative: Reder's curve, ww') is obtained by fitting it to the scatter diagram traced out by those variations, and is a "mongrel" surface. This lack of clarity seems to be due to the attempt to make the discussion "primarily theoretical," in the sense of omitting explicit treatment of random fluctuations. However, Reder's analysis seems to point in the right direction.

§45

Bronfenbrenner's [3] interpretation of the "empirical production function" (i.e., the function fitted to the data on output, labor, and capital of a number of firms) seems different. He seems to consider the empirical equation not as a "mongrel equation" at all; but rather as one of the equations of the system. The system he suggests consists, for each firm, of our equations (1), (2), (3), correctly assumed to change from firm to firm; and, in addition, of the equations, one for each firm, expressing the absence of profits,

$$x_0 p_0 = x_1 p_1 + x_2 p_2, \tag{5.1}$$

and, further, of two aggregate supply equations, one for each production factor (cf. §§41–42 above). Thus perfect competition is assumed not only in the sense of the prices being independent of any single firm's action; but also in the sense of instantaneous free entry or exit of firms, destroying instantaneously profits and losses. (As shown above, such a system, to be consistent, implies certain restrictions upon the functions involved.) Equation (5.1) which expresses this assumption, does not seem to contain, in Bronfenbrenner's view (if we interpret him correctly), any terms fluctuating from firm to firm. Hence the empirically fitted function must exactly coincide with (5.1). We do not think this is realistic. Entry and exit of firms, and the consequent wiping out of profits and losses (and the adjustment of the functions involved, to make the system consistent), can hardly be regarded as rapid enough to ensure, at any time, actual absence of profits and losses among the statistically observed firms. If, therefore, the "zero-profits" equation can be included at all in the system of equations describing a firm, with a random term to denote a profit (or loss), that random term is certainly subject to strong fluctuations from firm to firm. There is, therefore, little likelihood that a function fitted, e.g., by the least squares, would give even an approximation of the equation (5.1).[37]

[37] Unfortunately, certain statements of Bronfenbrenner make it very difficult to understand him. It seems to us that to interpret the "interfirm" function as coinciding with (5.1) is not compatible with Reder's interpretation of it as what we have called a "mongrel" function. Yet, Bronfenbrenner expresses partial agreement with Reder, and merely points out that monopsony in the labor market, which Reder (in our opinion, rightly) regards as compatible with Douglas' empirical findings, is not thus compatible. Those empirical findings had stated that the regression coefficient of $\log x_0$ on $\log x_1$ happened to come close to the ratio of the aggregate pay roll of all firms to their aggregate output (cf. our §32). Supposing that this ratio was nearly the same for all firms, this finding would indeed imply absence of monopsony, provided the empirical regression function did really represent the production function. But if the empirical function is thought to be (5.1) then Douglas' finding would follow in any case, regardless of absence or presence of monopsony, since from (5.1)

$$\partial \log x_0 / \partial \log x_1 = (\partial x_0 / \partial x_1)(x_1/x_0) = p_1 x_1 / p_0 x_0.$$

Bronfenbrenner's use of those findings leaves us, therefore, uncertain whether,

VI. APPENDIX 3: CONSISTENT ESTIMATES OF PARAMETERS OF SYSTEMS OF LINEAR EQUATIONS WITHOUT TIME LAGS

§46

At the end of §22 it was stated that the sample variances and covariances s_{jk} are consistent estimates of the corresponding true values σ_{jk}. This statement will now be elaborated on the lines of [31] with respect to the general case (1.36), which is more symmetrical and therefore easier to handle than our special case (1.34).

It is required to estimate γ_j, γ_{jk}, σ_{jk} ($j, k=0, 1, \cdots, v$) in the equations

$$\gamma_j + \sum_{k=0}^{v} \gamma_{jk} z_k = \epsilon_j \qquad (j = 0, \cdots, v), \tag{1.36}$$

using n sets of observed values $(z_{0f}, \cdots, z_{vf})$ $(f=1, \cdots, n)$. No assumption is made as to the form of the distribution of the ϵ_j's, except that (1) the random sets $(\epsilon_{0f}, \cdots, \epsilon_{vf})$, $(\epsilon_{0f'}, \cdots, \epsilon_{vf'})$, $\cdots$, affecting the sets of observations numbered $f, f', \cdots$, are independently distributed, each having the same distribution, and $E(\epsilon_j)=0$ for all values of j; (2) that all moments of ϵ_j are finite; and (3) that $|\sigma_{jk}| \neq 0$.

Write $|\sigma| =$ determinant $|\sigma_{jk}|$, $\|\sigma^{jk}\| = \|\sigma_{jk}\|^{-1}$,

$|\gamma| =$ absolute value of the determinant $|\gamma_{jk}|$,[38]

$$q_{jk} = \frac{1}{n} \sum_{f=1}^{n} \epsilon_{jf}\epsilon_{kf}.\text{[39]} \tag{6.1}$$

Form the expression

$$L = \log |\gamma| + \frac{1}{2} \log |\sigma|^{-1} - \frac{1}{2} \sum_{j=0}^{v} \sum_{k=0}^{v} \sigma^{jk} q_{jk}, \tag{6.2}$$

after all, the empirical equation is, in his opinion, (1), or (5.1), or a "mongrel" one. Also, though the explanation he gives verbally to his Figure 1 leaves little doubt that (5.1) must be meant [represented in the (x_0, x_1)-plane by a straight line through the origin, with slope p_1/p_0] his algebraic expressions again create doubt: cf. [3], p. 40, equation (7) and the accompanying text.

[38] Conditions (1) and (3) above are not compatible with $|\gamma| = 0$, e.g., the case of "constant returns" under perfect competition is excluded; cf. second paragraph of footnote 35.

Condition (1) follows from §14.

[39] Thus, while s_{jk} in (2.2) is the mean sum-product of ϵ's in the sample, measured from the sample means, q_{jk} is the mean sum-product of ϵ's in the sample measured from zero (i.e., from the population means since $E\epsilon_k=0$). Finally σ_{jk} is the mean sum-product of ϵ's in the population, measured from zero. Symbols s^, q^, σ^ are used when all parameters have such values as would make L in (6.2) a maximum. Typographical limitations prevented the use of symbols like $\hat{a}$.

the logarithm (apart from a constant) of the likelihood function of a normal distribution.[40] Suppose that, from knowledge additional to the observations, we have w relationships

$$F_u(\gamma_0, \cdots, \gamma_v, \gamma_{00}, \cdots, \gamma_{vv}, \sigma_{00}, \cdots, \sigma_{vv}) = 0 \qquad (u = 1, \cdots, w). \tag{6.3}$$

Further use symbol $P_n(A)$ for the words "probability that, when the sample has size n, A is true."

It has been proved that *the values* $\gamma_j = \gamma_j^\wedge$, $\gamma_{jk} = \gamma_{jk}^\wedge$, $\sigma_{jk} = \sigma_{jk}^\wedge$ *that make L the highest maximum, subject to the side conditions* (6.3) *are consistent estimates of the* γ*'s and* σ*'s*; i.e., for any positive η, the probabilities

$$P_n(|\gamma_j - \gamma_j^\wedge| \leqq \eta), \quad P_n(|\gamma_{jk} - \gamma_{jk}^\wedge| \leqq \eta), P_n(|\sigma_{jk} - \sigma_{jk}^\wedge| \leqq \eta),$$

all have limit 1 as $n \to \infty$.

If the joint probability distribution of the ϵ_j's is normal, these estimates are maximum-likelihood estimates. For that reason, according to a well-known theorem (see Wilks [49], pp. 138–139), they are consistent estimates. If the distribution of the ϵ_j's is not normal, the estimates are not maximum-likelihood estimates, but may be called, because of the method of their derivation, "quasi-maximum-likelihood estimates." In this case, their consistency has been proved by Mann and Wald [31], Part II, even under conditions more general than those here assumed (see last sentence of our §16).

We have to differentiate L with respect to γ_g, γ_{gh}, and σ^{gh} (g, h

[40] The density of the normal joint probability distribution of the ϵ_{jf}'s ($j = 0, \cdots, v; f = 1, \cdots, n$) is—see, for example, Wilks [49]—:

$$C|\sigma|^{-n/2} \exp\left(-\frac{1}{2}\sum_{j=0}^{v}\sum_{k=0}^{v}\sum_{f=1}^{n}\sigma^{jk}\epsilon_{jf}\epsilon_{kf}\right)d\epsilon_{01}\cdots d\epsilon_{vn}$$

$$= C|\sigma|^{-n/2} \exp\left(-\frac{n}{2}\sum_{j=0}^{v}\sum_{k=0}^{v}\sigma^{jk}q_{jk}\right)d\epsilon_{01}\cdots d\epsilon_{vn},$$

where C is a constant. To transform this into the joint distribution density of the z_{jf}'s ($j = 0, \cdots, v; f = 1, \cdots, n$) we have to multiply by the absolute value of the Jacobian of our equation (1.36) (for $f = 1, \cdots, n$), i.e., by the absolute value of

$$\frac{\partial(\epsilon_{01}, \cdots, \epsilon_{vn})}{\partial(z_{01}, \cdots, z_{vn})} = |\gamma|^n.$$

Thus the product

$$C \cdot |\gamma'| \cdot |\sigma|^{-n/2} \exp\left(-\frac{n}{2}\sum_{j=0}^{v}\sum_{k=0}^{v}\sigma^{jk}q_{jk}\right) \tag{6.2a}$$

gives the likelihood function to be maximized. Its logarithm (apart from a constant) is L in (6.2).

$=0, \cdots, v$); the latter is more convenient than differentiation with respect to σ_{jk} as it gives at once (the sign $\wedge$ indicating the values that maximize L)

$$(6.4)\qquad \left(\frac{\partial L}{\partial \sigma^{gh}}\right)^{\wedge} = \frac{1}{2}(2 - \delta_{gh})(\sigma_{gh}{}^{\wedge} - q_{gh}{}^{\wedge}) = 0 \quad (g, h = 0, \cdots, v),$$

$$(6.5)\qquad \sigma_{jk}{}^{\wedge} = q_{jk}{}^{\wedge} \qquad (j, k = 0, \cdots, v);$$

or in matrix form

$$(6.5a)\qquad \|\sigma^{\wedge}\| = \|q^{\wedge}\|.$$

For the other derivatives, we shall need the identities—from definitions (2.2), (6.1)—:

$$(6.6)\qquad q_{jk} = s_{jk} + \bar{\epsilon}_j\bar{\epsilon}_k,$$

$$(6.7)\qquad s_{jk} = \sum_{p=0}^{v}\sum_{q=0}^{v} \gamma_{jp} m_{pq} \gamma_{kq},$$

$$(6.7a)\qquad \|s\| = \|\gamma\|\cdot\|m\|\cdot\|\gamma\|'.$$

Hence by (6.6)

$$\frac{\partial q_{jk}}{\partial \gamma_g} = \frac{\partial \bar{\epsilon}_j}{\partial \gamma_g}\bar{\epsilon}_k + \frac{\partial \bar{\epsilon}_k}{\partial \gamma_g}\bar{\epsilon}_j = \delta_{jg}\bar{\epsilon}_k + \delta_{kg}\bar{\epsilon}_j,$$

$$(6.8)\qquad \left(\frac{\partial L}{\partial \gamma_g}\right)^{\wedge} = -\frac{1}{2}\sum_j\sum_k \sigma^{jk\wedge}\frac{\partial q_{jk}}{\partial \gamma_g}$$

$$= -\frac{1}{2}\left(\sum_k \sigma^{gk}\bar{\epsilon}_k{}^{\wedge} + \sum_j \sigma^{jg\wedge}\bar{\epsilon}_j{}^{\wedge}\right)$$

$$= -\sum_k \sigma^{gk\wedge}\bar{\epsilon}_k{}^{\wedge} = 0 \qquad (g = 0, \cdots, v),$$

a system of $(v+1)$ homogeneous linear equations in $\bar{\epsilon}_k{}^{\wedge}$ $(k=0, \cdots, v)$. Since its determinant σ^{-1} is not zero,

$$(6.9)\qquad \bar{\epsilon}_k{}^{\wedge} = 0 \qquad (k = 0, \cdots, v);$$

hence by (6.6)

$$(6.10)\qquad \|q^{\wedge}\| = \|s^{\wedge}\|,$$

and by (1.36) and (2.1)

$$(6.11)\qquad \gamma_j{}^{\wedge} = -\sum_{k=0}^{v} \gamma_{jk}{}^{\wedge}\bar{z}_k,$$

which is the usual kind of equation for estimating the constant term.

Finally, to obtain $\partial L/\partial\gamma_{gh}$, calculate, by (6.6), (6.9),

$$\begin{aligned}\left(\frac{\partial q_{jk}}{\partial\gamma_{gh}}\right) &= \left(\frac{\partial s_{jk}}{\partial\gamma_{gh}}\right)^\wedge + \frac{\partial\bar\epsilon_j}{\partial\gamma_{gh}}\,\bar\epsilon_k\wedge + \frac{\partial\bar\epsilon_k}{\partial\gamma_{gh}}\,\bar\epsilon_h\wedge \\ &= \left(\frac{\partial s_{jk}}{\partial\gamma_{gh}}\right)^\wedge \qquad \text{by (6.7)} \\ &= \sum_p\sum_q (\delta_{gj}\delta_{hp}m_{pq}\gamma_{kq}\wedge + \delta_{gk}\delta_{hq}m_{pq}\gamma_{jp}\wedge) \\ &= \sum_q \delta_{gj}m_{hq}\gamma_{kq}\wedge + \sum_p \gamma_{jp}\wedge m_{ph}\delta_{gk};\end{aligned}$$

hence

$$\left(\frac{\partial}{\partial\gamma_{gh}}\sum_j\sum_k \sigma^{jk}q_{jk}\right)^\wedge = 2\sum_k\sum_q \sigma^{gk}\wedge\gamma_{kq}\wedge m_{qh},$$

$$\left(\frac{\partial L}{\partial\gamma_{gh}}\right)^\wedge = \gamma^{hg}\wedge - \sum_k\sum_q \sigma^{gk}\wedge\gamma_{kq}\wedge m_{qh} = 0, \tag{6.12}$$

$$(\|\gamma\wedge\|')^{-1} - \|\sigma\wedge\|^{-1}\cdot\|\gamma\wedge\|\cdot\|m\| = 0. \tag{6.12a}$$

If there were no side conditions (6.3), the unknown γ's and σ's would have to be determined from (6.5), (6.11), and (6.12). However, this would be impossible since equations (6.12) are not independent of the rest. This is seen from the above matrix equations. We have by (6.5a), (6.7a), (6.10),

$$\begin{aligned}\|\sigma\wedge\| &= \|q\wedge\| = \|s\wedge\| = \|\gamma\wedge\|\cdot\|m\|\cdot\|\gamma\wedge\|', \\ \|\sigma\wedge\| &- \|\gamma\wedge\|\cdot\|m\|\cdot\|\gamma\wedge\|' = 0.\end{aligned} \tag{6.13}$$

Premultiplying this by $\|\sigma\wedge\|^{-1}$ and postmultiplying by $(\|\gamma\wedge\|')^{-1}$, we obtain (6.12a).[41] The system contains as many dependent equations as there are unknown γ_{jk}'s.[42]

§47

If we have as many side conditions (6.3) as there are unknown γ_{jk}'s (counting also each known σ_{jk} as one side condition), the necessary conditions for the maximum of L are of the form

$$\left(\frac{\partial L}{\partial\theta_t} + \sum_{u=1}^{w}\lambda_u\frac{\partial F_u}{\partial\theta_t}\right)^\wedge = 0,$$

$$F_u\wedge = 0 \ (t = 1, \cdots, w';\ u = 1, \cdots, w),$$

[41] See, for example, Aitken [1].

[42] "It is impossible to have estimates of all the parameters . . . on the basis of observations alone. However, if we have some a priori knowledge as to (their) values, then it may be possible to estimate all the unknown parameters." Mann and Wald [31], p. 201.

where the θ's denote w' unknown consistent estimates (all σ's and some γ's) and where the λ's are Lagrange multipliers. If the number of side conditions is exactly equal to the number of unknown γ_{jk}'s, then one solution is provided by all λ's being zero, since we then obtain the equations

$$(6.14) \qquad \left(\frac{\partial L}{\partial \theta_t}\right)^\wedge = 0, \qquad F_u = 0 \; (t = 1, \cdots, w'; u = 1, \cdots, w),$$

of which the first group is equivalent to (6.4), (6.8), and implies (6.12), so that we have the system

$$(6.15) \qquad \begin{aligned} \sigma_{jk}^\wedge &= \sum_{p=0}^{v} \sum_{q=0}^{v} \gamma_{jp}^\wedge m_{pq} \gamma_{kq}^\wedge, \\ \gamma_j^\wedge &= - \sum_{p=0}^{v} \gamma_{jp}^\wedge \bar{z}_p, \\ F_u^\wedge &= 0 \; (j, k = 0, \cdots, v; u = 1, \cdots, w). \end{aligned}$$

The problem has thus become that of a nonrestricted maximum of L, regarded as a function of σ's only. For any set of γ's, the equations (6.15) will provide a set of σ's such as will satisfy the necessary conditions for L maximum, or minimum. We have to see whether such a set of σ's makes L a maximum (and not, for example, a minimum); and whether this maximum is higher than other maxima, to fulfill the consistency property of the maximum-likelihood estimates.

§48

To show that the roots of (6.15) make L a maximum we have to show that the matrix

$$\left\| \frac{\partial^2 L}{\partial \sigma^{gh} \partial \sigma^{rs}} \right\| = \left\| (2 - \delta_{gh}) \frac{\partial \sigma_{gh}}{\partial \sigma^{rs}} \right\| \qquad \text{[by (6.4)]}$$

is negative definite, as it will be shown later (§49) that L will then be a maximum for the problem. We may omit a discussion of the constant terms γ_j as trivial.

Differentiate with respect to σ^{rs} the identity

$$\|\sigma\| \cdot \|\sigma\|^{-1} = I,$$

$$\frac{\partial \|\sigma\|}{\partial \sigma^{rs}} \cdot \|\sigma\|^{-1} + \|\sigma\| \cdot \frac{\partial \|\sigma\|^{-1}}{\partial \sigma^{rs}} = 0,$$

$$\frac{\partial \|\sigma\|}{\partial \sigma^{rs}} \cdot \|\sigma\|^{-1} = - \|\sigma\| \cdot \|\eta\|,$$

where any element of $\|\eta\|$ is one when $j=r$, $k=s$, and is zero otherwise:

$$\frac{\partial\|\sigma\|}{\partial\sigma^{rs}} = -\|\sigma\|\cdot\|\eta\|\cdot\|\sigma\|,$$

$$\frac{\partial^2 L}{\partial\sigma^{gh}\partial\sigma^{rs}} = -\frac{1}{2}(2-\delta_{gh})(2-\delta_{rs})(\sigma_{gr}\sigma_{hs}+\sigma_{hr}\sigma_{gs})$$

$$= -a_{gh,rs}, \quad \text{say.}[43]$$

If $g = h$, and $r = s$, $a_{gh,rs} = \sigma_{gr}\sigma_{hs}$;

if $g \neq h$, and $r = s$, $a_{gh,rs} = \sigma_{gr}\sigma_{hs} + \sigma_{hr}\sigma_{gs}$;

if $g = h$, and $r \neq s$, $a_{gh,rs} = \sigma_{gr}\sigma_{hs} + \sigma_{hr}\sigma_{gs}$;

if $g \neq h$, and $r \neq s$, $a_{gh,rs} = 2\,\sigma_{gr}\sigma_{hs} + 2\,\sigma_{hr}\sigma_{gs}$.

Let matrix $A=\|a_{gh,rs}\|$. Construct another matrix $B=\|b_{gh,rs}\|$ as follows: in the (g, h)th row and the (r, s)th column place the element $\sigma_{gr}\sigma_{hs}$. Then A can be obtained from B in the following way: if $h>g$, add the (h, g)th row to the (g, h)th row and delete the (h, g)th row; if $s>r$, add the (s, r)th row to the (r, s)th row and delete the (s, r)th row. The resulting matrix is A. Hence A is a diagonal minor of a Hermitian matrix cogredient with B. Therefore, if we succeed in proving that B is positive definite, it will follow that A is also positive definite,[44] and therefore $\|\partial^2 L/\partial\sigma^{gh}\partial\sigma^{rs}\|$ negative definite.

It remains to prove that the matrix B, whose elements are of the form $\sigma_{gr}\sigma_{hs}$, is positive definite. This follows from the assumptions of §46: the random sets $\{\epsilon_{jf}\}$, $\{\epsilon_{kf'}\}$ $(j, k=0, \cdots, v)$, are independently distributed, and $|\sigma_{jk}|\neq 0$. Since a moment matrix can be either positive definite or singular, the latter condition implies that $\|\sigma_{jk}\|$ is positive definite. On the other hand, the independence of the random ϵ-sets implies that

$$E(\epsilon_{gf}\epsilon_{hf'})(\epsilon_{rf}\epsilon_{sf'}) = E\epsilon_{gf}\epsilon_{rf}\cdot E\epsilon_{hf'}\epsilon_{sf'} = \sigma_{gr}\sigma_{hs},$$

so that B, too, can be considered a moment matrix. Since $|\sigma_{jk}|\neq 0$, each of the sets $\{\epsilon_{jf}\}$, $\{\epsilon_{kf'}\}$ is linearly independent. Hence the sets $\{\epsilon_{jk}\epsilon_{kf'}\}$ are linearly independent and therefore $|B|=|\sigma_{gr}\sigma_{hs}|\neq 0$, and B, too, is positive definite.

§49

It follows from the results of §46 that, whenever $\sigma_{jk}=s_{jk}$ $(j, k$

[43] The proof that follows and that in §49 are due to Herman Rubin, Cowles Commission.

[44] See, for example, Wedderburn [47], p. 100, Theorem 17.

$=0, \cdots, v$) L has the same constant value $L^\wedge$, whatever the values of the γ's. We have then, in fact, by (6.2)

$$L^\wedge = \log | \gamma^\wedge | + \frac{1}{2} \log | \sigma^\wedge |^{-1} - \frac{1}{2} \sum_{j=0}^{r} \sum_{k=0}^{r} \sigma^{jk\wedge} q_{jk}^\wedge ;$$

but since by (6.13),

$$| \sigma^\wedge | = | q^\wedge | = | s^\wedge | = | \gamma^\wedge | \cdot | m | \cdot | \gamma^\wedge | = | \gamma^\wedge |^2 \cdot | m |,$$

$$L^\wedge = - \frac{1}{2} \log | m | - \frac{(v+1)}{2}. \tag{6.16}$$

Since, by §48, $\|\partial^2 L / \partial\sigma^{ik} \partial\sigma^{pq}\|$ is negative definite, $L^\wedge$ is a maximum with respect to the σ's. We shall show it is the highest maximum. Denote by $\boldsymbol{\sigma}$ the set of σ_{jk}'s, and by $\boldsymbol{\gamma}$ the set of *unknown* γ_{jk}'s. Denote by $\mathbf{s}$ the set of s_{jk}'s; and since each s_{jk} is a function of γ_{jk}'s (6.7) we can write

$$\mathbf{s} = \mathbf{s}(\boldsymbol{\gamma}).$$

Whenever $\boldsymbol{\sigma} = \mathbf{s}(\boldsymbol{\gamma})$, $L = L^\wedge$:

$$L(\boldsymbol{\sigma}, \boldsymbol{\gamma}) = L[\mathbf{s}(\boldsymbol{\gamma}), \boldsymbol{\gamma}] = L^\wedge .$$

Change $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma}'$; and change $\boldsymbol{\sigma}$ to $\boldsymbol{\sigma}'$. Then either $\boldsymbol{\sigma}' = \mathbf{s}(\boldsymbol{\gamma}')$, in which case $L(\boldsymbol{\sigma}', \boldsymbol{\gamma}') = L^\wedge$. Or $\boldsymbol{\sigma}' \neq \mathbf{s}(\boldsymbol{\gamma}')$, in which case, by §48, $L(\boldsymbol{\sigma}', \boldsymbol{\gamma}') < L^\wedge$. This can be extended to the boundaries of the $\boldsymbol{\sigma}$-region, i.e., wherever $\|\sigma_{jk}\|$ and hence $\|\partial^2 L / \partial\sigma^{ik} \partial\sigma^{pq}\|$ is definite. In that region $L < L^\wedge$ if $\boldsymbol{\sigma} \neq \mathbf{s}(\boldsymbol{\gamma})$, and $L = L^\wedge$ if $\boldsymbol{\sigma} = \mathbf{s}(\boldsymbol{\gamma})$.

§50

In our particular case, (1.34), the unknown parameters are:

γ_0, γ_1, γ_2, corresponding to the γ_j's of the general case;

six covariances σ_{rk} (j, $k = 0, 1, 2$), corresponding to the σ_{jk}'s of the general case;

four coefficients α_1', α_2', β_1, β_2, corresponding to the γ_{jk}'s of the general case.

We should thus need four side conditions to obtain a system corresponding to (6.15) and have

$$\mathrm{F}_u = 0 \qquad (u = 1, \cdots, 4),$$

$$\sigma_{jk}^\wedge = \sum_{p=0}^{2} \sum_{q=0}^{2} \gamma_{jp}^\wedge m_{pq} \gamma_{kq}^\wedge \qquad (j, k = 0, 1, 2),$$

$$\gamma_j^\wedge = - \sum_{p=0}^{2} \gamma_{jp}^\wedge \bar{z}_p .$$

In the second group of equations one will recognize the equations (2.3), with $\sigma^\wedge$'s replacing the s's, and where $\gamma_{jp}{}^\wedge$'s become the estimates of the coefficients of z's in (1.34), forming the matrix

$$\left\| \begin{matrix} 1 & -\alpha_1'^\wedge & -\alpha_2'^\wedge \\ 1 & -\beta_1^\wedge & 0 \\ 1 & 0 & -\beta_2^\wedge \end{matrix} \right\| = \|\hat{\gamma}_{jk}\|. \tag{6.17}$$

Of the four side conditions two can be given by

$$\begin{aligned} \beta_1 &= c_1 \text{ (a constant)}, \\ \beta_2 &= c_2 \text{ (a constant)}; \end{aligned} \tag{6.18}$$

that is, we can combine trial values to β_1 and β_2. Four such combinations were tried (see Figure 2; the second combination falls outside of the relevant area):

$$\beta_1 = \beta_2 = 1; \quad \beta_1 = \beta_2 = 1.5; \quad \beta_1 = 1,\ \beta_2 = 1.5; \quad \beta_1 = 1.5,\ \beta_2 = 1.$$

In addition we have several inequalities, derived in §§18, 19, 20 from economic considerations. We can use any of them in the form of equalities: e.g., from (1.37)[45]

$$\sigma_{00}{}^\wedge = c_{00} - d^2,$$

where d is an arbitrary real number. As d changes, the solutions move within a region bounded by the condition

$$\sigma_{00}{}^\wedge = c_{00}; \tag{6.19}$$

i.e., (since $\sigma_{00}{}^\wedge = s_{00}{}^\wedge$), by the ellipse

$$\begin{aligned} c_{00} = {} & m_{00} + \alpha_1'^{\wedge 2} m_{11} + \alpha_2'^{\wedge 2} m_{22} \\ & - 2\alpha_1'^\wedge m_{01} - 2\alpha_2'^\wedge m_{02} + 2\alpha_1'^\wedge \alpha_2'^\wedge m_{02}, \end{aligned} \tag{6.20}$$

corresponding to the first of the equations (2.3). Similarly with the other inequalities of §§18–20.

Since by §49 the condition $\sigma_{jk} = s_{jk}$ makes the likelihood function the highest maximum regardless of the values of the α's and β's, any number of additional inequalities can be imposed as further side conditions.[46] If the four or more side conditions become incompatible with

[45] See, for example, Hancock [23].

[46] If there exists a point $x' = (x_1', x_2', \cdots, x_n')$ where a function $f(x) = f(x_1, x_2, \cdots, x_n)$ under the restrictions

$$\text{(i)} \qquad \phi_a(x) = 0 \quad (a = 1, \cdots, A), \qquad \psi_b(x) \geqq 0 \quad (b = 1, \cdots, B),$$

reaches an absolute maximum, and if that point x' is found to satisfy certain additional restrictions the observations, the boundary will not form a region on the diagram. This may be just what we want to know: for example, certain values of β_1 (labor monopsony as measured by the elasticity of a firm's pay roll with respect to its manpower) may not be compatible with: (1) the observed values of labor, capital, and output, as resulting in the computed moments m_{ik} and hence in the curves of the Figures 1, 2: (2) other economic considerations resulting in the boundary lines.

On the other hand, insofar as we do get regions of compatible estimates of our parameters, judgment may be formed about the limits within which our parameter sets may lie. This is done in §37 but merely as an illustration: the shortcomings of data, described in Chapter III, are too large.

§51

The results of Mann and Wald would, furthermore, permit, for any chosen significance level, the construction of confidence regions, one for each set of side *equations* (not inequalities) (6.3). If one or more nonlinear equations—such as (6.20)—occur in the system, we have, in general, one or more pairs of sets of roots, each set giving the same maximum value $L^\wedge$ to L (§49). Hence, we shall have two or more sets of consistent estimates (e.g., two points of intersections between ellipse and straight line): and confidence regions would have to be built around each of them. Correspondingly, the boundaries of the regions based on certain (two or more) inequalities will be widened somewhat.

As explained in §26, published data on manufacturing industries are such as make it difficult to decide what is the number measuring the size of the sample to be properly inserted into the confidence-region formula. In the example used, this number (at least 85) is probably large enough to make consistent estimates useful, and hence to make the construction of regions like those in our diagrams useful enough. But it is not certain enough to make it worth while to compute numerically the confidence regions around each set of estimates.

Cowles Commission for Research in Economics
The University of Chicago

(ii) $$\psi_c(x) \geqq 0 \qquad (c = 1, \cdots, C),$$

then evidently every point x'' where an absolute maximum of $f(x)$ under the combined restrictions

(iii) $$\phi_a(x) = 0, \qquad \psi_b(x) \geqq 0, \qquad \psi_c(x) \geqq 0,$$

is reached, is a point where $f(x)$ reaches an absolute maximum restricted by (i) only, and which satisfies (ii), and conversely (H. Rubin).

REFERENCES

[1] A. C. Aitken, *Determinations and Matrices*, Second edition, New York, Interscience Publishers, 1942, 135 pp.

[2] R. G. D. Allen, *Mathematical Analysis for Economists*, London, Macmillan, 1938, 548 pp.

[3] M. Bronfenbrenner, "Production Functions: Cobb-Douglas, Interfirm, Intrafirm," Econometrica, Vol. 12, January, 1944, pp. 35–44.

[4] Colin Clark, *The Conditions of Economic Progress*, London, Macmillan, 1940, 504 pp.; esp. references to work of Max Brown on New Zealand, pp. 384–385, 407.

[5] Charles W. Cobb and Paul H. Douglas, "A Theory of Production," *American Economic Review*, Vol. 18, Supplement, March, 1928, pp. 139–165.

[6] F. N. David and J. Neyman, "Extension of the Markoff Theorem on Least Squares," *Statistical Research Memoirs*, Vol. II, 1938, pp. 105–116.

[7] Harold T. Davis, *The Theory of Econometrics*, Bloomington, Indiana, The Principia Press, 1941, 482 pp., esp. pp. 159 ff.

[8] Paul H. Douglas, *The Theory of Wages*, New York, Macmillan, 1934, 639 pp., esp. pp. 113–158, 159–166 (Massachusetts), 167–172 (New South Wales).

[9] ———, (with Marjorie L. Handsaker), "The Theory of Marginal Productivity Tested by Data for Manufacturing in Victoria," *Quarterly Journal of Economics*, Vol. 52, November, 1937, pp. 1–36; February, 1938, pp. 215–254.

[10] ———, (with M. Bronfenbrenner), "Cross-Section Studies in the Cobb-Douglas Function," *Journal of Political Economy*, Vol. 47, December, 1939, pp. 761–785.

[11] ———, (with Grace T. Gunn), "Further Measurements of Marginal Productivity," *Quarterly Journal of Economics*, Vol. 54, May, 1940, pp. 399–428.

[12] ———, (with Grace T. Gunn), "The Production Function for American Manufacturing in 1919," *American Economic Review*, Vol. 31, March, 1941, pp. 67–80.

[13] ———, (with Grace T. Gunn), "The Production Function for Australian Manufacturing," *Quarterly Journal of Economics*, Vol. 56, November, 1941, pp. 108–129.

[14] ———, (with Grace T. Gunn), "The Production Function for American Manufacturing for 1914," *Journal of Political Economy*, Vol. 50, August, 1942, pp. 595–602.

[15] ———, (with Patricia Daly and Ernest Olson), "The Production Function for Manufacturing in the United States, 1904," *Journal of Political Economy*, Vol. 51, February, 1943, pp. 61–65.

[16] ———, (with Patricia Daly), "The Production Function for Canadian Manufactures," *Journal of American Statistical Association*, Vol. 39, June, 1943, pp. 178–186.

[17] David Durand, "Some Thoughts on Marginal Productivity, with Special Reference to Professor Douglas' Analysis," *Journal of Political Economy*, Vol. 45, December, 1937, pp. 740–758.

[18] Ragnar Frisch, *Pitfalls in the Statistical Construction of Demand and Supply Curves*, Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung, Neue Folge, Heft 5, Leipzig, Buske, 1933, 39 pp.

[19] N. GEORGESCU-ROEGEN, "Fixed Coefficients of Production and the Marginal Productivity Theory," *Review of Economic Studies*, Vol. 3, October, 1935, pp. 40–49.
[20] R. GIBRAT, *Les inégalités économiques*, Paris, Sirey, 1931, 296 pp.
[21] TRYGVE HAAVELMO, "The Statistical Implications of a System of Simultaneous Equations," ECONOMETRICA, Vol. 11, January, 1943, pp. 1–12.
[22] ———, "The Probability Approach in Econometrics," ECONOMETRICA, Vol. 12, Supplement, July, 1944, 118 pp.; also issued as Cowles Commission Papers, New Series, No. 4.
[23] HARRIS HANCOCK, *Theory of Maxima and Minima*, Boston, Ginn, 1917, 193 pp., esp. pp. 149 ff.
[24] J. R. HICKS, *The Theory of Wages*, London, Macmillan, 1935, 247 pp., esp. Appendix, pp. 233-239.
[25] ———, "Distribution and Economic Progress, A Revised Version," *Review of Economic Studies*, Vol. 4, October, 1936, pp. 1–12, esp. equation (30.3), pp. 2–3.
[26] ———, *Théorie mathématique de la valeur en régime de libre concurrence*, Paris, Hermann, 1937, 55 pp., esp. p. 47.
[27] HAROLD HOTELLING, "Problems of Prediction," *American Journal of Sociology*, Vol. 48, July, 1942, pp. 61–76.
[28] LAWRENCE R. KLEIN, "Pitfalls in the Statistical Determination of the Investment Schedule," ECONOMETRICA, Vol. 11, July–October, 1943, pp. 246–258.
[29] FRANK H. KNIGHT, *Risk, Uncertainty and Profit*, Boston, Houghton Mifflin, 1921, 381 pp. (reprinted by London School of Economics, 1933), esp. pp. 100–101.
[30] FRITZ MACHLUP, *International Trade and the National Income Multiplier*, Philadelphia, Blakiston, 1943, 237 pp.
[31] H. B. MANN and ABRAHAM WALD, "On the Statistical Treatment of Linear Stochastic Difference Equations," ECONOMETRICA, Vol. 11, July–October, 1943, pp. 173–220.
[32] J. MARSCHAK, "Personal and Collective Budget Functions," *Review of Economic Statistics*, Vol. 21, November, 1939, pp. 161–170.
[33] ———, "Income Inequality and Demand Studies: A Note," ECONOMETRICA, Vol. 11, April, 1943, pp. 163–166.
[34] H. L. MOORE, *Economic Cycles: Their Law and Cause*, New York, Macmillan, 1914, 149 pp., esp. Chapters IV and V.
[35] JACOB L. MOSAK, "Interrelations of Production, Price, and Derived Demand," *Journal of Political Economy*, Vol. 46, December, 1938, pp. 761–787, esp. p. 772, footnote 8.
[36] A. C. PIGOU, "A Method of Determining the Numerical Value of Elasticities of Demand," *Economic Journal*, Vol. 20, 1910, pp. 636–640; and revised in his *Economics of Welfare*, Fourth edition, Appendix II, pp. 782–788, London, Macmillan, 1932.
[37] ———, *The Theory of Unemployment*, London, Macmillan, 1933, 319 pp.
[38] M. W. REDER, "An Alternative Interpretation of the Cobb-Douglas Function," ECONOMETRICA, Vol. 11, July–October, 1943, pp. 259–264.
[39] HENRY SCHULTZ, *The Theory and Measurement of Demand*, Chicago, The University of Chicago Press, 1938, 817 pp.
[40] T. DE SCITOVSZKY, "A Note on Profit Maximisation and Its Implications," *Review of Economic Studies*, Vol. 11, Winter, 1943, pp. 57–60.

[41] SUMNER H. SLICHTER, "Present Savings and Postwar Markets," *Harvard Business Review*, Vol. 22, Autumn, 1943, Part 2, 73 pp.

[42] GEORGE J. STIGLER, *Production and Distribution Theories, The Formative Period*, New York, Macmillan, 1941, 392 pp., especially Chapter XII, "Euler's Theorem and the Marginal Productivity Theory," pp. 320–387.

[43] ———, *The Theory of Competitive Price*, New York, Macmillan, 1942, 197 pp., esp. p. 130.

[44] GERHARD TINTNER, "A Note on the Derivation of Production Functions from Farm Records," ECONOMETRICA, Vol. 12, January, 1944, pp. 26–34.

[45] UNITED STATES DEPARTMENT OF COMMERCE, BUREAU OF THE CENSUS, *Thirteenth Census of the United States Taken in the Year 1910*, Vol. VIII, *Manufactures, 1909*, Washington, 1913, 845 pp.

[46] LÉON WALRAS, *Éléments d'économie politique pure*, Paris, Pichon, 1926, 487 pp., esp. pp. 374–376 and footnote.

[47] J. H. M. WEDDERBURN, *Lectures on Matrices*, New York, American Mathematical Society, Colloquium Publications, Vol. XVII, 1934, 200 pp.

[48] KNUT WICKSELL, *Lectures on Political Economy*, Vol. I, London, Routledge, 1935, 299 pp., esp. pp. 127 ff.

[49] S. S. WILKS, *Mathematical Statistics*, Princeton, Princeton University Press, 1943, 284 pp., esp. §3.23, and §6.24.

[50] ELMER J. WORKING, "What Do Statistical 'Demand Curves' Show?" *Quarterly Journal of Economics*, Vol. 41, 1927, pp. 212–235.