Cowles Foundation Paper 3

STOCHASTIC MODELS OF ECONOMIC FLUCTUATIONS*

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I. THE PROBLEM

1. Dynamical systems serving as models for explanation of economic fluctuations are often described by sets of linear difference equations with constant coefficients. These equations are usually of the stochastic type: that is, they are nonhomogeneous and the nonhomogeneous term (the "disturbance") is a random variable.1

Since the economist's interest is frequently focused on one of the variates of the system, e.g., the level of employment, and also because some of the variates cannot be observed with the desired degree of accuracy, it is common practice to reduce the original system of equations to an equivalent single equation in one of the variates.2

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1 One example would be Haavelmo's model (Econometrica, Vol. 11, January, 1943, p. 4):

\[ u_t = \alpha u_t + \beta + \varepsilon_t, \]
\[ r_t = u_t = \varepsilon(u_t - u_{t-1}) + \eta_t, \]

where \( u \) is consumption, \( r \) is income, and \( x \) and \( y \) are random variables.

Koopmans' model (Annals of Mathematical Statistics, Vol. 3, March, 1942, p. 14) has similar formal properties, although it refers to only one sector of the economy:

\[ x_t = \alpha - \beta x_t + \varepsilon_t', \]
\[ y_t = \gamma + \delta x_t + \varepsilon_t'', \]

where \( x \) and \( y \) are respectively the price and supply of hogs and the \( x \)'s are again random variables.

Samuelson's system (Review of Economic Statistics, Vol. 21, May, 1939, p. 26) belongs to the same class, although the effects of random variables have not been shown explicitly:

\[ Y_t = \alpha Y_{t-1} + \eta_t, \]
\[ C_t = \alpha Y_{t-1}, \]
\[ I_t = \beta(C_t - C_{t-1}), \]

where \( Y \) is national income, \( C \) consumption expenditure, \( I \) private investment, and \( \eta \) governmental expenditure.

The references are selected without consideration for historical sequence. Earlier papers, e.g., that of Frisch, """Propagation Problems and Impulse Problems in Dynamic Economies," Economic Essays in Honor of Gustav Cassel, 1933, pp. 171-205), are not always explicit enough for the purposes of our discussion.

Reprinted from Econometrica, 12, 1944
The process of reduction yields, in general, a higher-order difference equation, to be called the “reduced equation,” which is also nonhomogeneous. The nonhomogeneous term will contain, possibly in addition to other terms, a linear combination of the “disturbances” of the initial system of equations. This “composite disturbance” of the “reduced equation” differs in its properties from the original disturbances. In particular, it may be autocorrelated even though the disturbances of the original system were nonautocorrelated.

2. Before proceeding to more general investigation of the properties of the reduced equation, it appears advisable to consider a simple example. Let there be given the system

\[
\begin{align*}
X_t &= aX_{t-1} + b_1Y_{t-1} + b_2Y_{t-2} + \epsilon_t', \\
Y_t &= cX_{t-1} + \epsilon_t'',
\end{align*}
\]

where the \(\epsilon\)'s are the “disturbances.” By substituting \(Y\) from the second equation into the first equation we obtain the “reduced equation”

\[
X_t = aX_{t-1} + b_1cX_{t-2} + b_2cX_{t-3} + \epsilon_t' + b_1\epsilon_{t-1}'' + b_2\epsilon_{t-2}'',
\]

which can be rewritten as

\[
X_t + \alpha_1X_{t-1} + \alpha_2X_{t-2} + \alpha_3X_{t-3} = \eta_t,
\]

where

\[
\eta_t \equiv \epsilon_t' + b_1\epsilon_{t-1}'' + b_2\epsilon_{t-2}''
\]

is the “composite disturbance.”

\textsuperscript{2} Thus Koopmans eliminates \(y\) from his system (supra, footnote 1) and obtains the equation

\[
x_t = \epsilon - \xi x_{t-1} + z_t,
\]

where \(\epsilon\) and \(\xi\) are functions of \(\alpha, \beta, \gamma, \delta\), while

\[
x_t = z_t' - \beta z_t''.
\]

Similarly, Samuelson obtains

\[
Y_t = g_t + \alpha(1 + \beta)Y_{t-1} - \alpha\beta Y_{t-2},
\]

where, again, the disturbance is not introduced explicitly.

\textsuperscript{2} In Koopmans’ model, equation (f2.1).

\textsuperscript{4} In Koopmans’ model, \(z_t = z_t' - \beta z_t''\).

\textsuperscript{5} A random variable is said to be nonautocorrelated when all of its autocorrelation coefficients \(r_k\) (except for \(k=0\)) are zero. \(r_k\) for a random variable \(z_t\) is defined as follows:

\[
r_k = \frac{\mathbb{E}(x_t x_{t+k}) - \mathbb{E}(x_t) \mathbb{E}(x_{t+k})}{\mathbb{E}(x_t^2) - \mathbb{E}(x_t)^2},
\]

where \(\mathbb{E}\) is the symbol of mathematical expectation. (Cf. Herman Wold, A Study in the Analysis of Stationary Time Series, Uppsala 1938, p. 12.)
Now if the $\varepsilon$'s are nonautocorrelated random variables with zero means and a finite covariance matrix $||\sigma_{ij}||$, we find that the lag product moment

$$E(\eta_i \eta_{i+1}) = E[(\varepsilon_{i} + b_1 \varepsilon_{i-1} + b_2 \varepsilon_{i-2})(\varepsilon_{i+1} + b_1 \varepsilon_{i+2} + b_2 \varepsilon_{i+1})]$$

$$= b_1 \sigma_{22} + b_2 \sigma_{12} + R \neq 0,$$

where $R$ is due to correlation among the $\varepsilon$'s. Hence $\eta_i$ may be autocorrelated even though the $\varepsilon$'s were nonautocorrelated. It is of interest to note that the autocorrelation properties of $\eta$ are partly due to correlations between $\varepsilon_i$ and $\varepsilon_i''$. However, even if we had $\sigma_{ij} = 0$ for all $i \neq j$ and all lags, $\eta$ would still remain autocorrelated because it has the nature of a "moving average" and pairs of its lagged values contain common elements, provided the lag does not exceed a certain integer. [For instance, it is easily seen that in case of $\eta_i$, defined by (1.2.4) we should have

$$(1.2.6) \quad r_h = 0 \quad \text{for} \quad k \geq 3.$$ 

In economic investigations the interest centers as a rule on the variate itself [$(X$ in Eq. (1.2.3)] rather than the "disturbance" $(\eta)$. In particular we may want to examine the autocorrelation properties of $X$, say its "periodicity," and also obtain a "forecast equation" (or "forecast curve"):

$$(1.2.7) \quad E(X_{t+k} | X_t, X_{t-1}, \ldots) = f(X_t, X_{t-1}, \ldots) \quad (k = 1, 2, \ldots),$$

where $E(\cdot)$ is the customary symbol for conditional expectations. Now if $\eta_i$ in (1.2.3) were nonautocorrelated, as it is assumed to be in Wold's treatment of stochastic difference equations, the autocorrelation properties of $X$ would depend on the ("structural") coefficients $\alpha_1, \alpha_2, \alpha_3$, and, similarly, the forecast equation would involve the values of $\alpha_1, \alpha_2, \alpha_3$ only. For instance, we should have

$$(1.2.8) \quad E(X_{t+k} | X_t, X_{t-1}, X_{t-2}) = - \alpha_1 X_t - \alpha_2 X_{t-1} - \alpha_3 X_{t-2} + \bar{E}.$$ 

As soon, however, as $\eta_i$ is assumed to be autocorrelated we must consider the conditional expectations of future disturbances, since the latter are stochastically dependent on their predecessors. Thus instead of (1.2.8) we should have

$$(1.2.81) \quad E(X_{t+1} | X_t, X_{t-1}, X_{t-2}) = - \alpha_1 X_t - \alpha_2 X_{t-1} - \alpha_3 X_{t-2} + E(\eta_t | X_t, X_{t-1}, X_{t-2}),$$

where the last term depends on the autocorrelation properties of $\eta$ and need not vanish if $\eta$ is autocorrelated.

* Cf. Wold, op. cit., p. 121.
3. Another important point is that in a stochastic difference equation with
an autocorrelated "disturbance," of which (1.2.3) is an example, the autocorrelation pattern (the "correlogram") of the observed variate depends not only on the "structural coefficients" [the \( \alpha \)'s in (1.2.3)] but also on the correlogram of the "disturbance."

This can easily be shown in the simple case \( [\mathcal{E}(\eta_t) = 0] \)

\[
\begin{align*}
X_1 &= \eta_1, \\
(1.3.11) \\
X_t &= \alpha X_{t-1} + \eta_t \\
(1.3.12)
\end{align*}
\]

where \( |\alpha| < 1 \) and the autocorrelation coefficients of \( \eta_t \), to be denoted by \( \rho_k \) are given by

\[
\begin{cases}
\rho_1 \neq 0, \\
\rho_k = 0 & \text{(for } k = 2, 3, \ldots \text{)}. \\
\end{cases}
(1.3.2)
\]

From (1.3.11) and (1.3.12) it follows that \( X_t \) can be expressed as a linear combination of the \( \eta \)'s:

\[
X_t = \sum_{j=0}^{t-1} \alpha^j \eta_{t-j} \\
(1.3.3)
\]

Hence by multiplying together the respective members of (1.3.12) and (1.3.3) and taking the expectation of the products we obtain

\[
(1.3.4) \quad \mathcal{E}(X_t X_s) - \alpha \mathcal{E}(X_{t-1} X_s) = \sum_{j=0}^{t-1} \alpha^j \mathcal{E}(\eta_t \eta_{t-j}).
\]

Now if \( \eta \) were nonautocorrelated, we could choose \( \tau = t-1 \), and, by dividing by \( \mathcal{E}(X^2) \), obtain the autocorrelation coefficient \( r_t \) of \( X \) from

\[
(1.3.5) \quad r_t - \alpha = 0,
\]

since all the expectations on the right-hand side would vanish. Then by setting

\[
\tau = t - 2, t - 3, \ldots
\]

we should obtain

\[
(1.3.6) \quad r_k - \alpha r_{k-1} = 0 \quad (k = 2, 3, \ldots),
\]

which is the difference equation satisfied by the correlogram of \( X \).

Since, however, we have assumed that the first autocorrelation coefficient \( (\rho_1) \) of \( \eta \) does not vanish, (1.3.5) no longer holds though (1.3.6) still does. For in this case we have, again setting \( \tau = t-1 \),

\[
(1.3.7) \quad r_t - \alpha = \sum_{j=0}^{t-2} \alpha^j \frac{\mathcal{E}(\eta_t \eta_{t-j})}{\mathcal{E}(X^2)}.
\]
where

$$
E(X_r) = E\left( \sum_{j=0}^{r-1} \alpha^j \eta_{r-j} \right)^2
$$

(1.3.8)

$$
= (1 + \alpha^2 + \ldots + \alpha^{2(r-1)}) E(\eta^2)
+ 2(\alpha + \alpha^3 + \ldots + \alpha^{2(r-1)-1}) E(\eta^2) \rho_1.
$$

Hence

$$
r_1 - \alpha = \frac{\rho_1 E(\eta^2)}{E(\eta^2) \left[ \sum_{j=0}^{r-1} \alpha^{2j} + 2 \rho_1 \sum_{j=0}^{r-2} \alpha^{2j+1} \right]}
$$

(1.3.9)

$$
= \frac{\rho_1}{\sum_{j=0}^{r-1} \alpha^{2j} + 2 \rho_1 \sum_{j=0}^{r-2} \alpha^{2j+1}}
$$

and $r_1$ depends on both $\alpha$ and $\rho_1$.

4. The examples given in Sections 2 and 3 of this paper served to show (a) that the “reduced equation” may have an autocorrelated “disturbance” and (b) that in a stochastic difference equation with an autocorrelated “disturbance” the correlogram of the observed variate is partly determined by the correlogram of the “disturbance.” That this phenomenon has not been given enough attention is probably due to the fact that the original set of difference equations is usually treated as a homogeneous one and the “disturbance” is introduced as a deus ex machina only after the reduction process has been completed. When this deficiency is remedied, certain implications for the theory of economic fluctuations become apparent. The “forecast curve” for one variate, based on its own correlogram, is still a combination of superimposed harmonics, but with entirely different initial values. The important point is that the forecasts must be based not only on the coefficients of lagged values of the variate (“structural coefficients”) but also on the autocorrelation properties of the “disturbance.” It might well be that some of the distrust with which the “literary” economists have viewed the “mathematical” business-cycle theory has arisen from their opposition to the unrealistic postulate of nonautocorrelated “disturbances.”

7 This procedure may in some cases prove to be quite harmless. For example, in the case of Koopmans’ system the “composite disturbance” $x_t = x_t' - \beta t'$ is nonautocorrelated if $x_t'$ and $z$ are nonautocorrelated. This is a general property of “composite disturbances” consisting of simultaneous values of the original “disturbances.”
II. NOTATION

In order to facilitate mathematical operations it will be found convenient to adopt the following notation.

The Boolean operator $\mathcal{E}$ will be equivalent to translation by one time unit:

\[(2.1) \quad \mathcal{E}X_t = X_{t+1},\]

and the subscript $t$ will be omitted.

Thus

\[(2.2) \quad X_{1,t+1} = aX_{1,t} + bX_{2,t} + \epsilon_{1,t}\]

will be written as

\[(2.3) \quad \mathcal{E}X_1 = aX_1 + bX_2 + \epsilon_1.\]

Also, for the sake of brevity, we shall write

\[(2.4) \quad \mathcal{E}X = X',\]

\[(2.5) \quad \mathcal{E}^nX = X^{(n)},\]

so that (2.3) could also be written as

\[(2.5) \quad X_1' = aX_1 + bX_2 + \epsilon_1\]

(since no differential operations are used, there should be no danger of confusion).

III. TRANSFORMATION TO A FIRST-ORDER SYSTEM

1. Now consider the $n$th-order system

\[(3.1.1) \quad \mathcal{E}^nX^{(0)} + A_{1}^{(0)}\mathcal{E}^{n-1}X^{(0)} + \cdots + A_{n}^{(0)}X^{(0)} = \mathcal{E}\epsilon^{(0)},\]

where

\[(3.1.2) \quad X^{(0)} = \{x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}\},\]

\[(3.1.3) \quad \epsilon^{(0)} = \{\epsilon_1^{(0)}, \epsilon_2^{(0)}, \ldots, \epsilon_N^{(0)}\},\]

\[(3.1.4) \quad A_s^{(0)} = \|a_{ij}^{(0)}\| \quad (i, j = 1, 2, \ldots, N)\]

and $\{\}$ denotes a column vector. As in differential equations,\(^8\) this

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can be transformed into a first-order system

\begin{equation}
X' + AX = \epsilon',
\end{equation}

where

\begin{equation}
A = \begin{bmatrix} a_{ij} \end{bmatrix} \quad (i, j = 1, 2, \ldots, N; \ldots, Nh),
\end{equation}

by defining a set of variables

\begin{equation}
x_{(pN)+q} = E^p x_q^{(0)} = \sum_{s=1}^{\infty} E^s x_s \quad (q = 1, 2, \ldots, N; p = 0, 1, \ldots, h - 1)
\end{equation}

so that

\begin{equation}
X = \{x_1, x_2, \ldots, x_{Nh}\} = \{X^{(0)}, EX^{(0)}, \ldots, E^nX^{(0)}\},
\end{equation}

and adjoining the defining equations as a part of the "normal system."

It must be observed, however, that the new vector \(\epsilon\) will acquire a set of zero components.

\begin{equation}
\epsilon = \{0, 0, \ldots, 0, \epsilon_1^{(0)}, \epsilon_2^{(0)}, \ldots\}.
\end{equation}

This will imply special properties of "normal" first-order system derived by the above process from a higher-order system. Otherwise, in what follows the "normal" system

\begin{equation}
X' + AX = \epsilon'
\end{equation}

will be discussed because the higher-order systems can be regarded as its special cases.

2. As an example, consider the system (1.2.1) discussed above. In order to conform with the new notation we shall rewrite it as

\begin{equation}
\begin{cases}
E^2x_1^{(0)} = a_{111}^{(0)}Ex_1^{(0)} + a_{112}^{(0)}Ex_2^{(0)} + a_{112}^{(0)}x_1^{(0)} \\
E^2x_2^{(0)} = a_{211}^{(0)}Ex_1^{(0)} + a_{212}^{(0)}Ex_2^{(0)} + a_{212}^{(0)}x_1^{(0)} \\
\quad + a_{222}^{(0)}x_2 + E\epsilon_1^{(0)},
\end{cases}
\end{equation}

or, in matrix form,

\begin{equation}
E^2 \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} = \begin{bmatrix} a_{111}^{(0)} & a_{112}^{(0)} \\ a_{211}^{(0)} & a_{222}^{(0)} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} + \begin{bmatrix} a_{112}^{(0)} & a_{122}^{(0)} \\ a_{212}^{(0)} & a_{222}^{(0)} \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} + E \begin{bmatrix} \epsilon_1^{(0)} \\ \epsilon_2^{(0)} \end{bmatrix}.
\end{equation}

\(^4\) The example in Section III-2 contains such a "partly empty" random vector [cf. eq. (3.2.10)].
The relationships between the old and new symbols are as follows:

\[
\begin{align*}
X &= x_1^{(0)}, \\
Y &= x_2^{(0)}, \\
\epsilon' &= E\epsilon_1^{(0)}, \\
\epsilon'' &= E\epsilon_2^{(0)}, \\
a &= a_{111}^{(0)}, \\
b_1 &= a_{112}^{(0)}, \\
c &= a_{211}^{(0)}, \\
b_2 &= a_{222}^{(0)}, \\
o &= a_{211}^{(0)} = a_{111}^{(0)} = a_{212}^{(0)} = a_{222}^{(0)}.
\end{align*}
\]

(3.2.3)

Now we redefine our system as follows:

\[
\begin{align*}
x_1 &= x_1^{(0)}, \\
x_2 &= x_2^{(0)}, \\
x_3 &= E x_1^{(0)} = E x_1, \\
x_4 &= E x_4^{(0)} = E x_2.
\end{align*}
\]

(3.2.4)

Then we can rewrite (3.1.1) in terms of \(x_1, x_2, x_3, x_4\) and so avoid the \(E^2\) operator. Thus we obtain a first-order system with 4 unknowns:

\[
\begin{align*}
E x_1 &= x_3, \\
E x_2 &= x_4, \\
E x_3 &= a^{(0)} x_1 + a^{(0)} x_2 + a^{(0)} x_3 + a^{(0)} x_4 + E\epsilon_1^{(0)}, \\
E x_4 &= a_{212}^{(0)} x_1 + a_{222}^{(0)} x_2 + a_{211}^{(0)} x_3 + a_{221}^{(0)} x_4 + E\epsilon_2^{(0)}.
\end{align*}
\]

(3.2.5)

This, written in matrix form, becomes

\[
EX = AX + \epsilon
\]

(3.2.6)

or

\[
X' = AX + \epsilon',
\]

(3.2.7)

where

\[
X = \{x_1, x_2, x_3, x_4\},
\]

(3.2.8)

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a_{112}^{(0)} & a_{122}^{(0)} & a_{111}^{(0)} & a_{121}^{(0)} \\
a_{212}^{(0)} & a_{222}^{(0)} & a_{211}^{(0)} & a_{221}^{(0)}
\end{bmatrix},
\]

(3.2.9)

and

\[
\epsilon = \{0, 0, \epsilon_1^{(0)}, \epsilon_2^{(0)}\}.
\]

(3.2.10)
IV. "REDUCTION" TO AN EQUATION IN ONE VARIABLE

After having shown how any system of equations of order \( k \) in \( N \) variates can be reduced to a system of first order in \( Nk \) variates, we shall now show how any system of first order in \( N \) variates can be reduced to one equation of higher order (the "reduced equation") in one of the variates. Thus, it will have been shown that any system of order \( k \) in \( N \) variables can be reduced to one equation in one variate.

In order to investigate the properties of the "reduced equation" in one of the variates of the normal system, it will suffice to show how the process of reduction is performed.\(^\text{10}\) In order to be able to eliminate all but one variate (say \( x_1 \)), the operator \( E \) must be applied \( N \) times to the matrix difference equation

\[
X' + AX = \epsilon'
\]

so that the following system of \( N \) matrix equations is obtained:

\[
E^{j+1}X + A^jEX = E^{j+1} \epsilon \\
(j = 0, 1, 2, \ldots, N - 1).
\]

As pointed out by Samuelson,\(^\text{11}\) this is a system in \((N-1)(N+1) = N^2 - 1\) unknowns

\[
\begin{pmatrix} x_1 & x_2 & \cdots & x_N \\
x_1' & x_2' & \cdots & x_N' \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{(N)} & x_2^{(N)} & \cdots & x_N^{(N)}
\end{pmatrix}
\]

since \((x_1, x_1', \cdots, x_1^{(N)})\) are treated as known. Now the number of equations available is \(N^2\) since \( A \) is an \( N \)-rowed matrix and \( j = 0, 1, 2, \cdots, N - 1 \). Hence it is possible to eliminate all the unknowns listed in (4.3) and in this manner to obtain an \( N \)th-order\(^\text{12}\) difference equation in \( x_1 \), say

\[
E^N x_1 + a_1 E^{N-1} x_1 + \cdots + a_{N-1} E x_1 + a_N x_1 = E^N \eta
\]

where \( E^N \eta \) is a linear combination of the components of the matrix\(^\text{13,14}\).


\(^\text{11}\) See reference in footnote 10.

\(^\text{12}\) If the resulting matrix is singular, (4.4) might turn out to be of lower order.

\(^\text{13}\) If the system was obtained by a transformation of the type (3.1.7) then some columns of this matrix would be zero vectors.
\[
\begin{pmatrix}
\epsilon_1', & \epsilon_2', & \cdots, & \epsilon_N' \\
\epsilon_1'', & \epsilon_2'', & \cdots, & \epsilon_N'' \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_1^{(N)}, & \epsilon_2^{(N)}, & \cdots, & \epsilon_N^{(N)}
\end{pmatrix}
\]

say

\[
\mathbb{E}^N \eta_t = \sum_{i,j=1}^N c_{ij} \xi_t^{(i)}.
\]

This is a sum of the "moving average" type of variables with a correlogram characterized by zero autocorrelation coefficients for lags of \( N \) or more units provided the \( \epsilon \)'s are nonautocorrelated.

In this special case, discussed by A. Wald and H. B. Mann\(^{16}\) where the \( \epsilon \)'s are nonautocorrelated, a part of the correlogram of \( x_1 \) will depend on the elements of the matrix \( A \) as well as the covariance matrix of the \( \epsilon \)'s.

V. AUTOCORRELATION PATTERN IN "REDUCED EQUATIONS"

It has been shown that the system

\[
X' + AX = \epsilon'
\]

is in general equivalent to a single equation in one variable (we now drop the subscript "1" and revert to customary notation), say

\[
x_t + a_1 x_{t-1} + \cdots + a_N x_{t-N} = \eta_t,
\]

where \( \eta_t \) is a sum of moving averages of order \( N - 1 \). In order to obtain the autocorrelation coefficients of \( x \) (denoted by \( r_k \), \( k \) being the lag) in terms of \( a \)'s and of the autocorrelation coefficients of \( \eta \) (denoted by \( \rho_k \)), we follow the procedure of H. Wold,\(^{16}\) writing

\[
x_t = \eta_t + b_1 \eta_{t-1} + \cdots + b_{r-1} \eta_1,
\]

where \( b \)'s are functions of the \( a \)'s.\(^{17}\)

\(^{14}\) The explicit values of \( a \)'s in the reduced equation (4.4) and of the \( c \)'s in (4.6) can be obtained easily, but they are not needed in this context.


\(^{17}\) A simple example of this method was given in Section I-3 of this paper.
By multiplying the respective numbers of (5.2) and (5.3) and by taking the expectations we obtain the lag moments

\[ E(x_t x_t) + a_1 E(x_{t-1} x_t) + \cdots + a_N E(x_{t-N} x_t) = E(\eta_t \eta_t) + b_1 E(\eta_t \eta_{t-1}) + \cdots + b_{t-N} E(\eta_t \eta_0). \]  

(5.4)

It is immediately clear that if

\[ t \leq t - N \]

(5.5)

the right-hand member of (5.4) vanishes and we obtain

\[ r_k + a_1 r_{k-1} + \cdots + a_N r_{k-N} = 0 \quad \text{provided } k \geq N. \]

(5.6)

Thus the correlogram still satisfies the difference equation corresponding to (5.2) (as in the case of nonautocorrelated \( \eta_t \)), but the set \((r_1, r_2, \ldots, r_N)\) is affected by the autocorrelation properties of \( \eta \). The same is true of the forecast curve for \( x \): the initial values are affected by the \( \rho \)'s but the later ones only by \( \alpha \)'s.\(^{19}\)

\(^{19}\) The problems of interpreting an empirical correlogram of the observed variable in cases where the "disturbances" are autocorrelated will be treated in a later paper.