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# The Computation of Economic Equilibria

by Herbert Scarf

with the collaboration of Terje Hansen

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of the Class of 1894, Yale College.

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**Cowles Foundation for  
Research in Economics**

To the memory of my father  
Louis H. Scarf



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## Preface

One of the major triumphs of mathematical economics during the past quarter of a century has been the proof of the existence of a solution for the neoclassical model of economic equilibrium. This demonstration has provided one of the rare instances in which abstract mathematical techniques are indispensable in order to solve a problem of central importance to economic theory.

When cast in a mathematical form the general equilibrium model becomes a complex system of simultaneous equations and inequalities for the determination of all prices and output levels in the economy. Aside from an occasional special case, however, the system is so complex that the existence of a solution can be guaranteed only by an appeal to fixed point theorems rather than by more elementary and constructively oriented techniques. As a consequence, general equilibrium analysis has remained at a level of abstraction and mathematical theorizing far removed from its ultimate purpose as a method for the evaluation of economic policy.

The present monograph attempts to remedy this difficulty by providing a general method for the explicit numerical solution of the neoclassical model. The method should be of interest not only to economists concerned with the techniques of economic planning, but to applied mathematicians in a variety of fields whose work requires the solution of highly nonlinear systems of equations.

An elaborate history of my own involvement and that of others in the development of this method is given in chapter 1. It is appropriate, however, to emphasize the important contributions of my friend and collaborator Terje Hansen. Many of the ideas presented here were first suggested by Professor Hansen; and the entire monograph benefited substantially from our close association during the past six years.

I would like, also, to express my gratitude to Rolf Mantel, Marcus Miller, Donald Richter, John Shoven, John Spencer, and John Whalley, all of whom have been students of economics at Yale during the last ten years. In distinct ways they have participated in the elaboration and application of the methods of this volume.

The Cowles Foundation, with which I have been associated since 1963, has provided an exceptional atmosphere for the development of

this work. My colleagues, including Tjalling Koopmans, Martin Shubik, Alvin Klevorick, and Robert Aumann (who visited the Cowles Foundation in 1965), have supplied me with advice and continuing intellectual support. Gerard Debreu, who was for many years on the staff of the Cowles Foundation, has remained my good friend and valued advisor on all aspects of mathematical economics. The standards of mathematical rigor and clarity of thought which prevail at Cowles are well known to the economics profession. But perhaps more important is the persistent though subtle suggestion that the highest aim of even the most theoretical work in economics is an ultimate practical applicability.

Outside of the Cowles Foundation, the two professional colleagues with whom I have collaborated most closely on the development of fixed point methods are Harold Kuhn and Curtis Eaves. Their independent discoveries have been a source of stimulation and occasional drama. Both Kuhn and Gerard Debreu were kind enough to read the manuscript with unusual care and to provide many suggestions that have been incorporated into the text.

I should also like to offer thanks to the staff of the Cowles Foundation, including Althea Strauss, Laura Harrison, and the inspired typist of the manuscript, Glenna Ames. To the Yale University Press I am grateful for the fine editorial assistance provided.

I am indebted to the National Science Foundation and the Ford Foundation for their generous support over a period of many years. The book was written during my tenure as an Overseas Fellow at Churchill College, Cambridge University, in 1969–70. At Churchill, I found a stimulating environment and the leisure necessary in order for me to complete this work.

H. S.

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