APPENDIX

BASIC CONCEPTS OF ACTIVITY ANALYSIS

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For numerical analysis of the classical problem of production economics—allocation of limited resources among alternative uses—activity analysis is uniquely powerful. The activity analysis approach—and in particular, the simplex technique for linear programming calculations—can be applied to systems involving literally hundreds or even thousands of commodities. Activity analysis permits us to consider substitution and complementarity, as well as diminishing returns to scale in production processes. Optimization and economic choice are recognized explicity within this model. Moreover—unlike classical calculus methods—activity analysis will handle cases where "kinks," inequalities, and nonnegativity restrictions are important. Within an activity analysis framework, there is no need for production functions to be differentiable at all points.

It is needless to remind economists that this is a world in which every good thing has its price. The price of using the activity analysis and linear programming framework is this: that all production functions and all economic choices must be formulated in terms of *linear* relationship among the unknowns. At first glance this requirement of linearization appears highly restrictive. However, after examining the variety of empirical cases that can be handled within this framework, most readers will probably agree that linearization is not in itself an onerous requirement. The features that are likely to appear as more serious shortcomings in activity analysis are precisely those which are also troublesome in the more conventional models of production processes: the absence of economies of scale and the absence of stochastic elements. As of this date, the inclusion of stochastic elements within a *many*-commodity optimization model appears to be a formidable challenge to the mathematician.[1] Fortunately, economies of scale no longer appear as forbidding as they did prior to the discovery of integer programming.

AN ACTIVITY AS A "BLACK BOX"

Central to the models utilized throughout this volume is the concept of an "activity": a process for transforming inputs of goods and services into out-

[1] Computer simulation often enables us to find good solutions—although not necessarily optimal ones—even in cases involving many commodities, stochastic elements, and economies of scale. See the bibliography on simulation compiled by Shubik (1960).

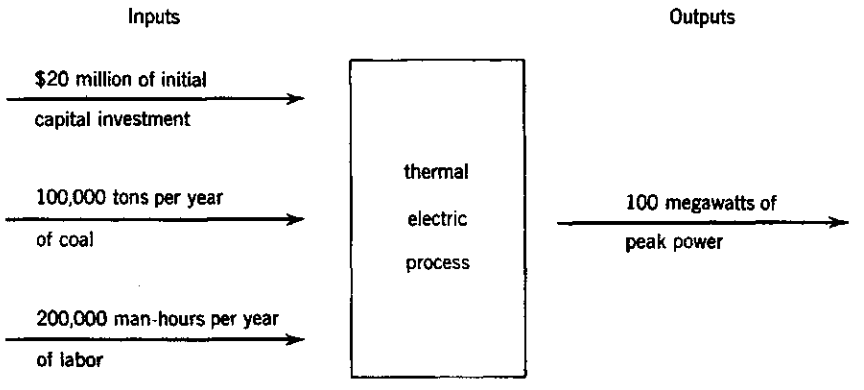

FIGURE 1

puts. If, for example, a coal-burning electric power plant were being analyzed from this viewpoint, we might be concerned with how much capital investment, how much coal, and how much labor would be required in order to produce the plant's rated output.[2] Suppose that in a plant capable of producing 100 megawatts of peak power, the required initial capital investment will be $20 millions; the annual rate of coal consumption will be 100,000 tons; and the annual labor inputs will amount to 200,000 man-hours. Then—assuming that these are the appropriate input and output categories for our purposes—the entire operation of the thermal electric plant can be summarized in terms of the black box shown in Figure 1.

According to this diagram, we note that the activity analysis viewpoint is one in which the producing sector of the economy is described in terms of: (*a*) "commodities" such as electric power, labor, coal, and capital equipment; and (*b*) "activities" such as the "thermal electric process" for transforming one group of economic commodities into another. Starting with these definitions, two axioms and one maxim are introduced: the axioms of proportionality and of additivity, and the maxim of economic efficiency.[3]

Beside these formal axioms, there is a further major assumption that is implicit here. Conventional activity analysis models are characterized by the absence of random elements and uncertainties. Randomness is regarded as a second-order effect, and is supposedly allowed for through converting all parameters into "certainty equivalents." Unfortunately, it is all too easy to cite cases where such deterministic shortcuts can be misleading. Both in the theory of inventory control and of waiting lines, the random nature of inputs and of outputs plays an essential role.

[2] A pioneering application of process analysis in the electric power industry is to be found in Masse and Gibrat (1957).

[3] For an axiomatization of activity analysis that is both precise and readable, see T. C. Koopmans (1957).

The Two Axioms and the Maxim of Economic Efficiency

AXIOM 1. PROPORTIONALITY. By "proportionality," we mean that if an activity can be operated at its base level, it can also be operated at any non-negative fraction or multiple of that level, with all inputs and outputs varying proportionately. Figure 1 already provides a listing of the inputs and outputs corresponding to the installation of 100 megawatts of peak power output. The axiom of divisibility says that if we wish to depart from this base level and build a 50 megawatt plant, all the inputs will be halved; that if we build a 300 megawatt plant, all the inputs will be tripled; and that if we install x hundreds of megawatts, all of the inputs will be multiplied by x. (The quantity x is known as the "intensity" or "level" at which the thermal power process is operated.)

Activity analysis—in its conventional form—is incompatible with what the economist terms "increasing returns to scale." (By increasing returns we mean that if, say, the inputs into an activity are tripled, the output will exceed triple its base level.) The real world abounds with cases of increasing returns —including, in particular, some cases of investment in thermal electric power generating stations. Despite the real world, the usual activity analysis model is one in which the possibility of increasing returns is completely ignored. There is, however, an extension of activity analysis—an extension known as "integer programming"—through which it *is* possible to obtain numerical solutions in cases that involve increasing returns. For the numerical technique itself, see Gomory (1958); and for a discussion of increasing and of decreasing returns within activity analysis models, see Markowitz and Manne (1957).

AXIOM 2. ADDITIVITY. The axiom of additivity rules out most cases of what an economist would call "external economies." In terms of the power industry, this axiom implies that if there are two processes utilized together for producing electricity—the first one operated at an intensity of x_1 and the second at an intensity of x_2—the inputs required and the outputs produced will consist of the *sum* of the inputs and outputs corresponding to the operation of the two individual activities at levels of x_1 and x_2 respectively.

Additivity rules out certain possibilites for interactions between the individual processes. Suppose that we are constructing a model in which process 1 refers to the installation of a hydroelectric power plant at a downstream site, and process 2 to the installation of a hydroelectric plant and reservoir at an upstream site. It does not take a profound knowledge of hydrology to recognize the importance of a nonmarketable service produced by the upstream plant and consumed below, namely streamflow regularization. This by-product of the upstream reservoir will have a major influence upon the value of the downstream plant. The axiom of additivity implies that any physical interactions between processes have already been allowed for—e.g., through defining the activities in terms of integrated upstream and downstream plants. Additivity implies that the net output of the entire system be equal to the sum of what is independently produced (or consumed) by the individual activities—no more and no less.

THE MAXIM OF ECONOMIC EFFICIENCY. By "economic efficiency," we mean much the same thing that the economist ordinarily takes for granted about his "production function": Whatever activity levels are selected, there exists no other set of activity levels which generates a greater amount of net output (or a smaller net input) of one commodity from the system without reducing the net output (or increasing the net input) of some other commodity. This maxim of efficiency provides a partial ordering over all possible combinations of inputs and outputs.

Traditionally, the economist has taken it for granted that the responsibility for constructing such a partial ordering falls exclusively within the domain of the industrial engineer. For his own part, the industrial engineer has usually been kept busy providing optimal solutions geared to the particular needs of his employer, and has had no incentive to spell out the set of all possible efficient allocations of inputs and outputs. It is the aim of process analysis to explore some of the territory that lies between the domain of the economist and the industrial engineer—to exploit the latter's detailed knowledge of production processes in order to provide the economist with a better characterization of the set of efficient production possibilities.

Linear Programming

"Linear programming" represents one particular form of activity analysis—a form which has proved particularly well suited for numerical calculations. In the case of linear programming, instead of attempting to construct a partial ordering over all possible combinations of inputs and outputs, we pose a much less ambitious question: Given the net input availabilities of certain commodities and the net output requirements of certain others, what is the maximum possible output of some item defined as the "maximand"? Or alternatively, what is the minimum possible input of the "minimand"?

It is typical for a linear programming model of an entire economy to be phrased in terms of maximizing some such physical quantity as the amount of a specific product mix or, alternatively, maximizing some such financial quantity as "national income" subject to possible side conditions on the product mix. An economy-wide model may also be phrased in terms of minimizing the input of investment or of foreign aid required to reach a predetermined national income and/or product mix target. If the system represents a single enterprise or an industry, the linear programming objective will often be phrased in terms of maximizing the output of a specific mix of commodities; or in terms of maximizing money profits; or minimizing the money costs of producing a certain product mix; or sometimes of minimizing the time elapsed before certain commodities have been produced.

For a concise introduction to linear programming computational methods, the reader should consult Gass (1958); or for a more comprehensive treatment, Dantzig (1963). A summary of the more important industrial applications is to be found in Vajda (1958). For economy-wide applications, see Chenery and Clark (1959); also Sandee (1960). For analogies between market mechanisms and linear programming computations, the economist will want to consult Koopmans (1951) and Dorfman, Samuelson, and Solow (1958).

The art of electronic computations is progressing altogether too rapidly for it to be safe to predict future developments in the numerical analysis of linear programs. In 1953, the IBM Card Programmed Calculator required eight hours to solve systems involving 27 equations. Inside the span of just four years—in 1957—the IBM 704 succeeded in solving a 195-equation system within a few hours (Orchard-Hays, 1958). By 1961, the IBM 704 system was already superseded by still more powerful machines and programs. By some date within the 1960's, it should be possible to handle systems involving upwards of 10,000 distinct equations. Note that success in computing large-scale models is not dependent solely upon improvements in computing machinery, but can also be achieved through improvements in the mathematical techniques that are employed. The "decomposition principle" represents one of the first practical attempts to take advantage of specialized matrix structures (Dantzig and Wolfe, 1961).

In planning linear programming computations, one further possibility should be borne in mind: Through the technique known as parametric programming, it is comparatively inexpensive to engage in sensitivity tests of a linear programming solution, to see what happens to the maximand or to the minimand as the availabilities of individual inputs or the requirements for individual outputs are varied. This makes it possible to end up with much the same result as activity analysis, a numerical description of the set of efficient combinations of inputs and outputs. Parametric programming is quite practical from a computational viewpoint, provided that the analysis is restricted to two or perhaps three dimensions, i.e., to the tradeoffs between just two or three groups of commodities.[4] Examples of parametric programming are to be found throughout the activity analysis chapters of this monograph.

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[4] For details on this technique, see Manne (1956, pp. 151–177) and Gass (1958, Chapter 8).

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