CHAPTER 7

UNCERTAINTY

7.1. INTRODUCTION

The analysis is extended in this chapter to the case where uncertain events determine the consumption sets, the production sets, and the resources of the economy. A contract for the transfer of a commodity now specifies, in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional. This new definition of a commodity allows one to obtain a theory of uncertainty free from any probability concept and formally identical with the theory of certainty developed in the preceding chapters.

7.2. EVENTS

An economy whose activity extends over T elementary time-intervals, or dates, will be studied. It is assumed that the uncertainty of the environment during that period originates in the choice that Nature makes among a finite number of alternatives. These alternatives will be called *events at T* and indicated by an index e_T running from 1_T to k_T. Once e_T is given, atmospheric conditions, natural disasters, technical possibilities, . . . are determined for the entire period.

At the beginning of date t, the agents of the economy have some information about the event at T which will obtain. This information can be formally presented as follows. The set of events at T is partitioned into non-empty subsets called *events at t* and indicated by an index e_t running from 1_t to k_t. At the beginning of date t, every agent knows to what event at t the event at T which will obtain belongs. At the beginning of date $t + 1$, further information is available, i.e., the partition which defines the events at $t + 1$ is derived by partitioning the events at t. The events at $t = 1, \cdots, T$ can be conveniently represented by the vertices of a tree with the vertex 1_0 corresponding to the absence of information prevailing initially. In fig. 1 such an event tree is drawn for the particular case where $T = 3$.

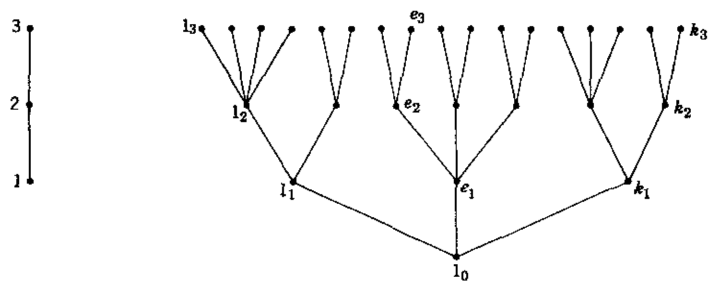

Fig. 1

7.3. Commodities and Prices

Wheat with specified physical characteristics available at location s, at date t will play entirely different economic roles according to the event at t which obtains (in particular, according to precipitation during the growing season). One is thus led to define a *commodity* in this new context by its physical characteristics, its location, and its event (or vertex of the event tree; this vertex defining implicitly the date of the commodity). A contract for delivery of wheat between two agents takes, for example, the form: the first agent shall deliver to the second agent, who shall accept delivery, five thousand bushels of wheat of a specified type at location s, at event e_t. If e_t does not obtain, no delivery takes place. It was remarked in Chapter 2 that the definition of a certain commodity might require several dates (and several locations). Therefore the definition of an uncertain commodity may require here several events (and several locations). Summing up, the concept of uncertain commodity is derived from the concept of certain commodity by substituting the tree structure of events for the line structure of dates and replacing everywhere "date" by "event."

It is assumed that there is only a finite number l of commodities; these are indicated by an index h running from 1 to l. It is also assumed that the quantity of any one of them can be any real number. Given a sign convention for the inputs and the outputs of an *agent*, a complete plan of action, or more briefly an *action*, for him is represented by a point a of the *commodity space* R^l. The plan of action a made initially for the whole future specifies for each good and service the quantity that he will make available, or that will be made available to him, at each location, at each date, and at each event.

The *price* p_h of the hth commodity is a real number interpreted as the amount paid (in the sense of 2.1) initially by (resp. to) the agent who commits himself to accept (resp. to make) delivery of one unit of that commodity. Payment is irrevocably made although delivery does not take place if specified events do not obtain. An agent who buys a bushel of No. 2 Red Winter Wheat available in Chicago at date *t in any event* buys in fact as many commodities as there are events at t. The usual futures "price" thus corresponds to a *sum* of prices of uncertain commodities. The *price system* is the l-tuple $p = (p_1, \cdots, p_h, \cdots, p_l)$. The *value* of an action a relative to the price system p is the inner product $p \cdot a$.

7.4. Producers

An action y_j of the jth *producer* is called a *production* (inputs are negative and outputs positive). Let $y_j(e_T)$ denote the vector of the components of y_j associated with the unicursal path from the vertex 1_0 of the event tree to the vertex e_T, and let $Y_j[e_T]$ be the certain production set associated with the same path. The production y_j is possible if and only if $y_j(e_T)$ belongs to $Y_j[e_T]$ for every event e_T at T. The set of productions y_j possible for the jth producer is a subset of the commodity space R^l denoted Y_j and called the *production set* of the jth producer.

It is easy to interpret the assumptions of 3.3 on production sets in this new context, and to relate them to the corresponding assumptions in the case of certainty. For example, if $Y_j[e_T]$ is convex for every event e_T at T, then Y_j is clearly convex.

Given a price system p and a production y_j, the *profit* of the jth producer is $p \cdot y_j$. Considering the price system as a datum, the jth producer tries to maximize his profit in his production set. For this he needs neither an appraisal (conscious or unconscious) of the likelihoods of the various events, nor an attitude toward risk. His behavior amounts to maximizing the value of the stock outstanding of the jth corporation. In other words, the jth corporation announces a production plan y_j; as a result, its share has a determined value on the stock market; it chooses its plan so as to maximize the value of its share.

7.5. Consumers

An action x_i of the ith *consumer* is called a *consumption* (inputs are positive and outputs negative). Exactly as for a producer, one defines the *consumption set* X_i of the ith consumer. It is assumed that the set X_i is completely preordered by the preferences $\underset{i}{\precsim}$ of the ith consumer. This *preference preordering* reflects the tastes of the consumer for goods and services (including, in particular, their spatial and temporal specifications), his personal appraisal of the likelihoods of the various events, and his attitude toward risk.

The assumptions of 4.3 on consumption sets, and of 4.5–4.7 on preference preorderings are again easily interpreted in this context of uncertainty and related to the corresponding assumptions in the case of certainty. Most interesting are the three convexity assumptions on preferences of 4.7. Attention will be focused on:

(a) If $x_i^2 \underset{i}{\succsim} x_i^1$, then $tx_i^2 + (1 - t)x_i^1 \underset{i}{\succsim} x_i^1$,

which is the weakest (when preferences satisfy the continuity assumption (a) of 4.6). This axiom for uncertain consumptions implies an attitude of risk-aversion for the ith consumer. To see this, consider the case of one date and two events which are the outcomes *Head* and *Tail* of the tossing of a coin. Let b and c be two certain consumptions, and denote by (b, c) the uncertain consumption which associates b with event *Head* and c with event *Tail*, by (c, b) the uncertain consumption which makes the reverse association. Assume moreover that (b, b) is not indifferent to (c, c), i.e., that the certain consumptions b and c are not indifferent. If (b, c) is indifferent to (c, b), i.e., if the ith consumer appraises *Head* and *Tail* as being equally likely, (a) asserts that $((b + c)/2, (c + b)/2)$, i.e., the certainty of consuming $(b + c)/2$, is at least as desired as the uncertain consumption (b, c) or (c, b).

Given a price system p and his *wealth* w_i, the ith consumer tries to satisfy his preferences $\underset{i}{\precsim}$ in the subset of X_i defined by the wealth constraint $p \cdot x_i \leqq w_i$.

7.6. Equilibrium

Finally the *total resources* are a given vector ω of R^l such that, for every event e_T at T, the vector $\omega(e_T)$ of the components of ω associated with the unicursal path from the vertex 1_0 of the event tree to the vertex e_T coincides with the certain total resources associated with that path. The formal description of an *economy* $E = ((X_i, \underset{i}{\precsim}), (Y_j), \omega)$ is thus identical with that given in 5.3. In particular, an *attainable state* of E is an $(m + n)$-tuple $((x_i), (y_j))$ of actions such that

$$x_i \in X_i \text{ for every } i,\ y_j \in Y_j \text{ for every } j,\ \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j = \omega.$$

The equality expresses that the actions of the agents are compatible with the total resources, i.e., for every event e_T at T,

$$\sum_i x_i(e_T) - \sum_j y_j(e_T) = \omega(e_T).$$

A *private ownership economy* $\mathscr{E}$ is described by an economy $((X_i, \underset{i}{\precsim}), (Y_j), \omega)$, the *resources* (ω_i) of the consumers and their *shares* (θ_{ij}). The ω_i are points of R^l satisfying $\sum_{i=1}^{m} \omega_i = \omega$, and the θ_{ij} are non-negative real numbers satisfying $\sum_{i=1}^{m} \theta_{ij} = 1$ for every j. Given a price system p and productions (y_j) for the n producers the wealth of the ith consumer is $w_i = p \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij}\, p \cdot y_j$.

The formal identity of this theory of uncertainty with the theory of certainty developed earlier allows one to apply here all the results established in the preceding chapters. In particular, sufficient conditions for the existence of an equilibrium for the private ownership economy $\mathscr{E}$ are given by theorem (1) of 5.7.

7.7. Optimum

In the same fashion, theorems (1) of 6.3 and (1) of 6.4 applied to the economy E yield sufficient conditions for an equilibrium relative to a price system to be an optimum, and for an optimum to be an equilibrium relative to a price system.

Notes

1. This chapter is based on the mimeographed paper, "Une économie de l'incertain," written by the author at Electricité de France in the summer of 1953. The analysis of the theory of value under uncertainty in terms of choices of Nature originated in K. J. Arrow [2], where the risk-aversion implication of weak-convexity of preferences is established. The definition of the preference preordering in 7.5 has been suggested by the work of L. J. Savage [1].

A similar approach has been taken by E. Baudier [1]. A different attack has been tried by M. Allais [2].

2. The assumption that markets exist for all the uncertain commodities introduced in 7.3 is a natural extension of the usual assumption that markets exist for all the certain commodities of Chapter 2 (see in particular 2.6).