APPENDIX[1]

The computational procedures used to obtain the numerical results of the foregoing chapters are illustrated in this section by a presentation of the detailed steps involved in all of the calculations pertaining to equation 4.1. Limited-information results were obtained by the limited-information single-equation (LISE) method which was developed by Rubin and Anderson[2] and has been explained in some detail by Chernoff and Divinsky[3] and by Klein.[4] Least squares procedures are well known; the calculations are included here mainly for completeness.

Insofar as possible, the notation of the Chernoff-Divinsky paper is also used here, so that the reader may easily refer to their exposition of the method. The relation to be estimated was given in Chapter 4.

$$(4.1)\quad Y_{1t} + \beta_{16}Y_{6t} + \beta_{17}Y_{7t} + \gamma_{11}Z_{1t} + \gamma_{12}z_{2t} + \gamma_{13}Z_{3t} + \gamma_{10} = U_{1t}$$

In the following, the time subscript will be omitted wherever practicable. The variables that enter the limited-information calculation are classified as:

(i) Current endogenous or jointly dependent variables entering the relation, denoted by $y_\Delta = (Y_1\ Y_6\ Y_7)$.

(ii) Predetermined variables entering the relation and denoted by $z_* = (Z_1\ z_2\ Z_3)$.

(iii) Predetermined variables absent from the relation but present in the model, denoted by $z_{**} = (Z_4\ Z_5\ Z_7\ Z_8\ Z_9\ Z_{10})$.

It will also be convenient sometimes to let

(iv) $z = (z_*\ z_{**})$ and

(v) $x = (y_\Delta\ z) = (x_1\ x_2 \cdots x_{12})$; i.e,

$$x_1 = Y_1\,,\qquad x_2 = Y_6\,,\qquad x_3 = Y_7$$

$$x_4 = Z_1\,,\qquad x_5 = z_2\,,\qquad x_6 = Z_3\,,\quad \text{etc.}$$

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[1] This appendix was prepared by Mrs. Jagna Zahl and Mr. Francis Bobkoski with the assistance of the authors.

[2] See T. W. Anderson and H. Rubin, Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations of *Annals, Mathematical Statistics*, Vol. 20, pp. 46–63, 1949.

[3] Herman Chernoff and Nathan Divinsky, The Computation of Maximum Likelihood Estimates of Linear Structural Equations, Ch. X of *Studies in Econometric Method,* Cowles Commission Monograph 14, William C. Hood and T. C. Koopmans, editors, John Wiley & Sons, 1953.

[4] Lawrence R. Klein, *A Textbook of Econometrics*, Row Peterson & Co., 1953. See pp. 169–183.

The values taken by these variables in our sample period and their arithmetic means, $m^{(0)}$, are given in Table 1.1.[5] The data are given to six significant figures. The computations were carried out to nine decimal places; for brevity, the numbers appearing in later tables in this article have been rounded to four places, which will mean that a reader who checks particular steps in the computations may obtain results that differ slightly from those presented here.

The first step in the computation is to calculate the matrix of moments of these variables, $\bar{M}_{zz}^{(0)}$, as defined in the Chernoff-Divinsky article in Section 2. A sample element of this matrix, say the moment involving Y_6 and Z_3, is given by

$$(1) \qquad \bar{m}_{Y_6 Z_3}^{(0)} = T \sum_t Y_{6t} Z_{3t} - \left(\sum_t Y_{6t}\right)\left(\sum_t Z_{3t}\right)$$

where $T = 30$, the number of observations.

These moments[6] are given in Table 1.2, together with the row sums, $\sum_{j=1}^{12} \bar{m}_{x_i x_j}^{(0)}$ (for row i). The sums include those terms under the main diagonal that need not be written down because the matrix is symmetric. This convention will be followed throughout this appendix. The calculation of this moment matrix is checked by the identity $\sum_{j=1}^{12} \bar{m}_{x_i x_j}^{(0)} = \bar{m}_{x_i x.}^{(0)}$, where $x. = \sum_{j=1}^{12} x_j$ for each t, which is the column headed $\sum$ in Table 1.1. In other words, the sum of the ith row of the moment matrix must be equal to the moment of x_i and $\sum$.

Next the moment adjustments are determined (Section 2, Chernoff-Divinsky) and entered in Table 2.1. In the same table are the means of each variable, copied from Table 1.1, and the adjusted means, which are the products of the means and the corresponding adjustment factors.

Next is the computation of a matrix of residuals, $\bar{W}_{\Delta\Delta}$, presented in Table 3.[7] $\bar{M}_{zz}$ and $\bar{M}_{z\Delta}$ are formed. These are blocks of the adjusted

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[5] We deviate here from the Chernoff-Divinsky notation by keeping our symbols for the variables throughout the exposition rather than relabeling them in standard notation. Since our original notation already distinguishes between jointly dependent and predetermined variables, this should not cause confusion and will facilitate comparisons between the appendix and our text.

[6] The three preliminary calculations illustrated by Chernoff-Divinsky for each moment given in Table 1.2 are here omitted and only the final moments entered.

[7] This represents a deviation from the Chernoff-Divinsky article; in preparation for the solution of the other equations of the model, their Table 3 is the computation of $\bar{W}_{yy} = \bar{M}_{yy} - \bar{M}_{yz}\bar{M}_{zz}^{-1}\bar{M}_{zy}$ which includes the moments of all the variables in the model. If LISE estimates only are desired, it is sufficient to find the submatrix $W_{\Delta\Delta}$, which excludes the y's that do not occur in the equation under consideration.

$\bar{M}_{zz}^{(0)}$ matrix corresponding to the variables z, z and z, y_Δ, respectively. A sample element of $\bar{M}_{z\Delta}$ is the adjusted moment of Y_6 and z_2 which is $\bar{m}_{Y_6z_2} = \bar{m}_{Y_6z_2}^{(0)}k_{Y_6}k_{z_2} = \bar{m}_{Y_6z_2}^{(0)} \cdot 1 \cdot 10^{-2}$. Then $\bar{M}_{\Delta z}\bar{M}_{zz}^{-1}\bar{M}_{z\Delta}$ is computed by the Doolittle method. The row sums of $\bar{M}_{\Delta z}\bar{M}_{zz}^{-1}\bar{M}_{z\Delta}$ are obtained and copied in the column headed $\sum_1$; these are used in the sum check for the computation of $\overline{W}_{\Delta\Delta}$. The column headed $\sum_2$ is a sum check for

TABLE 1.1
TIME SERIES FOR VARIABLES
Part I

Year	Y_1	Y_6	Y_7	Z_1	z_2	Z_3
1920	9.748084	8.068389	6.856327	9.774997	1	8.379150
1921	9.765719	8.105950	6.822490	9.769460	2	8.363938
1922	9.796348	8.102376	6.834405	9.778758	3	8.382668
1923	9.810109	8.114904	8.871142	9.796037	4	8.383129
1924	9.809222	8.098474	6.902960	9.776513	5	8.372399
1925	9.802213	8.072969	6.929292	9.748548	6	8.354120
1926	9.815773	8.098248	6.957628	9.742323	7	8.356553
1927	9.829126	8.109833	6.964877	9.747289	8	8.370315
1928	9.828799	8.105673	6.956671	9.751579	9	8.375821
1929	8.833779	8.100195	6.975930	9.754352	10	8.369334
1930	9.838925	8.091019	6.978998	9.755504	11	8.335463
1931	9.850433	8.082708	6.953046	9.758315	12	8.337076
1932	9.852929	8.130834	6.933601	9.782538	13	8.352973
1933	9.861783	8.127156	6.958915	9.802675	14	8.354061
1934	9.820362	8.012879	6.943129	9.802718	15	8.342169
1935	9.801663	7.971032	6.952794	9.744170	16	8.358042
1936	9.830960	8.064606	7.009480	9.755272	17	8.336144
1937	9.823768	7.996518	7.026471	9.745581	18	8.339627
1938	9.846623	8.094853	7.026197	9.746278	19	8.376573
1939	9.877329	8.118528	7.053128	9.773336	20	8.368374
1940	9.883906	8.124569	7.082682	9.791397	21	8.403127
1941	9.897390	8.142696	7.123580	9.789017	22	8.435870
1942	9.955978	8.218703	7.188769	9.824941	23	8.458267
1943	9.991362	8.295371	7.218770	9.865975	24	8.455265
1944	9.969697	8.235548	7.218875	9.864963	25	8.452174
1945	9.969946	8.244758	7.217291	9.831898	26	8.467254
1946	9.954799	8.233562	7.185296	9.821342	27	8.441883
1947	9.952042	8.201303	7.197108	9.810597	28	8.437942
1948	9.946647	8.171542	7.213443	9.792172	29	8.424071
1949	9.971076	8.236400	7.249224	9.799996	30	8.425557
Σ	295.936790	243.771596	210.802519	293.498541	465	251.609339
$m^{(0)}$	9.864560	8.125720	7.026751	9.783285	15.5	8.386978

Y_1 = logarithm of production of livestock products in dollars.
Y_6 = logarithm of the quantity of feed grains fed in 1000 lb. TDN.
Y_7 = logarithm of the quantity of protein feeds fed in 1000 lb. TDN.
Z_1 = logarithm of beginning inventory of livestock in dollars.
Z_3 = logarithm of the quantity of roughages fed in 1000 lb. TDN.

Part II

Year	Z_4	Z_5	Z_7	Z_8	Z_9	Z_{10}	Σ
1920	9.769387	1.840106	2.189771	1.763428	8.150955	6.886952	74.427546
1921	9.752454	1.740363	2.000434	1.574031	8.219585	6.878292	74.992716
1922	9.755776	1.771587	1.988113	1.568202	8.156801	6.870803	76.005837
1923	9.780821	1.834421	2.003891	1.628389	8.143769	6.903104	77.269716
1924	9.810186	1.839478	1.987219	1.638489	8.127418	6.949356	78.311714
1925	9.819966	1.863917	2.006038	1.643453	8.102729	6.988243	79.331488
1926	9.807323	1.879669	2.000000	1.648360	8.166857	7.017608	80.490342
1927	9.816910	1.881385	1.975891	1.648360	8.138763	7.023753	81.506502
1928	9.820507	1.892651	1.976808	1.648360	8.148182	6.990449	82.495500
1929	9.822525	1.916454	1.969882	1.653213	8.145911	7.008210	83.549785
1930	9.824031	1.867467	1.933993	1.623249	8.115113	6.994172	84.357934
1931	9.826662	1.799341	1.872739	1.505150	8.092900	6.978083	85.056453
1932	9.833841	1.679428	1.834421	1.380211	8.174903	6.975705	85.931384
1933	9.829947	1.655138	1.838849	1.332438	8.171566	6.979986	86.912511
1934	9.839881	1.712650	1.885926	1.380211	8.026765	6.957531	87.724221
1935	9.851221	1.763428	1.904174	1.423246	7.998172	6.960296	88.728238
1936	9.793733	1.820201	1.906874	1.454845	8.074944	7.012236	90.059295
1937	9.829708	1.851870	1.935507	1.505150	8.029853	7.035188	91.119241
1938	9.820828	1.816241	1.906335	1.491362	8.175991	7.046625	92.347906
1939	9.835368	1.846337	1.900367	1.491362	8.198353	7.054429	93.516911
1940	9.855692	1.879096	1.907411	1.498311	8.206962	7.064997	94.698150
1941	9.879714	1.963788	1.945961	1.585461	8.223555	7.121547	96.108579
1942	9.880941	2.067071	1.986772	1.698970	8.266015	7.160933	97.707360
1943	9.936738	2.121888	1.994317	1.819544	8.282105	7.215318	99.196653
1944	9.976315	2.167317	1.998259	1.897627	8.239977	7.215042	100.235794
1945	9.987021	2.179264	2.003461	1.944483	8.255596	7.211921	101.312893
1946	9.977232	2.201124	2.060320	1.982271	8.259415	7.190973	102.308217
1947	9.961137	2.229170	2.162863	2.008600	8.241154	7.209551	103.411467
1948	9.964474	2.275081	2.203577	2.029384	8.229313	7.223097	104.472801
1949	9.948048	2.272770	2.182985	2.012247	8 325092	7.256021	105.679416
Σ	295.608387	57.628701	59.463158	49.478407	245.088714	211.380421	2679.266573
$m^{(0)}$	9.853613	1.920957	1.982105	1.649280	8.169624	7.046014	

Z_4 = logarithm of the quantity of livestock products sold in dollars, lagged one year, i.e., Y_{t-1}.
Z_5 = logarithm of disposable personal income in the United States in billions of current dollars.
Z_7 = logarithm of wholesale price index, excluding farm products (1935–39 = 100).
Z_8 = logarithm of cost of farm labor in cents per hour.
Z_9 = logarithm of supply of feed grains in 1000 lb. TDN.
Z_{10} = logarithm of production of protein feeds in 1000 lb. TDN.

the computation of $\bar{M}_{\Delta z}\bar{M}_{zz}^{-1}\bar{M}_{z\Delta}$.[8] $\bar{M}_{\Delta\Delta}$ is written down, analogously to $\bar{M}_{zz}$, and the difference $\bar{W}_{\Delta\Delta} = \bar{M}_{\Delta\Delta} - \bar{M}_{\Delta z}\bar{M}_{zz}^{-1}\bar{M}_{z\Delta}$ formed.

[8] $\bar{M}_{yz}$ is the matrix $\bar{M}_{zy}$ with rows written as columns. This is, in general, the meaning of an interchange of subscripts of a matrix.

This particular product of three matrices can be computed directly by a variation of the standard Doolittle method. Although the general Doolittle procedure is explained in statistics literature, we will sketch the technique for this computation, as well as another that will be needed later in this discussion of LISE. To change the notation, suppose A is an $n \times n$ nonsingular symmetric matrix and B and $n \times m$ matrix. We desire $B'A^{-1}B$ (as in this case) and also $A^{-1}B$ (this will be used later).

(*a*) *Forward solution* (same for both cases). Let the element (i, j) of A be a_{ij} and of B be $a_{i,n+j}$. The two matrices are written side by side, and added across:

$$\begin{array}{cccccccc}
 & & A & & & B & & \Sigma \\
a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} & \cdots & a_{1,n+m} & a_{1\cdot} \\
a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} & \cdots & a_{2,n+m} & a_{2\cdot} \\
\cdot & & & & \cdot & & & \\
\cdot & & & & \cdot & & & \\
\cdot & & & & \cdot & & & \\
a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} & \cdots & a_{n,n+m} & a_{n\cdot}
\end{array}$$

Define

$$\alpha_{ri} = a_{ri} - \sum_{k=1}^{r-1} \frac{\alpha_{kr}}{\alpha_{kk}} \alpha_{ki}$$

Let β_{kr} be the coefficient α_{kr}/α_{kk}.
Then we may write

$$\alpha_{ri} = a_{ri} - \sum_{k=1}^{r-1} \beta_{kr}\alpha_{ki}$$

All the elements α_{ri} form a triangular matrix, as well as the elements β_{ri}. This means that they are zero under the main diagonal. They are computed alternately, in the order and layout indicated below. After an entire row has been computed, it is checked by the relation, for a row of α's,

$$\alpha_{r\cdot} = a_{r\cdot} - \sum_{k=1}^{r-1} \beta_{kr}\alpha_{k\cdot} \quad \text{and for a row of } \beta\text{'s,} \quad \beta_{r\cdot} = \frac{\alpha_{r\cdot}}{\alpha_{rr}}$$

(We have used the notation $\alpha_{r\cdot} = \sum_{i=1}^{n+m} \alpha_{ri}$, $\beta_{r\cdot} = \sum_{i=1}^{n+m} \beta_{ri}$, $a_{r\cdot} = \sum_{i=1}^{n+m} a_{ri}$):

$$\begin{array}{cccccccc}
 & & & & & & & \Sigma \\
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \alpha_{1,n+1} & \cdots & \alpha_{1,n+m} & \alpha_{1\cdot} \\
1 & \beta_{12} & \cdots & \beta_{1n} & \beta_{1,n+1} & \cdots & \beta_{1,n+m} & \beta_{1\cdot} \\
0 & \alpha_{22} & \cdots & \alpha_{2n} & \alpha_{2,n+1} & \cdots & \alpha_{2,n+m} & \alpha_{2\cdot} \\
0 & 1 & \cdots & \beta_{2n} & \beta_{2,n+1} & \cdots & \beta_{2,n+m} & \beta_{2\cdot} \\
 & \cdot & & & & & & \\
 & \cdot & & & & & & \\
 & \cdot & & & & & & \\
0 & 0 & \cdots & \alpha_{nn} & \alpha_{n,n+1} & \cdots & \alpha_{n,n+m} & \alpha_{n\cdot} \\
0 & 0 & \cdots & 1 & \beta_{n,n+1} & \cdots & \beta_{n,n+m} & \beta_{n\cdot}
\end{array}$$

(*b*) *Computation of $B'A^{-1}B$* (backward solution): Let

$$\begin{array}{rccc} & & \Sigma_1 & \Sigma_2 \\ B'A^{-1}B = & \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix} & \begin{array}{c} c_{1\cdot} \\ \\ \\ \\ c_{m\cdot} \end{array} & \begin{array}{c} c_{1\cdot} + a_{\cdot,n+1} \\ \\ \\ \\ c_{m\cdot} + a_{\cdot,n+m} \end{array} \end{array}$$

It is symmetric. Adjoining the matrix, as indicated, are two sum columns; for row k the first sum is $c_{k\cdot} = \sum_{i=1}^{m} c_{ki}$, the second $c_{k\cdot} + a_{\cdot,n+k} = c_{k\cdot} + \sum_{i=1}^{n} a_{i,n+k} =$ the sum of the kth row of $B'A^{-1}B$ and the kth column of B.

Then

$$c_{ij} = \sum_{k=1}^{n} \alpha_{k,n+i}\,\beta_{k,n+j} = \alpha_{1,n+i}\,\beta_{1,n+j} + \alpha_{2,n+i}\,\beta_{2,n+j} + \cdots + \alpha_{n,n+i}\,\beta_{n,n+j}$$

Each row of $B'A^{-1}B$, say, the kth, is checked by the relation

$$c_{k\cdot} + a_{\cdot,n+k} = \sum_{i=1}^{n} \alpha_{1,n+k}\beta_{i\cdot}$$

(*c*) *Computation of* $A^{-1}B$ (alternate backward solution): Let

$$\begin{array}{rcc} & & \Sigma \\ A^{-1}B = & \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ d_{n1} & d_{n2} & \cdots & d_{nm} \end{pmatrix} & \begin{array}{c} d_{1\cdot} \\ \cdot \\ \cdot \\ \cdot \\ d_{n\cdot} \end{array} \end{array}$$

The last row is computed first, then the preceding row, etc., as follows:

$$d_{ni} = \beta_{n,n+i}$$

Check: $d_{n\cdot} = \beta_{n\cdot} - 1$

$$d_{n-1,i} = \beta_{n-1,n+i} - \beta_{n-1,n}d_{ni}$$

Check: $d_{n-1,\cdot} = \beta_{n-1,\cdot} - \beta_{n-1,n}d_{n\cdot} - 1$

$$d_{n-2,i} = \beta_{n-2,n+i} - \beta_{n-2,n}d_{ni} - \beta_{n-2,n-1}d_{n-1,i}$$

Check: $d_{n-2,\cdot} = \beta_{n-2,\cdot} - \beta_{n-2,n}d_{n\cdot} - \beta_{n-2,n-1}d_{n-1,\cdot} - 1$

$\cdot$

$\cdot$

$\cdot$

$$d_{1,i} = \beta_{1,n+i} - \beta_{1n}d_{ni} - \beta_{1,n-1}d_{n-1,i} - \cdots - \beta_{12}d_{2i}$$

Check: $d_{1\cdot} = \beta_{1\cdot} - \beta_{1n}d_{n\cdot} - \beta_{1,n-1}d_{n-1,\cdot} - \cdots - \beta_{12}d_{2\cdot} - 1$

The inversion of A alone is a special case of either method; it is obtained by letting B be the identity matrix. Method b is the simpler. These computations become clear if we observe that the matrix consisting of the elements $\alpha_{i,n+j}$, and the matrix whose elements are $\beta_{i,n+j}$ correspond to matrices $(B'Q)'$ and PB where $PAQ = I$, the identity matrix, and PA is a triangular matrix whose elements on the main diagonal are ones. The other elements of the matrix PA are the β_{ij}. Method b of the backward solution corresponds to the matrix product $B'QPB = B'A^{-1}B$, and method c corresponds to $PB + [I - PA]A^{-1}B = A^{-1}B$.

TABLE 1.2

MOMENTS $\overline{M}^{(0)}_{xx}$

	Y_1	Y_6	Y_7	Z_1	z_2	Z_3	Z_4	Z_6	Z_7	Z_8	Z_9	Z_{10}	Σ
Y_1	4.0764	3.8846	7.4685	1.5631	472.8347	2.0532	3.8039	8.9707	1.6522	7.7248	3.4573	6.5324	524.0218
Y_6		5.0023	6.4885	1.8208	347.5748	2.3808	3.3439	9.2228	2.6585	9.6122	4.8047	5.8858	402.6797
Y_7			14.8409	2.5258	949.3906	3.7841	7.3076	17.9673	3.9648	15.2824	5.9185	12.9399	1047.8789
Z_1				1.0237	149.1434	.9936	1.4395	3.1355	.7041	3.0561	1.5462	2.1357	169.0875
z_2					67425.0000	212.6307	472.1944	1027.6748	161.8000	762.2073	317.8088	811.3831	73109.6426
Z_3						1.5190	1.9769	5.4554	1.7948	5.5140	2.3362	3.3131	243.7518
Z_4							4.1498	9.4261	2.3008	8.7767	2.8286	6.4832	524.0314
Z_6								28.3392	10.8826	29.5819	8.6868	16.2892	1175.6323
Z_7									8.1978	14.6186	3.2235	3.9762	215.7739
Z_8										35.2625	9.2871	14.4666	915.3902
Z_9											5.3842	5.4046	370.6865
Z_{10}												11.5024	900.3122

TABLE 2.1

TRANSLATION PAGE

Variable	Adjustment Factor, k	Mean, $m^{(0)}$	Adjusted Mean, $m = km^{(0)}$
Y_1	1	9.8646	9.8646
Y_6	1	8.1257	8.1257
Y_7	10^{-1}	7.0268	0.7027
Z_1	1	9.7833	9.7833
z_2	10^{-2}	15.5	0.1550
Z_3	1	8.3870	8.3870
Z_4	1	1.920957	1.920957
Z_5	10^{-1}	9.853613	0.9853613
Z_7	1	1.982105	1.982105
Z_8	10^{-1}	1.649280	0.1649280
Z_9	1	8.169624	8.169624
Z_{10}	10^{-1}	7.046014	0.7046014

In Table 4.1, $\bar{M}_{**}$, the submatrix of $\bar{M}_{zz}$ corresponding to the variables Z_1, z_2, Z_3, is given, and its inverse, $\bar{M}_{**}^{-1}$ is computed. Since $\bar{M}_{zz}$ was formed so that $\bar{M}_{**}$ constitutes its upper left corner, half of the Doolittle forward solution in Table 4.1 is the same as in Table 3 and could conveniently be copied.

Table 4.2 consists of three matrices:

1. $\bar{M}_{*\Delta}$ which is the matrix of adjusted moments of Z_1, z_2, Z_3, each with Y_1, Y_6, Y_7; they are copied from $\bar{M}_{z\Delta}$ in Table 3.
2. $P_{\Delta *}^{*\prime} = \bar{M}_{**}^{-1} M_{*\Delta}$, product of the two matrices,[9] and
3. $\bar{M}_{\Delta *} P_{\Delta *}^{*\prime}$.

The matrices $\bar{M}_{**}^{-1}$ and $P_{\Delta *}^{*\prime}$ are used later in the computation; consequently, the method of computing $\bar{M}_{\Delta z} \bar{M}_{zz}^{-1} \bar{M}_{z\Delta}$ is not used in computing $\bar{M}_{\Delta *} P_{\Delta *}^{*\prime}$.

From the above we compute $R_{\Delta\Delta} = \bar{M}_{\Delta z} \bar{M}_{zz}^{-1} \bar{M}_{z\Delta} - \bar{M}_{\Delta *} P_{\Delta *}^{*\prime}$; the first matrix is in Table 3, the second in Table 4.2. Our equation is overidentified, and so the procedure outlined in Section 3 of Chernoff-Divinsky is applicable. Next to $R_{\Delta\Delta}$ is written $\overline{W}_{\Delta\Delta}$ from Table 3. Then $Q_{\Delta\Delta} = R_{\Delta\Delta}^{-1} \overline{W}_{\Delta\Delta}$ is computed.[10] This is the content of Table 4.3.

Since the equation for the largest latent root $k_1^{\dagger}$ of $R_{\Delta\Delta}^{-1} \overline{W}_{\Delta\Delta}$ is of the third degree, the iterative method of solution described in Section 3 of Chernoff-Divinsky was used. To conserve space we omit the first few iterations from our original arbitrary vector. Also the iterations were

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[9] Matrix multiplication and its check are described in Section 2 of Chernoff-Divinsky.

[10] Cf. footnote 6.

TABLE 3

MATRIX $\overline{W}_{\Delta\Delta}$

	$\overline{M}_{zz}$									$\overline{M}_{z\Delta}$			Σ
	Z_1	z_2	Z_3	Z_4	Z_5	Z_7	Z_8	Z_9	Z_{10}	Y_1	Y_6	Y_7	
Z_1	1.0237	1.4914	0.9936	1.4395	0.3136	0.7041	0.3056	1.5462	0.2136	1.5631	1.8208	0.2526	11.6678
z_2		6.7425	2.1263	4.7219	1.0277	1.6180	0.7622	3.1781	0.8114	4.7283	3.4757	0.9494	31.6330
Z_3			1.5190	1.9769	0.5455	1.7948	0.5514	2.3362	0.3313	2.0532	2.3808	0.3784	16.9875
Z_4				4.1498	0.9426	2.3008	0.8777	2.8286	0.6483	3.8039	3.3439	0.7308	27.7647
Z_5					0.2834	1.0883	0.2958	0.8687	0.1629	0.8971	0.9223	0.1797	7.5274
Z_7						8.1978	1.4619	3.2285	0.3976	1.6522	2.6585	0.3965	25.4939
Z_8							0.3526	0.9287	0.1447	0.7725	0.9612	0.1528	7.5671
Z_9								5.3842	0.5405	3.4573	4.8047	0.5919	29.6886
Z_{10}									0.1150	0.6532	0.5886	0.1294	4.7365
	1.0237	1.4914	0.9936	1.4395	0.3136	0.7041	0.3056	1.5462	0.2136	1.5631	1.8208	0.2526	11.6678
	1.	1.4569	0.9706	1.4061	0.3063	0.6878	0.2985	1.5104	0.2086	1.5269	1.7786	0.2467	11.3974
		4.5697	0.6788	2.6248	0.5709	0.5922	0.3170	0.9255	0.5002	2.4511	0.8231	0.5814	14.6346
		1.	0.1485	0.5744	0.1249	0.1296	0.0694	0.2025	0.1095	0.5364	0.1801	0.1272	3.2025
			0.4538	0.1899	0.1564	1.0234	0.2077	0.6980	0.0497	0.1721	0.4913	0.0469	3.4893
			1.	0.4184	0.3447	2.2550	0.4577	1.5381	0.1096	0.3791	1.0826	0.1033	7.6885
				0.5386	0.1084	0.5424	0.1790	−0.1691	0.0399	0.1261	0.1053	0.0220	1.4926
				1.	0.2012	1.0071	0.3323	−0.3140	0.0740	0.2342	0.1955	0.0409	2.7711
					0.0403	0.3367	0.0550	0.0729	0.0098	0.0274	0.0713	0.0091	0.6226
					1.	8.3515	1.3645	1.8087	0.2437	0.6804	1.7671	0.2253	15.4410
						1.9703	0.1025	0.0273	−0.0484	−0.4847	−0.5095	−0.0564	1.0011
						1.	0.0520	0.0138	−0.0246	−0.2460	−0.2586	−0.0286	0.5081
							0.0045	0.0388	−0.0007	0.0030	0.0300	−0.0011	0.0744
							1.	8.6667	−0.1527	0.6614	6.7094	−0.2568	16.6281
								1.2666	0.0414	0.3064	0.7835	0.0217	2.4916
								1.	0.0327	0.2419	0.6186	0.0171	1.9103
									0.0023	0.0025	0.0060	0.0018	0.0126
									1.	1.0875	2.6716	0.7976	5.5567

TABLE 3 (Continued)

$\overline{M}_{\Delta z}\overline{M}_{zz}^{-1}\overline{M}_{z\Delta}$

	Y_1	Y_6	Y_7	Σ_1	Σ_2
Y_1	4.0128	3.8224	0.7470	8.5822	28.1631
Y_6		4.8990	0.6502	9.3716	30.3282
Y_7			0.1478	1.5450	5.3063

$\overline{M}_{\Delta\Delta}$

	Y_1	Y_6	Y_7	Σ
Y_1	4.0764	3.8846	0.7469	8.7078
Y_6		5.0023	0.6488	9.5357
Y_7			0.1484	1.5441

$\overline{W}_{\Delta\Delta} = \overline{M}_{\Delta\Delta} - \overline{M}_{\Delta z}\,\overline{M}_{zz}^{-1}\,\overline{M}_{z\Delta}$

	Y_1	Y_6	Y_7	Σ
Y_1	0.0636	0.0622	−0.0001	0.1257
Y_6		0.1033	−0.0014	0.1641
Y_7			0.0006	−0.0009

TABLE 4.1

	$\bar{M}_{**}$			I			Σ
	Z_1	z_2	Z_3				
Z_1	1.0237	1.4914	0.9936	1.			4.5088
z_2		6.7425	2.1263		1.		11.3602
Z_3			1.5190			1.	5.6389
	1.0237	1.4914	0.9936	1.			4.5088
	1.	1.4569	0.9706	0.9768			4.4043
		4.5697	0.6788	−1.4569	1.		4.7916
		1.	0.1485	−0.3188	0.2188		1.0486
			0.4538	−0.7542	−0.1485	1.	0.5511
			1.	−1.6618	−0.3273	2.2035	1.2144

	$\bar{M}_{**}^{-1}$			Σ
	Z_1	z_2	Z_3	
Z_1	2.6946	−0.0720	−1.6618	0.9608
z_2		0.2674	−0.3273	−0.1318
Z_3			2.2035	0.2144

TABLE 4.2

$\bar{M}_{*\Delta}$

	Y_1	Y_6	Y_7	Σ
Z_1	1.5631	1.8208	0.2526	3.6365
z_2	4.7283	3.4757	0.9494	9.1535
Z_3	2.0532	2.3808	0.3784	4.8125

$P_{\Delta*}^{*\prime} = \bar{M}_{**}^{-1}\bar{M}_{*\Delta}$

	Y_1	Y_6	Y_7	Σ
Z_1	0.4596	0.6997	−0.0166	1.1427
z_2	0.4801	0.0193	0.1119	0.6113
Z_3	0.3791	1.0826	0.1033	1.5651
Σ	1.3187	1.8017	0.1987	3.3190

$\bar{M}_{\Delta*} P_{\Delta*}^{*\prime}$

	Y_1	Y_6	Y_7	Σ
Y_1	3.7667	3.4079	0.7153	7.8900
Y_6		3.9187	0.6047	7.9314
Y_7			0.1411	1.4612

TABLE 4.3

$Q_{\Delta\Delta}$

	$R_{\Delta\Delta}$			$\overline{W}_{\Delta\Delta}$			Σ
	Y_1	Y_6	Y_7	Y_1	Y_6	Y_7	
Y_1	0.2462	0.4144	0.0317	0.0636	0.0622	−0.0001	0.8179
Y_6		0.9803	0.0455		0.1033	−0.0014	1.6043
Y_7			0.0067			0.0006	0.0829
	0.2462	0.4144	0.0317	0.0636	0.0622	−0.0001	0.8179
	1.	1.6834	0.1286	0.2582	0.2527	−0.0004	3.3224
		0.2827	−0.0078	−0.0448	−0.0014	−0.0012	0.2274
		1.	−0.0277	−0.1585	−0.0050	−0.0042	0.8046
			0.0024	−0.0095	−0.0094	0.0006	−0.0160
			1.	−4.0033	−3.9515	0.2458	−6.7090

$Q_{\Delta\Delta} = R^{-1}_{\Delta\Delta}\overline{W}_{\Delta\Delta}$

	Y_1	Y_6	Y_7	Σ
Y_1	1.2263	0.9533	−0.0365	2.1431
Y_6	−0.2692	−0.1143	0.0026	−0.3809
Y_7	−4.0033	−3.9515	0.2458	−7.7090

carried further than is indicated in the table where convergence of the elements of $k^{\dagger\prime}_{(n)}$ to only four places was required. In the actual computation, convergence to nine places required twelve steps instead of the five shown here.

The final vector of the iteration $Q_{(8)}$ is a characteristic vector b'_Δ of $R^{-1}_{\Delta\Delta}\overline{W}_{\Delta\Delta}$ associated with the largest root $k^{\dagger\prime}_{(8)} = k^{\dagger}_1$. Next is the product $c'_* = -P^{*\prime}_{\Delta *}b'_\Delta$, where $P^{*\prime}_{\Delta *}$ comes from Table 4.2. The iteraton for b'_Δ and the calculation of c'_* are in Table 4.4. The sum check for matrix multiplication is used in checking c'_*.

$c'_* =$	$-P^{*\prime}_{\Delta *}b'_\Delta$
Z_1	1.1216
z_2	0.2645
Z_3	−0.6850
Σ	0.7011

Table 4.5 concludes the estimation of coefficients. First the adjusted means are copied for each variable from Table 2.1. In the second column headed $[b_\Delta c_* e]'$ the first three numbers are the components of b_Δ, the second three are the components of c_*, the last number e is the estimate of γ_{10}. This estimate is equal to $-[b_\Delta c_*]m'$, the inner product of the two

Table 4.4

CALCULATION OF b'_Δ AND c'_*

	$Q'_{(0)}$	$Q'_{(1)}$	$Q'_{(2)}$	$Q'_{(3)}$	$Q'_{(4)}$	$Q'_{(5)}$	$Q'_{(6)}$	$Q'_{(7)}$	$10^{-1}Q'_{(8)} = b'_\Delta$
Y_1	−10.6465	−12.1964	−13.9311	−15.9135	−18.1784	−20.7659	−23.7216	−27.0980	−3.0955
Y_6	2.2819	2.7004	3.0867	3.5257	4.0275	4.6007	5.2556	6.0036	0.6858
Y_7	36.0835	42.4739	48.5957	55.5189	63.4220	72.4493	82.7615	94.5415	10.7998

	$k^\dagger_{(1)}$	$k^\dagger_{(2)}$	$k^\dagger_{(3)}$	$k^\dagger_{(4)}$	$k^\dagger_{(5)} = k^\dagger_1$
Y_1	1.1456	1.1422	1.1423	1.1423	1.1423
Y_6	1.1834	1.1430	1.1422	1.1423	1.1423
Y_7	1.1771	1.1441	1.1425	1.1423	1.1423

columns so far formed. The third column is the vector of adjustment factors k'; this also is copied from Table 2.1, with an adjustment 1 for e. The fourth column is the product of the preceding two; these are the estimates with the effect of the adjustments removed. The last column is obtained by dividing each element of the fourth by −3.0955, the coefficient of Y_1; these are the normalized estimates.

The next computation is for the estimate of the sampling variance-covariance matrix. We introduce the following notation: If y is a vector $(y_1\ y_2 \cdots y_n)$ then $_1y$ shall mean the vector y with the first element omitted: $_1y = (y_2 \cdots y_n)$. If A is a matrix, then $_{10}A$ is the matrix A with the first row deleted, $_{01}A$ is A with the first column deleted, $_{11}A$ is A with both the first row and first column deleted.

In Table 5.1 appear the vector b'_Δ copied from Table 4.4, $\overline{W}_{\Delta\Delta}$ copied from Table 3, and the products indicated in the table.

Table 5.2 shows the calculation of two constants, $l_1^\dagger/b_\Delta\overline{W}_{\Delta\Delta}b'_\Delta$ and C^*, that are used in subsequent calculations. In Table 5.3, the matrices

Table 4.5

ESTIMATES OF THE COEFFICIENTS

	m'	$[b_\Delta c_* e]'$	k'	$[b_\Delta^{(0)} c_*^{(0)} e^{(0)}]'$	$[b_\Delta^{(0)} c_*^{(0)} e^{(0)}]'^{(\mathrm{norm})}$
Y_1	9.8646	−3.0955	1	−3.0955	1
Y_6	8.1257	0.6858	1	0.6858	−0.2216
Y_7	0.7027	10.7998	10^{-1}	1.0800	−0.3489
Z_1	9.7833	1.1216	1	1.1216	−0.3623
z_2	0.1550	0.2645	10^{-2}	0.002645	−0.0009
Z_3	8.3870	−0.6850	1	−0.6850	0.2213
e		12.1051	1	12.1051	−3.9105

TABLE 5.1

	b'_Δ	$\overline{W}_{\Delta\Delta}$			Σ
		Y_1	Y_6	Y_7	
Y_1	-3.0955	0.0636	0.0622	-0.0001	0.1257
Y_6	0.6858		0.1033	-0.0014	0.1641
Y_7	10.7998			0.0006	-0.0009

$b_\Delta \overline{W}_{\Delta\Delta}$

	Y_1	Y_6	Y_7	Σ
	-0.1553	-0.1364	0.0059	-0.2858 -0.1305 (sum excluding Y_1)

$b_\Delta \overline{W}_{\Delta\Delta} b'_\Delta \quad = 0.4511$

${}_{11}[(b_\Delta \overline{W}_{\Delta\Delta})' \, (b_\Delta \overline{W}_{\Delta\Delta})] = ({}_1 b_\Delta \overline{W}_{\Delta\Delta})' \, ({}_1 b_\Delta \overline{W}_{\Delta\Delta})$

	Y_6	Y_7	Σ
Y_6	0.0186	-0.0008	0.0178
Y_7		0.0000	-0.0008

$F_{\beta\beta}$, $F_{\beta\gamma}$, and $F_{\gamma\gamma}$ are computed. The necessary steps should be clear from the table.

In Table 5.4, blocks of the variance-covariance matrix of the adjusted normalized coefficients are computed. These are

$$\overline{F} = C^* \begin{bmatrix} F_{\beta\beta} & -F_{\beta\gamma} \\ -F_{\gamma\beta} & F_{\gamma\gamma} \end{bmatrix} \qquad \overline{F}_1 m', \qquad {}_1 m \overline{F}_1 m' + \frac{C^*}{T^2}$$

The submatrices $F_{\beta\beta}$, $F_{\beta\gamma}$, $F_{\gamma\gamma}$ are from Table 5.3, C^* is from Table 5.2,

TABLE 5.2

$k_1^\dagger \quad = 1.1423$ (copied from Table 4.4)
$l_1^\dagger \quad = 1/k_1^\dagger = 0.8754$
$(l_1^\dagger / b_\Delta \overline{W}_{\Delta\Delta} b'_\Delta) = 1.9406$ (the denominator comes from Table 5.1)
$C \quad = (1 + l_1^\dagger) \, b_\Delta \overline{W}_{\Delta\Delta} b'_\Delta = 0.8460$
$T \quad = 30 =$ the number of observations
$F \quad = 6 =$ the number of y_Δ's and z_*'s
$k_1 \quad = 1 =$ the adjustment on Y_1
$C^* \quad = C/(T - F) \, (b_1 k_1)^2 = 0.003679$

TABLE 5.3

${}_{11}R_{\Delta\Delta}$ (from Table 4.3)

	Y_6	Y_7	Σ
Y_6	0.9803	0.0455	1.0258
Y_7		0.0067	0.0521

$$ {}_{11}H = {}_{11}R_{\Delta\Delta} - \frac{l_1^\dagger}{b_\Delta W_{\Delta\Delta} b'_\Delta} \; {}_{11}[(b_\Delta \overline{W}_{\Delta\Delta})' \, (b_\Delta \overline{W}_{\Delta\Delta})] $$

	Y_6	Y_7	Σ
Y_6	0.9442	0.0470	0.9913
Y_7		0.0066	0.0536

$$ F_{\beta\beta} = ({}_{11}H)^{-1} $$

	Y_6	Y_7
Y_6	1.6425	−11.7107
Y_7		235.0580

$({}_{01}P^*_{\Delta *})'$ (from Table 4.2)

	Y_6	Y_7	Σ
Z_1	0.6997	−0.0166	0.6832
z_2	0.0193	0.1119	0.1312
Z_3	1.0826	0.1033	1.1859
Σ	1.8017	0.1987	

$$ F_{\beta\gamma} = F_{\beta\beta}({}_{01}P^*_{\Delta *}) $$

	Z_1	z_2	Z_3	Σ
Y_6	1.3434	−1.2785	0.5680	0.6329
Y_7	−12.0900	26.0730	11.6130	25.5960

$$ F^\dagger_{\gamma\gamma} = ({}_{01}P^*_{\Delta *})' \, F_{\beta\gamma} $$

	Z_1	z_2	Z_3	Σ
Z_1	1.1404	−1.3267	0.2050	0.0187
z_2		2.8924	1.3103	2.8760
Z_3			1.8150	3.3303

$$F_{\gamma\gamma} = F^{\dagger}_{\gamma\gamma} + \overline{M}^{-1}_{**} \quad \text{(the last from Table 4.1)}$$

	Z_1	z_2	Z_3	Σ
Z_1	3.8350	−1.3987	−1.4568	0.9795
z_2		3.1599	0.9830	2.7442
Z_3			4.0185	3.5447

and ${}_1m'$ is the vector of adjusted means with the mean of Y_1 deleted. $\overline{F}$ corresponds to $V^*(b_\Delta, c_*)^{(\text{norm})}$ in the Chernoff-Divinsky paper.

The variance-covariance matrix $V^*(b_\Delta^{(0)}, c_*^{(0)}, e^{(0)})^{(\text{norm})}$ of the normalized, unadjusted coefficients is computed in Table 5.5. If we let D be a diagonal matrix whose diagonal elements are k_{Y_6}, k_{Y_7}, k_{Z_1}, k_{z_2}, k_{Z_3}, 1 then

TABLE 5.4

$$\overline{F} = C^* \begin{bmatrix} F_{\beta\beta} & -F_{\beta\gamma} \\ -F_{\gamma\beta} & F_{\gamma\gamma} \end{bmatrix}$$

	Y_6	Y_7	Z_1	z_2	Z_3	Σ
Y_6	0.0060	−0.0431	−0.0049	0.0047	−0.0021	−0.0394
Y_7		0.8647	0.0445	−0.0959	−0.0427	0.7274
Z_1			0.0141	−0.0051	−0.0054	0.0431
z_2				0.0116	0.0036	−0.0811
Z_3					0.0148	−0.0318

	${}_1m'$(from Table 2.1)	$\overline{F}({}_1m')$
Y_6	8.1257	−0.0463
Y_7	0.7027	0.3195
Z_1	9.7833	0.0834
z_2	0.1550	−0.0474
Z_3	8.3870	0.0251
Σ		0.3343

${}_1m[V^*(b_\Delta, c_*)/(b_1^{(0)})^2]{}_1m'$, the inner product of the two columns above, = 0.8670

$$\frac{C^*}{T^2} = 0.000004087$$

$${}_1m\overline{F}{}_1m' + \frac{C^*}{T^2} = 0.8670$$

TABLE 5.5

$V^*(b_\Delta^{(0)}, c_*^{(0)}, e^{(0)})^{(\text{norm})}$

	Y_6	Y_7	Z_1	z_2	Z_3	1
Y_6	0.0060	−0.00431	−0.0049	0.00047	−0.0021	0.0463
Y_7		0.008647	0.00445	−0.0000959	−0.00427	−0.03195
Z_1			0.0141	0.000051	−0.0054	−0.0834
z_2				0.00000116	0.00036	0.000474
Z_3					0.0148	−0.0251
1						0.8670

$$V^*(b_\Delta^{(0)}, c_*^{(0)}, e^{(0)})^{(\text{norm})} =$$

$$D\begin{pmatrix} \bar{F} & \bar{F}_1 m' \\ {}_1m'\bar{F} & {}_1m\bar{F}_1 m' + \dfrac{C^*}{T^2} \end{pmatrix} D$$

where the matrix consisting of the blocks $\bar{F}$, $\bar{F}_1 m'$, ${}_1m\bar{F}$ and ${}_1m\bar{F}_1 m' + C^*/T^2$ is the variance-covariance matrix of the adjusted normalized coefficients. This is equivalent to multiplying the estimated covariance between two adjusted coefficients by the adjustments on those coefficients.

The last step is to test for serial independence, using the Durbin-Watson statistic.[11]

$$d = \sum_{t=2}^{T} v_{1t}^2 \Big/ \sum_{t=1}^{T} \tilde{u}_{1t}^2 ,$$

where $\tilde{u}_{1t}$ is the calculated residual, and $v_{1t} = \tilde{u}_{1t} - \tilde{u}_{1,t-1}$. A simplification is given by the identity

$$\sum_t \tilde{u}_{1t}^2 = C^* \left(\frac{T - F}{T} \right), \qquad \text{so that} \qquad d = \frac{T \sum v_{1t}^2}{C^*(T - F)} .$$

This yields 1.11 for the value of the Durbin-Watson statistic.

The calculated residuals are sometimes useful for casual inspections of the results (such as seeing how closely they correspond to movements in omitted variables or whether striking historical events have occasioned particularly large residuals) and are therefore shown in Table 6.

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[11] J. Durbin and G. S. Watson, Testing for Serial Correlation in Least Squares Regression I, *Biometrika*, Vol. 37, Parts 3 and 4, pp. 409–428, Dec. 1950.

J. Durbin and G. S. Watson, Testing for Serial Correlation in Least Squares Regression II, *Biometrika*, Vol. 38, Parts 1 and 2, pp. 159–178, June 1951.

This statistic was not used in our other references, Chernoff-Divinsky and Klein.

TABLE 6
CALCULATED RESIDUALS BY LISE

Year	Calculated Residual ($\tilde{u}_t$)
1920	−0.030655
1921	−0.011751
1922	0.015434
1923	0.006589
1924	0.002086
1925	−0.003226
1926	−0.003212
1927	0.005435
1928	0.007704
1929	0.003884
1930	0.001225
1931	0.022113
1932	0.014616
1933	0.007543
1934	−0.006553
1935	0.004520
1936	−0.016416
1937	−0.011023
1938	−0.002789
1939	0.000803
1940	−0.003977
1941	−0.001526
1942	0.008565
1943	0.000108
1944	−0.009510
1945	0.003713
1946	−0.000433
1947	0.002002
1948	0.000254
1949	−0.005531

Equation 4.1 was also estimated by the least squares method, with Y_1 chosen as the dependent variable, and $v = (Y_6\ Y_7\ Z_1\ z_2\ Z_3)$ the independent variables. The adjustment factors are the same as before.

First, the matrix of the adjusted moments of the independent variables, $\bar{M}_{vv}$, is formed, analogously to Table 3, and inverted (we omit presenting the intermediate forward solution).

TABLE 7

	Y_6	Y_7	Z_1	z_2	Z_3	Σ
			$\bar{M}_{vv}$			
Y_6	5.0023	0.6488	1.8208	3.4757	2.3808	14.3285
Y_7		0.1484	0.2526	0.9494	0.3784	3.3776
Z_1			1.0237	1.4914	0.9936	6.5822
z_2				6.7425	2.1263	15.7854
Z_3					1.5190	8.3981
			$\bar{M}_{vv}^{-1}$			
Y_6	1.2256					−6.9426
Y_7	−7.4375	182.6998				152.3685
Z_1	−0.9809	8.2322	3.5173			8.3440
z_2	0.8085	−20.2975	−0.9741	2.5228		−17.0453
Z_3	−0.5583	−10.8284	−1.4506	0.8950	3.9268	−8.0154

Table 8 gives the computation of the coefficient estimates; first $\bar{M}_{Y_1v}$ is formed, which is the column of the adjusted moments of Y_1 with each of the v's. Then the adjusted means m' are copied for each variable from Table 2.1. The third column, headed $[\beta, e]'$, consists of two calculations; $\beta' = \bar{M}_{vv}^{-1}\bar{M}_{vY_1}$, a column of five numbers, and $e = m_{Y_1} - \Sigma_{i=1}^{5}\beta_{v_i}m_{v_i}$, where we consider $v_1 = Y_6$, $v_2 = Y_7$, $v_3 = Z_1$, etc. The sum indicated is of β' only and is used for a check of the first calculation. The next column is k', the vector of adjustment factors copied from Table 2.1. The last column consists of the desired estimates with adjustments canceled; it is obtained by multiplying each element of column 3 by its corresponding adjustment and dividing by the adjustment on Y_1, here 1. From the form in which equation 4.1 is written, we actually find $-\beta_{16} = \beta_{Y_6} - \beta_{17} = \beta_{Y_7}$, etc.

TABLE 8

ESTIMATION OF COEFFICIENTS

	$\bar{M}_{Y_1v}$	m'	$[\beta e]'$	k'	$[\beta^{(0)}e^{(0)}]$
Y_1		9.8646		1	
Y_6	3.8846	8.1257	0.3495	1	0.3495
Y_7	0.7469	0.7027	2.2191	10^{-1}	0.2219
Z_1	1.5631	9.7833	0.2518	1	0.2518
z_2	4.7283	0.1550	0.2250	10^{-2}	0.002250
Z_3	2.0532	8.3870	−0.2286	1	−0.2286
e			4.8846	1	4.8846

TABLE 9

CALCULATED RESIDUALS $\tilde{u}_t$ BY LEAST SQUARES ESTIMATES

Year	$\tilde{u}_t$
1920	−0.025691
1921	−0.018010
1922	0.010916
1923	0.005649
1924	0.003658
1925	0.000330
1926	−0.001360
1927	0.005980
1928	0.006856
1929	0.005047
1930	0.002435
1931	0.020018
1932	0.005293
1933	0.002743
1934	−0.000214
1935	0.009688
1936	−0.016350
1937	−0.002530
1938	−0.007963
1939	−0.002446
1940	−0.003390
1941	0.000515
1942	0.011898
1943	0.000561
1944	−0.002921
1945	0.003981
1946	−0.005544
1947	−0.000094
1948	0.000504
1949	−0.009554

The residuals $\tilde{u}_t$ calculated from the least squares estimates are in Table 9.

The Durbin-Watson statistic, using the formula given previously, is $d = 1.28$.

To obtain estimates of the sampling covariance matrix, we find (Table 10A).

TABLE 10*A*

1. $\overline{M}_{Y_1 v}\, \overline{M}_{vv}^{-1} \overline{M}_{v Y_1} = \overline{M}_{Y_1 v} \beta' = 4.0033$
2. $\overline{M}_{Y_1 Y_1} = 4.0764$
3. $\overline{W}_{Y_1 Y_1} = 2. - 1. = 0.0732$
4. $\overline{W}_{Y_1 Y_1}/T - F = 0.003048$; T and F have the same meaning and value as in LISE.

$m_v \overline{M}_{vv}^{-1}$

Y_6	Y_7	Z_1	z_2	Z_3	Σ
−9.4201	154.5174	19.9083	−9.3254	6.7363	162.4164

and $m_v \overline{M}_{vv}^{-1} m_v' = 281.8506$ where m_v is the vector of means of Y_6, Y_7, Z_1, z_2, Z_3.

TABLE 10*B*
ADJUSTED COVARIANCE MATRIX

$$V^*(\beta, e) = \frac{W_{Y_1 Y_1}}{T - F} \left(\begin{array}{c|c} \overline{M}_{vv}^{-1} & -\overline{M}_{vv}^{-1} m_v' \\ \hline & m_v \overline{M}_{vv}^{-1} m_v' + \frac{1}{T^2} \end{array} \right)$$

	Y_6	Y_7	Z_1	z_2	Z_3	e
Y_6	0.003736	−0.02267	−0.002990	0.002464	−0.001702	0.02871
Y_7		0.5569	0.02509	−0.06187	−0.03301	−0.4710
Z_1			0.01072	−0.002969	−0.004422	−0.06068
z_2				0.007690	0.002728	0.02843
Z_3					0.01197	−0.02053
e						0.8591

To obtain the final covariance matrix, multiply each element of $V^*(\beta, e)$ by the two adjustment factors corresponding to the row and column in which it is located, and divide by $(k_{Y_1})^2$, in this case 1.

TABLE 10*C*
COVARIANCE MATRIX
$V^*(\beta^{(0)}, e^{(0)})$

	Y_6	Y_7	Z_1	z_2	Z_3	e
Y_6	0.003736	−0.002267	−0.002990	0.00002464	−0.001702	0.02871
Y_7		0.005569	0.002509	−0.00006187	−0.003301	−0.04710
Z_1			0.01072	−0.00002969	−0.004422	−0.06068
z_2				0.0000007690	0.00002728	0.0002843
Z_3					0.01197	−0.02053
e						0.8591

We also obtain the estimated variance s^2 of the residual $\tilde{u}$, and the correlation coefficient R:

$$s^2 = \frac{\overline{W}_{Y_1Y_1}}{T(T-F)} \cdot \frac{1}{(k_{Y_1})^2} = 0.001016; \qquad s = 0.03187$$

$$R^2 = \frac{\bar{M}_{Y_1v}\bar{M}_{vv}^{-1}\bar{M}_{vY_1}}{\overline{M}_{Y_1Y}} = 0.9821; \qquad R = 0.9910$$