CHAPTER XXV

COMPUTATIONAL SUGGESTIONS FOR MAXIMIZING A LINEAR FUNCTION SUBJECT TO LINEAR INEQUALITIES

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It is the purpose of this chapter to record suggestions that arose from discussions between the authors and G. B. Dantzig regarding iterative computational procedures for maximizing a linear function,

$$y = c'x \tag{1}$$

(y scalar, c and x vectors), subject to linear inequalities,

$$\alpha_k + a_k'x \geqq 0 \quad (k = 1, \cdots, K, \alpha_k \text{ scalar}, a_k \text{ vector}). \tag{2}$$

We distinguish two main cases (1 and 2) and record in each case two suggestions. At present, insufficient experience or theoretical knowledge is available to assess the possible usefulness of these suggestions. No proofs of convergence are offered. The general idea underlying the suggestions is an attempt to make big jumps rather than "crawling along the edges" of the convex set (2), as in the simplex method. It depends on the set (2) whether in fact faster convergence is obtained. In comparing two different methods, it is usually possible to construct sets (2) so as to favor one method as compared with the other. All methods indicated are based on some idea of steepest ascent and thus depend on the units of measurement of the variables x.

1. THE CASE IN WHICH AN INITIAL POINT x_0 SATISFYING (2) IS KNOWN

1.1. *Traversal method.* Find the largest value θ_0 of the scalar θ such that

$$x = x_0 + \theta c \tag{3}$$

satisfies (2), and write

$$\bar{x}_0 = x_0 + \theta_0 c. \tag{4}$$

Then, if we insert $\bar{x}_0$ for x in (2), we must for at least one value, k_0, say, of k, have an equality

$$\alpha_{k_0} + a'_{k_0}\bar{x}_0 = 0 \tag{5}$$

because otherwise values $\theta > \theta_0$ could be found for which x satisfies (2). If (5) is true for only one value of k, determine scalars λ_0, μ_0, such that

$$c'(\lambda_0 a_{k_0} + \mu_0 c) = 0. \tag{6}$$

This is impossible only if $c = -\nu a_{k_0}$, ν a positive scalar, in which case $\bar{x}_0$ already maximizes y. Determine $\bar{\theta}_0$ as the largest value of θ for which

$$x = \bar{x}_0 + \theta(\lambda_0 a_{k_0} + \mu_0 c) \tag{7}$$

satisfies (2), and write

$$x_1 = \bar{x}_0 + \tfrac{1}{2}\bar{\theta}_0(\lambda_0 a_{k_0} + \mu_0 c). \tag{8}$$

Proceed with x_1 as previously with x_0. If at the nth step more than one value of k_n of k satisfies, or nearly satisfies, an equation like (5), select one arbitrarily, or use an average of all a_{k_n} that satisfy (5) exactly, or within a small amount ϵ, under some rule of normalization used for the vectors a_k.

1.2. *Plane intersection method.* Having obtained the point (4) above, intersect the plane

$$x = \bar{x}_0 + \lambda a_{k_0} + \mu c \tag{9}$$

(λ and μ freely variable scalar parameters) successively with each of the hyperplanes

$$\alpha_k + a'_k x = 0. \tag{10}$$

The intersections consist of K straight lines inside (9). The segments of these lines on which (2) is satisfied form a convex polygon. On the polygon select a point on which y reaches its maximum. Generally there is just one such point, for which write $\bar{x}_1$. Now there are two variants.

1.2a. *Plane determined by normal to the convex set* (2). If x_1 is unique, there are at least two values of k for which (10) is satisfied. Take any one of these, or take their average, as a_{k_1} and proceed as in (9) with $\bar{x}_0$ replaced by $\bar{x}_1$.

1.2b. *Plane determined by normal within boundary of the convex set* (2). Having arrived close to the maximum, it may be desirable to attempt not to lose any of the equalities (10) once they are satisfied. Let $\bar{x}_n$ be such that

$$\alpha_k + a'_k\bar{x}_n = 0 \qquad (k = k_1, k_2, \cdots, k_{r_n}). \tag{11}$$

In the space of the vectors d such that

$$d'a_{k_r} = 0 \qquad (r = 1, \cdots, r_n), \tag{12}$$

choose the vector of steepest ascent, i.e., the vector d satisfying

$$d'd = 1, \qquad d'c = \text{maximum}, \tag{13}$$

and use that vector as a_{k_n} in (9). This will have been wasted effort if the resulting $\bar{x}_{n+1}$ fails to satisfy (11). Since iterations on this principle become computationally more expensive as r_n grows, the present variant should only be employed toward the end of a sequence of iterations.

2. How to Obtain an Initial Point Satisfying (2)

2.1. *Successive penetration method.* Take an arbitrary initial point, x_0. This point partitions the set S of inequalities (2) into two subsets, S_0 and S'_0, those of S_0 being satisfied by x_0, those of S'_0 not being satisfied by x_0. If S'_0 is empty, the goal has been achieved. If it is not, select arbitrarily an inequality of S'_0, numbered k_0, say. Use a_{k_0} as the vector c in (1) indicating the "desired direction." Use the inequalities of S_0 instead of the full set of conditions (2), and apply any variant of the method in Section 1 until a point is reached in which the inequality number k_0 is satisfied. Call that point x_1 and proceed with a new subset S_1 of the inequalities (2). Obviously

$$S_0 \subset S_1 \subset S_2 \cdots. \tag{14}$$

If for any n a maximum x_n of $a'_{k_n}x$ subject to the inequalities of S_n fails to satisfy (2) for $k = k_n$, no point satisfying (2) exists.

2.2. *Guided penetration method.* Instead of selecting an arbitrary a_k of S'_0 to be the c in (1), take

$$c = \sum_{k \in S'_0} (\alpha_k + a'_k x_0) \frac{a_k}{a'_k a_k}. \tag{15}$$

This is the vector sum of the normals dropped from x_0 onto the planes

$$(\alpha_k + a_k x) = 0 \qquad (k \in S'). \tag{16}$$

Two alternative modes of proceeding from here are worth considering.

2.2a. Keep the c so selected constant while making a number of iterative improvements to x_0 by method A, always requiring that the inequalities S_0 be preserved.

2.2b. Be willing to sacrifice some inequalities of S_0 if thereby a larger number of inequalities of S_0' can be satisfied. In this case determine θ_0 in (4) in such a way as to minimize the number of inequalities in S_1'.

For neither of these variants certain attainment of the objective (if attainable) has been proved. They might, however, work faster than the successive penetration method. The second alternative is suspect if the convex set (2) is not bounded.