XIX. NOTE ON RANDOM COEFFICIENTS

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Let us consider an equation of the form

$$(1) \qquad y_t = \sum_{k=1}^{K} a_{kt} x_{kt}, \qquad t = 1, \ldots, T,$$

which defines the dependent variable y_t in terms of the fixed variables x_{kt} and the coefficients a_{kt} which are assumed to be normally and independently distributed with mean α_k and variance $\sigma_k^2 \varphi_{kt}$, where the inverse weights φ_{kt} are known functions of the fixed variates.[1] We could estimate the parameters α_k by least squares, i.e., by the formula

$$\overline{\alpha}_k = \sum_{l=1}^{L} m_{yx_l} m^{lk},$$

where

$$m_{yx_l} = \sum_{t=1}^{T} y_t x_{lt},$$

and m^{lk} is the element in the lth row and kth column of

$$(m_{kl})^{-1} = \left(\sum_{t=1}^{T} x_{kt} x_{lt}\right)^{-1}.$$

Although this method can be shown, under certain conditions, to be consistent, it would not be efficient.

We shall derive equations defining maximum-likelihood estimates of the α_i and σ_j^2. Let us consider the distribution of the y_t. We see that the quantities

$$(2) \qquad z_t = y_t - \sum_{k=1}^{K} \alpha_k x_{kt}, \qquad t = 1, \ldots, T,$$

[1] See also [XVIII].

are normally and independently distributed with mean 0 and variance

$$\sum x_{kt}^2 \sigma_k^2 \varphi_{kt}^2 . \tag{3}$$

Then the likelihood function of the observation becomes

$$\begin{aligned} L = {} & (2\pi)^{-\frac{T}{2}} \prod_{t=1}^{T} \left(\sum x_{kt}^2 \sigma_k^2 \varphi_{kt}\right)^{-\frac{1}{2}} \\ & \times \exp\left\{-\frac{1}{2} \sum_{t=1}^{T} \frac{\left(y_t - \sum_{k=1}^{K} \alpha_k x_{kt}\right)^2}{\sum_{k=1}^{K} x_{kt}^2 \sigma_k^2 \varphi_{kt}}\right\}. \end{aligned} \tag{4}$$

Let us maximize $\log L$ with respect to α_i and σ_j^2, subject to the restriction $\sigma_j^2 \geq 0$. We obtain

$$\sum_{t=1}^{T} \frac{x_{it}\left(y_t - \sum_{k=1}^{K} \alpha_k x_{kt}\right)}{\sum_{k=1}^{K} x_{kt}^2 \sigma_k^2 \varphi_{kt}} = 0, \tag{5}$$

$$\sum_{t=1}^{T} \frac{x_{jt}^2 \varphi_{jt}}{\sum_{k=1}^{K} x_{kt}^2 \sigma_k^2 \varphi_{kt}} = \sum_{t=1}^{T} \frac{x_{jt}^2 \varphi_{jt}\left(y_t - \sum_{k=1}^{K} \alpha_k x_{kt}\right)^2}{\left(\sum_{k=1}^{K} x_{kt}^2 \sigma_k^2 \varphi_{kt}\right)^2} - \lambda_j, \tag{6}$$

$$\lambda_j \sigma_j^2 = 0, \tag{7}$$

$$\sigma_j^2 \geq 0. \tag{8}$$

It is necessary to introduce the Lagrange multipliers λ_j because the solutions of (5) and (6) with $\lambda_j = 0$ might give negative σ_j^2. We take that solution of (5), (6), (7), and (8) which gives to (4) its highest value. It should be noted that it is unnecessary to consider solutions having λ_j's $\neq 0$ if there is a solution with λ_j's $= 0$, and, in general, if there is a solution with only $\lambda_{j_1}, \ldots, \lambda_{j_\beta} \neq 0$, it is not necessary to consider solutions with those λ_j's $\neq 0$ and other λ_j's $\neq 0$.