XVI. MODELS INVOLVING A CONTINUOUS TIME VARIABLE

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1. In the opinion of this author, the simultaneous-equations method to which the first part of this volume is devoted constitutes an important advance over single-equation methods in the measurement of economic relations. Reasons for this opinion have been stated elsewhere [Koopmans, 1945]. In this note it is my intention to point out certain shortcomings of the new methods in their present stage.

Limitations to the usefulness of the new methods arise from the combination of the following two aspects in the specification of the distribution of the variables:

(a) the treatment of time as a discrete variable,

(b) the assumption that disturbances at different points in time are independent.

These aspects are also found in the single-equation methods that have been generally used, and some of the limitations to be mentioned below therefore apply equally to simultaneous-equations and single-equation methods. However, the development of the simultaneous-equations method has brought to light further disadvantages of the specifications (a) and (b) above, and thus has made their revision even more desirable than before.

2. It was found in another article[1] that in the simultaneous-equations method the procedure for estimating the coefficient α_{11} of the endogenous variable x_1 in the first structural equation, say, is different according to whether that variable is "predetermined" or is one of the "jointly dependent variables." A given variable may be in one category or the other depending solely on the timing of that variable in the equation concerned. If an endogenous variable x_1 appears in the first equation only with a lag of one unit behind the most recent timing of that variable in the

[1] [II-*3.1.3*]. For the definition of "endogenous," "predetermined," "dependent," variables, see also [XVII].

equation system as a whole, then in the first equation it will be classified as predetermined. If the time lag is reduced to zero, the variable is in general to be treated as one of the jointly dependent variables. This situation can be reduced *ad absurdum* by making the time unit of measurement smaller and smaller and at the same time reducing the time lag to zero. Then suddenly at the moment the time lag reaches zero the status of the variable x_1 in the first equation is changed, and with it the "unbiased" estimate of the coefficient in question is changed. The solution of this paradox is, of course, that such a procedure is illegitimate. The assumption of independent disturbances in successive observations can be maintained only if the size of the time unit to which these observations refer is not made too small. Therefore, the independence assumption makes the distinction between predetermined and dependent variables appear as absolute instead of a matter of degree, which it would be in a more refined model.

The main source of disturbances is the erratic element in economic behavior. Some causes of erratic behavior (not already represented by measurable variables), like the weather affecting the amount and direction of consumers' expenditure, may be so variable as to reverse themselves in a few days. Other causes like fads and fashions affecting consumption, confidence or lack of confidence affecting investment, may lead to deviations in the same direction for a whole year or even longer. Therefore, as the time unit of observation is reduced in size, a situation in which serial correlation of the disturbances in a given equation can no longer be neglected is bound to arise at some stage.

Methods based on the independence assumption thus involve a lower bound on the permissible size of the time unit of observation. This precludes adequate treatment of a number of important statistical problems in the measurement of economic relations. The time lags occurring in economic behavior are not always integral multiples of one time unit of a size compatible with the independence assumption. They are almost always distributed lags, with the lower limit to the range of lags sometimes practically equal to zero. There is therefore a need for methods of estimating the parameters that characterize lag distributions.

Another problem that cannot be studied adequately under the independence assumption is that of the most economic time unit of observation. Whether it is best to use annual, quarterly, or monthly, data depends on a comparison of the cost of collection of such data (if not already available), the cost of calculating the necessary estimates, and the information gained (in the sense of smaller sampling errors of estimated parameters) by a given reduction in the size of the unit period of observation. The latter gain in information is likely to be reduced by serial correlation in the disturbances. For time units below a certain size, this gain in information can therefore not be analyzed theoretically on the basis of the independence assumption.

3. An adequate model for the study of the foregoing problem is obtained by considering the disturbances (and therefore the economic variables) as generated by a stochastic process with a continuous time variable. Let us for simplicity assume that this process is normal and stationary, i.e., the values of the variables $u_g(t)$ at any set of time points $t_1, t_2, \ldots, t_S$ have a joint normal distribution depending only on the differences $t_2 - t_1, \ldots, t_S - t_{S-1}$. In that case the process is entirely characterized [Doob, Theorem 4.3] by the elements

$$\mathcal{E}\, u_g(t)\, u_h(t) = \sigma_{gh}(0) \tag{1}$$

of the covariance matrix $\Sigma(0)$, and by a matrix Ξ, determining the lagged covariance matrix

$$\mathcal{E}\, u_g(t)\, u_h(t+\tau) = \sigma_{gh}(\tau) \tag{2}$$

through the formula

$$\Sigma(\tau) = \Sigma(0)\, e^{-\tau\Xi}, \qquad \tau > 0. \tag{3}$$

If economic variables $x_n(t)$ are considered as determined by equations in which quantities $u_g(t)$ of this nature are the only random elements, it is necessary to define further the way in which these variables are observed. In practice, the method of observation is again a discrete procedure. One method of observation is to make readings

$$\begin{gathered} x_n(t_0),\ x_n(t_1),\ \ldots,\ x_n(t_S), \\ t_s - t_{s-1} = h, \qquad s = 1, \ldots, S, \end{gathered} \tag{4}$$

at equidistant points in time. Price variables are sometimes observed in this way. Quantities of goods and flows of money are usually observed through averages

(5) $$\bar{x}_n(t_s) = \int_{t_s - h}^{t_s} x_n(\tau)\,d\tau$$

over a period of observation of length h. For any specified method of obtaining a finite number of observations for each variable in the system, the set of all observations on all variables becomes subject to a joint probability distribution derivable from (3) (or from whatever other process of a continuous time variable t is specified).

Although the mathematical difficulties involved may be considerable, a model of this kind would provide a means of studying the estimation of lag distributions and the choice of the most economic time unit of observation.

4. Perhaps the most important advantage of a continuous treatment of time has not yet been mentioned. It is explained in another article [II-2.5.6] that, in the discrete case, as soon as the independence assumption for successive disturbances is dropped, the problem of identification of the structural equations is greatly complicated. For the removal of that assumption may open up a new group of transformations of the equations (involving shifts along the time axis) that preserve the probability distribution of the variables.

The introduction of a continuous time variable is perhaps the best way to study fully all aspects of the identification problem of relations between economic time series. Consider for instance a system of one linear equation containing only one variable x. In the discrete formulation this equation would be of the type

(6) $$\mathcal{L}x(t) \equiv x(t) + \alpha_1 x(t-1) + \cdots + \alpha_\tau x(t-\tau) = u(t).$$

Under the independence assumption for $u(t)$,

(7) $$\mathcal{E}u(t)u(t+\theta) = 0 \quad \text{if} \quad \theta \neq 0,$$

there is no identification problem. For any linear combination of the type

(8) $$\lambda_0 \mathcal{L}x(t) + \lambda_1 \mathcal{L}x(t-1) + \cdots + \lambda_K \mathcal{L}x(t-K) = \bar{v}(t) = \sum_{k=0}^{K} \lambda_k u(t-k)$$

introduces serial correlation into $v(t)$ unless all but one of the quantities λ_k vanish.

In the continuous formulation the equation is of the type

$$(9) \qquad \mathcal{L}^* x(t) \equiv x(t) - \int_{-\infty}^{t} \varphi(t-\tau)\, x(\tau)\, d\tau = u(t)$$

where the disturbance process, if normal and stationary, is described by

$$(10) \qquad \mathcal{E} u(t)\, u(t+\theta) = f(|\theta|).$$

Now there are infinitely many transformations in the space of the functions φ and f which preserve the form of (9). The simplest of these is obtained by substituting

$$(11) \qquad \int_{-\infty}^{t} \varphi(t-\tau)\, x(\tau)\, d\tau + u(t)$$

for $x(t)$ under the integral sign in (9). Another possibility is first to write

$$(12) \qquad \varphi(t-\tau) = \varphi_1(t-\tau) + \varphi_2(t-\tau),$$

and to substitute (11) only in one of the two integrals so obtained, etc. The only invariants of all these transformations are the auto-covariance function

$$(13) \qquad g(\theta) = \mathcal{E} x(t)\, x(t+\theta)$$

of the variable $x(t)$ and all its functions and functionals. These are therefore the only identifiable characteristics of the process (9), and only these are subject to estimation.

The identification difficulties just described are absent from the process

$$(14) \qquad x(t) = \int_{-\infty}^{t} \varphi(t-\tau)\, x(\tau)\, d\tau + \alpha y(t) + u(t)$$

containing an observable exogenous variable $y(t)$. They reappear if $y(t)$ occurs in the form

$$(15)\quad x(t) = \int_{-\infty}^{t} \varphi(t-\tau)\, x(\tau)\, d\tau + \int_{-\infty}^{t} \psi(t-\tau)\, y(\tau)\, d\tau + u(t)$$

with unknown lag-distribution functions φ and ψ.

The process (9) becomes a system of equations if $x(t)$ and $u(t)$ are interpreted as column vectors, $\varphi(t-\tau)$ as a matrix. In this case, the identification of individual equations is aided if economic considerations require or permit the specification that all diagonal elements of $\varphi(t-\tau)$ shall vanish for all values of $t-\tau$. In two dimensions this leads to the system

$$(16)\quad \begin{aligned} x_1(t) &= \int_{-\infty}^{t} \varphi_{12}(t-\tau)\, x_2(\tau)\, d\tau + u_1(t), \\ x_2(t) &= \int_{-\infty}^{t} \varphi_{21}(t-\tau)\, x_1(\tau)\, d\tau + u_2(t). \end{aligned}$$

The equations (16) are completely identified, i.e., there is no transformation other than the identity, in the space of the functions φ_{12}, φ_{21} and f_1, f_2 [defined as in (10)] which preserves the form of (16).