III. NOTE ON THE IDENTIFICATION OF ECONOMIC RELATIONS

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T. C. Koopmans and H. Rubin have discussed the problem of identification of economic relations in [II - 2], and have obtained a number of very interesting results. In this note the problem treated by Koopmans and Rubin is somewhat generalized and a different approach to its solution is briefly discussed.

1. Definitions and Formulation of the Problem

Let $x_1, \ldots, x_K$ be a set of K variables[1] and let $A = [\alpha_{gk}]$ $(g = 1, \ldots, G;\ k = 1, \ldots, K)$ be a given matrix of rank G. Denote the linear form $\sum_{k=1}^{K} \alpha_{gk} x_k$ by l_g $(g = 1, \ldots, G)$ and let $\Sigma = [\ \sigma_{gh}\ ]$ $(g, h = 1, \ldots, G)$ be a given symmetric and positive definite matrix. Furthermore, let

$$(1.1)\qquad \varphi_r(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{GK}, \sigma_{11}, \sigma_{12}, \ldots, \sigma_{GG}) = 0 \qquad (r = 1, \ldots, R)$$

be a given system of equations, called a priori restrictions, that are satisfied by the quantities α_{gk} and σ_{gh}. For any nonsingular matrix $\Upsilon = [\ \upsilon_{gh}\ ]$ $(g, h = 1, \ldots, G)$ we shall denote the matrix ΥA by $A(\Upsilon)$ and the elements of $A(\Upsilon)$ by $\alpha_{gk}(\Upsilon)$ $(g = 1, \ldots, G;$ $k = 1, \ldots, K)$. Thus, $\alpha_{gk}(\Upsilon) = \sum_{h=1}^{G} \upsilon_{gh}\alpha_{hk}$. Furthermore, we shall

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[1] The integer K corresponds to what was denoted K_x in [II] .

denote the matrix $\Upsilon \Sigma \Upsilon'$ (Υ' is the transpose of Υ) by $\Sigma(\Upsilon)$ and the elements of $\Sigma(\Upsilon)$ by $\sigma_{gh}(\Upsilon)$ $(g, h = 1, \ldots, G)$. Finally, the linear form $\sum_{h=1}^{G} \upsilon_{gh} l_h$ will be denoted by $l_g(\Upsilon)$.

DEFINITION 1.1. *A nonsingular matrix* $\Upsilon = [\upsilon_{gh}]$ $(g, h = 1, \ldots, G)$ *is said to be an admissible transformation if and only if the equations*

$$\varphi_r\{\alpha_{11}(\Upsilon), \alpha_{12}(\Upsilon), \ldots, \alpha_{GK}(\Upsilon), \sigma_{11}(\Upsilon), \sigma_{12}(\Upsilon), \ldots, \sigma_{GG}(\Upsilon)\} = 0 \quad (r = 1, \ldots, R) \tag{1.2}$$

are fulfilled.

DEFINITION 1.2. *An element* α_{gk} *of the matrix* A *is said to be identifiable*[1] *if* $\alpha_{gk}(\Upsilon)$ *takes only a finite number of different values over the domain of all admissible transformations* Υ. *Similarly, an element* σ_{gh} *of* Σ *is said to be identifiable if* $\sigma_{gh}(\Upsilon)$ *takes only a finite number of different values over the domain of all admissible transformations* Υ.

DEFINITION 1.3. *The linear form* l_g *is said to be identifiable if the coefficients* $\alpha_{g1}, \ldots, \alpha_{gK}$ *are identifiable.*

The matrix A has GK elements and the matrix Σ has $(G^2 + G)/2$ elements. Thus, the total number of elements in the two matrices A and Σ is equal to $GK + (G^2 + G)/2 = P$ (say). Consider the elements of A and Σ arranged in an ordered sequence and denote them by $\theta_1, \ldots, \theta_P$, respectively. The set $\theta = (\theta_1, \ldots, \theta_P)$ can be represented by a point in the P-dimensional space, called parameter space. For any nonsingular transformation Υ we shall denote the point $(\theta_1(\Upsilon), \ldots, \theta_P(\Upsilon))$ by $\theta(\Upsilon)$.

DEFINITION 1.4. *A coordinate* θ_p *of a point* θ *will be said to be locally identifiable if there exists an open set* ω *containing* θ *such that for any admissible transformation* Υ *either* $\theta_p(\Upsilon) = \theta_p$ *or* $\theta(\Upsilon)$ *lies outside* ω.

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[1] This concept corresponds to what was called multiple identifiability in [II-2.4.4].

The problem considered in this note is to formulate conditions under which a coordinate θ_p of a point θ of the parameter space is identifiable or is locally identifiable.

2. *Two Lemmas*

In this section we shall prove two lemmas which will then be used for deriving necessary and sufficient conditions for the identification of θ_p.

Consider the quadratic form

$$X = \sum_{h=1}^{G} \sum_{g=1}^{G} \sigma^{gh} l_g l_h , \tag{2.1}$$

where $[\sigma^{gh}]$ is the inverse of $[\sigma_{gh}]$. Let ξ_{kl} $(k, l = 1, \ldots, K)$ denote the coefficient of $x_k x_l$ in X. For any nonsingular transformation Υ we shall put

$$X(\Upsilon) = \sum_{h=1}^{G} \sum_{g=1}^{G} \sigma^{gh}(\Upsilon) l_g(\Upsilon) l_h(\Upsilon) , \tag{2.2}$$

where $[\sigma^{gh}(\Upsilon)]$ denotes the inverse of $[\sigma_{gh}(\Upsilon)]$. We shall denote by $\xi_{kl}(\Upsilon)$ the coefficient of $x_k x_l$ in $X(\Upsilon)$.

LEMMA 2.1. *For any nonsingular transformation* Υ *we have* $\xi_{kl}(\Upsilon) = \xi_{kl}$ $(k, l = 1, \ldots, K)$.

Proof: Denote by l the row vector $[l_1 \ldots l_G]$. Using matrix notation we can write

$$X = l \Sigma^{-1} l' \tag{2.3}$$

and

$$X(\Upsilon) = l(\Upsilon) \Sigma^{-1}(\Upsilon) l'(\Upsilon) , \tag{2.4}$$

where l' is the transpose of l and Σ^{-1} is the inverse of Σ. We have

$$l'(\Upsilon) = \Upsilon l' , \tag{2.5}$$

$$l(\Upsilon) = l\,\Upsilon', \tag{2.6}$$

and

$$\Sigma^{-1}(\Upsilon) = \Upsilon'^{-1}\,\Sigma^{-1}\,\Upsilon^{-1}. \tag{2.7}$$

Hence, from (2.4) – (2.7) we obtain

$$X(\Upsilon) = l\,\Upsilon'\,\Upsilon'^{-1}\,\Sigma^{-1}\,\Upsilon^{-1}\,\Upsilon\,l = l\,\Sigma^{-1}\,l' = X, \tag{2.8}$$

and Lemma 2.1 is proved.

Let $\theta^* = (\theta_1^*, \ldots, \theta_p^*)$ be a parameter point different from θ and denote by l_g^*, X^*, and ξ_{kl}^* the expressions we obtain from l_g, X, and ξ_{kl}, respectively, by substituting θ^* for θ. Now we shall prove the following lemma.

LEMMA 2.2. *If* θ^* *is a point such that* $\xi_{kl}^* = \xi_{kl}$ $(k,\ l = 1, \ldots, K)$, *then there exists a nonsingular transformation* Υ *such that*

$$\xi_{kl}^* = \xi_{kl}(\Upsilon) \qquad (k,\ l = 1, \ldots, K). \tag{2.9}$$

Proof: From $\xi_{kl}^* = \xi_{kl}$ it follows that $X^* = X$ identically in $x_1, \ldots, x_K$. Thus we have

$$X^* = \sum\sum \sigma^{*gh}\, l_g^*\, l_h^* = \sum\sum \sigma^{gh}\, l_g\, l_h = X. \tag{2.10}$$

Since $l_1, \ldots, l_G$ are independent linear forms and since $[\sigma_{gh}]$ is nonsingular, the rank of the quadratic form X is equal to G. Hence, the rank of X^* is also equal to G and, therefore, $l_1^*, \ldots, l_G^*$ are independent linear forms. From this and (2.10) it follows that each linear form l_g^* is a linear combination of the forms $l_1, \ldots, l_G$. Hence there exists exactly one nonsingular transformation Υ such that

$$l_g(\Upsilon) = l_g^* \qquad (g = 1, \ldots, G). \tag{2.11}$$

From Lemma 2.1 it follows that

$$\sum\sum \sigma^{gh}(\Upsilon)\, l_g(\Upsilon)\, l_h(\Upsilon) = \sum\sum \sigma^{gh}\, l_g\, l_h. \tag{2.12}$$

From (2.10), (2.11), and (2.12) we obtain

$$\sum \sum \sigma^{gh}(\Upsilon)\, l_g^* \, l_h^* = \sum \sum \sigma^{*gh} \, l_g^* \, l_h^* . \tag{2.13}$$

Hence

$$\sigma^{gh}(\Upsilon) = \sigma^{*gh} \qquad (g,\ h = 1,\ \dots,\ G), \tag{2.14}$$

and, therefore,

$$\sigma_{gh}(\Upsilon) = \sigma^*_{gh} \qquad (g,\ h = 1,\ \dots,\ G). \quad 1 \tag{2.15}$$

Since (2.11) implies that $\alpha_{gk}(\Upsilon) = \alpha^*_{gh}$, Lemma 2.2 is proved.

The coefficients ξ_{kl} $(k,\ l = 1,\ \dots,\ K)$ depend, of course, on the parameter point θ. To make this evident, we shall occasionally replace ξ_{kl} by $\xi_{kl}(\theta)$, and $\xi_{kl}(\Upsilon)$ by $\xi_{kl}\{\theta(\Upsilon)\}$. Since $\xi_{kl}\{\theta(\Upsilon)\} = \xi_{kl}(\theta)$, we shall say that the functions $\xi_{kl}(\theta)$ are invariant under nonsingular transformations Υ.

Let $F(\theta)$ be a function of θ. We shall say that $F(\theta)$ is invariant under nonsingular transformations if for any nonsingular transformation Υ we have $F\{\theta(\Upsilon)\} = F(\theta)$. Clearly, if $F(\theta)$ is a function of $\xi_{11}(\theta)$, $\xi_{12}(\theta)$, $\dots$, $\xi_{KK}(\theta)$, then $F(\theta)$ is invariant under nonsingular transformations. We shall show that the converse is also true. Let $F(\theta)$ be a function such that $F\{\theta(\Upsilon)\} = F(\theta)$ for all nonsingular transformations Υ. Suppose that $F(\theta)$ is not a function of $\xi_{11}(\theta)$, $\xi_{12}(\theta)$, $\dots$, $\xi_{KK}(\theta)$. Then there exist two points θ'' and θ''' such that

$$\xi_{kl}(\theta'') = \xi_{kl}(\theta''') \qquad (k,\ l = 1,\ \dots,\ K) \tag{2.16}$$

and

$$F(\theta'') \neq F(\theta''') . \tag{2.17}$$

From Lemma 2.2 and (2.16) it follows that there exists a nonsingular transformation Υ such that

$$\theta''(\Upsilon) = \theta''' . \tag{2.18}$$

But then

$$F(\theta'') \neq F\{\theta''(\Upsilon)\} , \tag{2.19}$$

which contradicts our assumption about $F(\theta)$. Hence $F(\theta)$ must be a function of $\xi_{11}(\theta)$, $\xi_{12}(\theta)$, $\ldots$, $\xi_{KK}(\theta)$. Thus, the functions $\xi_{11}(\theta)$, $\xi_{12}(\theta)$, $\ldots$, $\xi_{KK}(\theta)$ form a fundamental set of invariants.

3. *Necessary and Sufficient Conditions for the Identification of a Coordinate θ_p of a Parameter Point θ*

Let θ be a parameter point satisfying the a priori conditions (1.1). And further, let $\theta^* = (\theta_1^*, \ldots, \theta_p^*)$ be an unknown parameter point and consider the equations in θ^*:

$$(3.1) \qquad \xi_{kl}(\theta^*) = \xi_{kl}(\theta) \qquad (k, l = 1, \ldots, K)$$

and

$$(3.2) \qquad \varphi_r(\theta^*) = 0 \qquad (r = 1, \ldots, R) \text{ (a priori conditions).}$$

The following two theorems are immediate consequences of Lemmas 2.1 and 2.2.

THEOREM 3.1. *A necessary and sufficient condition that θ_p be identifiable is that the equations* (3.1) *and* (3.2) *in the unknowns $\theta_1^*, \ldots, \theta_P^*$ should admit of only a finite number of solutions for θ_p^*.*

THEOREM 3.2. *A necessary and sufficient condition that θ_p be locally identifiable is that there exists a finite neighborhood ω of θ such that for any solution θ^* in ω of the equations* (3.1) *and* (3.2) *we have $\theta_p^* = \theta_p$.*

In what follows in this section we shall assume that the R equations (3.2) have unique solutions in R unknowns, i.e., in R coordinates of θ^*. We may assume without loss of generality that these R coordinates are the last ones, i.e., $\theta_{P-R+1}^*, \ldots, \theta_P^*$. Thus, equations (3.2) can be written as

$$(3.3) \qquad \theta_p^* = \psi_p(\theta_1^*, \ldots, \theta_{P-R}^*) \qquad (p = P - R + 1, \ldots, P).$$

We shall assume that the functions ψ_p admit of continuous first-order partial derivatives. For any parameter point $\theta = (\theta_1, \ldots, \theta_P)$ in the P-dimensional parameter space we shall denote by $\bar{\theta}$ the parameter point in the $(P - R)$-dimensional space obtained from θ by omitting the last R coordinates, i.e., $\bar{\theta} = (\theta_1, \ldots, \theta_{P-R})$.

Denote by $\bar{\xi}_{kl}(\bar{\theta})$ the function we obtain from $\xi_{kl}(\theta)$ by substituting $\psi_p(\bar{\theta})$ for θ_p $(p = P - R + 1, \ldots, P)$. Then the system of equations (3.1) and (3.2) is equivalent to the system

$$\bar{\xi}_{kl}(\bar{\theta}^*) = \bar{\xi}_{kl}(\bar{\theta}) \qquad (k, \ l = 1, \ldots, K) \tag{3.4}$$

and

$$\theta_p^* = \psi_p(\bar{\theta}^*) \qquad (p = P - R + 1, \ldots, P). \tag{3.5}$$

Denote the $(P - R)$-dimensional parameter space by $\bar{\Omega}$. For any point $\bar{\theta}$ of $\bar{\Omega}$ we shall denote by $\Delta(\bar{\theta})$ the Jacobian of the functions $\bar{\xi}_{11}(\bar{\theta})$, $\bar{\xi}_{12}(\bar{\theta})$, $\ldots$, $\bar{\xi}_{KK}(\bar{\theta})$ taken at the point $\bar{\theta}$. A point $\bar{\theta}$ of $\bar{\Omega}$ will be called regular if the following condition is satisfied: Any minor of the Jacobian of the $K^2 + P - R$ functions $\bar{\xi}_{11}(\bar{\theta})$, $\bar{\xi}_{12}(\bar{\theta})$, $\ldots$, $\bar{\xi}_{KK}(\bar{\theta})$, $d_p(\bar{\theta}) = \theta_p$ $(p = 1, \ldots, P - R)$ is either unequal to zero at $\bar{\theta}$ or is *identically* zero in some finite neighborhood of $\bar{\theta}$.

THEOREM 3.3. *Let $\bar{\theta}^0$ be a regular point and denote by $\delta_p(\bar{\theta})$ the Jacobian of the $K^2 + 1$ functions $\bar{\xi}_{11}(\bar{\theta})$, $\bar{\xi}_{12}(\bar{\theta})$, $\ldots$, $\bar{\xi}_{KK}(\bar{\theta})$, $d_p(\bar{\theta}) = \theta_p$ for any value of p satisfying $p \leq P - R$. A necessary and sufficient condition that θ_p be locally identifiable for any point $\bar{\theta}$ in a finite neighborhood $\bar{\theta}^0$ is that the rank of $\Delta(\bar{\theta}^0)$ be equal to the rank of $\delta_p(\bar{\theta}^0)$.*

Proof: Since $\bar{\theta}^0$ is a regular point, a necessary and sufficient condition that $\bar{\theta}_p$ be a single valued function of $\bar{\xi}_{11}(\bar{\theta})$, $\bar{\xi}_{12}(\bar{\theta})$, $\ldots$, $\bar{\xi}_{KK}(\bar{\theta})$ in a finite neighborhood of $\bar{\theta}^0$ is that the rank of $\Delta(\bar{\theta}^0)$ be equal to that of $\delta_p(\bar{\theta}^0)$. Theorem 3.3 follows from this and the fact that the functions $\bar{\xi}_{11}(\bar{\theta})$, $\bar{\xi}_{12}(\bar{\theta})$, $\ldots$, $\bar{\xi}_{KK}(\bar{\theta})$ form a fundamental set of invariants.