TARGETING IN ADVERTISING MARKETS: IMPLICATIONS FOR OFFLINE VS. ONLINE MEDIA

By

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Abstract

We develop a model with many heterogeneous advertisers (products) and advertising markets (media). Each advertiser has a different consumer segment for its product, and each medium has a different ability to target advertisement messages. We characterize the competitive equilibrium in the media markets and investigate the role of targeting for the price and allocation of advertisements across media markets.

An increase in the targeting ability leads to an increase in the total number of purchases (matches), and hence in the social value of advertisements. Yet, an improved targeting ability also increases the concentration of advertising firms in each market. Surprisingly, we find that the equilibrium price for advertisements is decreasing in the targeting ability over a large range of parameter values.

We trace out the implications of targeting for competing media markets. We distinguish offline and online media by their targeting ability: low versus high. We show that competition by an online medium lowers the revenue of the offline medium more than competition by another offline medium of the same size.

KEYWORDS: Targeting, Advertising, Online Advertising, Sponsored Search, Media Markets.

JEL CLASSIFICATION: D44, D82, D83.

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1 Introduction

Over the past decade the internet has become an increasingly important medium for advertising. The arrival of the internet has had important consequences on the market position of many traditional media, i.e. offline media such as print, audio and television. For some of these media, most notably the daily newspapers, the very business model is under the threat of extinction due to competition from the internet for the placement of advertising. The following chart shows the dramatic changes in aggregate spending for advertising on different media between 2004 and 2008.[1]
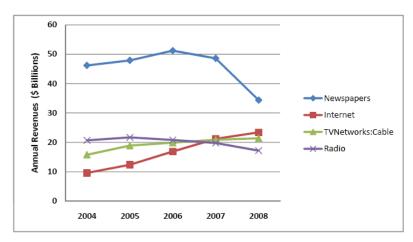
Figure 1: U.S. Advertising Markets: Revenue Comparison

At the same time, through a variety of technological advances, the internet has allowed many advertisers to address a targeted audience beyond the reach of traditional media. In fact, it has been argued that the distinguishing feature of internet advertising is its ability to convey information to a targeted audience. In particular, targeting improves the quality of the match between the consumer and the advertisement message, and enables smaller businesses to access advertising markets from which they were previously excluded.[2] While this holds for display advertising, it is even more true for sponsored search, where the individual consumer declares her intent or preference directly, by initiating a query.

The objective of this paper is to develop a model of competition between the offline (traditional) and the online (new) media, in which the distinguishing feature of the online

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[1] Source: Price Waterhouse Coopers annual reports for the Interactive Advertising Bureau.
[2] Anderson (2006) refers to this phenomenon as the "long tail of advertising."

media is the ability to (better) target advertisement messages to their intended audience. We investigate the role of targeting in the determination of (a) the allocation of advertisements across different media, and (b) the equilibrium price for advertising. For this purpose, we first develop a framework to analyze the role of targeting, and then use this to model to analyze the interaction between offline and online advertising.

We present a model in which advertising creates awareness for a product. We consider an economy with a continuum of buyers and a continuum of products. Each product has a potential market size which describes the mass of consumers who are contemplating to purchase it. Each consumer is contemplating only one of the available products, and the role of the advertisement is to generate a match between product and consumer. The placement of an advertisement constitutes a message from the advertiser to a group of consumers. If the message is received by a consumer who is interested in the advertiser's product, the potential customer turns into an actual customer and a purchase is realized. A message received by a customer who is not in the market for the product in question is irretrievably lost and generates no tangible benefit for the advertiser. At the same time, a potential customer might be reached by multiple and hence redundant messages from the same advertiser. Consequently, the probability that a potential customer is turned into an actual customer is an increasing but concave function of the number of messages sent.

We begin the analysis with a single advertising market in which all consumers are present and can be reached by any advertiser. It is useful to think of the single advertising market as a national platform, such as the nationwide newspapers or the major television networks. We show that in this market only the largest firms, measured by the size of their potential market, purchase any advertising space. We also show that the concentration of consumer types (i.e. the degree of asymmetry in the firms' potential market sizes) has an initially positive, but eventually negative effect on the equilibrium price of messages.

We then introduce the possibility of targeting by introducing a continuum of advertising markets. Each advertising market is characterized by the number of messages that it can send out to its audience. While each consumer is at most present in one advertising market, the likelihood of her presence in a specific market is correlated with her potential interest. For concreteness, the distribution of consumers across advertising markets is assumed to have a triangular structure. Namely, a consumer of type x is located with positive probability in one of the advertising markets labeled $k \leq x$, and is located with zero probability

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in advertising markets $k > x$. We assume that the distribution of the consumers across the advertising markets is given by an exponential distribution parametrized by γ. As each consumer segment becomes more concentrated in a few advertising markets, the probability of a match between consumers and advertisements increases. In consequence, the social welfare is increasing with the ability of the advertisers to reach their preferred audience. We then investigate the equilibrium advertising prices as the degree of targeting improves. While the marginal product of each message is increasing in the targeting ability, thus potentially increasing the prices for the advertisement, a second and more powerful effect appears. As consumers become more concentrated, the competition among different advertisers becomes weaker. In fact, each advertiser focuses his attention on a few important advertising markets and all but disappears from the other advertising markets. In consequence, the price of advertising is declining in the degree of targeting, even though the value of advertising is increasing. The number of participating advertisers shows a similarly puzzling behavior. While improved targeting increases the total number of advertisers participating across all markets – by allowing smaller advertisers to appear – it reduces the number of actively advertising firms in each specific market k.

In the second part of the paper we introduce competition among advertising markets (media) for the attention of the consumer. Thus, while each consumer is still only interested in one product, he can now receive a message from an advertiser on two different media markets. A single message received through either one of the markets is sufficient to create a sale. The "dual-homing" of the consumer across the two media markets may then lead to duplicative efforts by the advertisers, who therefore view messages in the two competing markets as strategic substitutes. We first describe the advertising allocation when the competitors are both traditional media without any targeting ability. In this case, messages on the two media are perfect substitutes, and the equilibrium prices are equalized. Furthermore, the allocation of messages only depends on the total supply, not on its distribution across media.

The competition among two offline media markets presents a useful benchmark when we next consider competition between an offline and an online market. We analyze the interaction of offline media – such as newspapers or TV – with online media, such as display (banner) and sponsored search advertisements. Display advertisements allow for targeting through superior knowledge of the consumer's preferences (attribute targeting). Sponsored

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keyword search advertisements allow advertisers to infer the consumer's preferences from her actions (behavioral targeting). As expected, competition lowers the price of advertisement on the traditional medium. However, the online medium lowers the revenue of the offline medium more than competition by another offline medium of the same size.

This paper is related to several strands in the literature on advertising. Anderson and Coate (2005) provide the first model of competing broadcasters, with exclusive assignment of viewers to stations; their setup is extended by Ferrando, Gabszewicz, Laussel, and Sonnac (2004), and Ambrus and Reisinger (2006) to the case of non-exclusive assignments. However, the role of targeting for the structure of advertising markets has received scant attention in the literature. The most prominent exception is Iyer, Soberman, and Villas-Boas (2005), who analyze the strategic choice of advertising in an imperfectly competitive market with product differentiation. In their model, the consumers are segmented into different audiences that the firms can target with advertising messages. However, Iyer, Soberman, and Villas-Boas (2005) are mostly concerned with the equilibrium product prices that result from the competitive advertising strategies. In contrast, we take the product prices as given, and focus our attention on the equilibrium prices of the advertising messages themselves. Our results on equilibrium advertising prices and competing media are in line with recent empirical work by Goldfarb and Tucker (2009), who exploit the variation in targeting ability generated by the legal framework across states. They show that prices for sponsored search advertising are higher when regulations limit the offline alternatives for targeted advertising. In another empirical study, Chandra (2009) relates the degree of segmentation (targeting) of a newspaper's subscriber base to the price it charges for the advertising space. The results imply a substantial benefit to advertisers and media firms from targeted advertising, and see Chandra and Kaiser (2010) for additional evidence from the magazine markets.

In this paper, we model the market for advertisements as a competitive market and the allocation of advertising messages is determined by the competitive equilibrium price. Each advertising message generates a match between a product and a potential customer. The present interpretation of advertising as matching products and users is shared with recent papers, such as Athey and Ellison (2008) and Chen and He (2006). Yet, in these contributions, the primary focus is on the welfare implications of position auctions in a search model where consumers are uncertain about the quality of the match. Similarly, several recent papers, Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) among

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others, focus on the specific mechanisms used in practice to sell advertising messages online, such as auctions for sponsored links in keyword searches.

Finally, in closely related work, Athey and Gans (2010) analyze the impact of targeting on the supply and price of advertising. In their model, targeting improves the efficiency of the allocation of messages, and leads to an increase in demand. However, as long as advertisement space can be freely expanded, the revenue effects of targeting could also be obtained by increasing the supply of (non targeted) messages, yielding an equivalence result. More generally, Athey and Gans (2010) show that supply-side effects mitigate the value of targeting.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 analyzes the equilibrium in a single advertising market. Section 4 generalizes the analysis to many advertising markets. Section 5 extends the analysis by allowing each consumer to be present in many advertising markets. Section 6 investigates the competition between offline and online media. Section 7 reconsiders the equilibrium in a single advertising market when one or more publishers control the supply of advertising messages. The Appendix collects the formal proofs of all propositions in the main body of the text.

2   Model

We consider a model with a continuum of products and a continuum of advertising markets. Each product x is offered by firm x with $x \in [0, \infty)$. The advertising markets are indexed by $k \in [0, \infty)$. There is a continuum of buyers with unit mass and each buyer is present in exactly one product market and one advertising market. The consumers population is jointly distributed across products x and advertising markets k according to $F(x, k)$, with a density $f(x, k)$. The market share of product x is given by the marginal distribution, integrating over all the advertising markets k :

$$s_x \triangleq \int_0^\infty f(x, k) \, dk. \tag{1}$$

Firms are ranked, without loss of generality, in decreasing order of market share, so s_x is decreasing in x. Similarly, the size of the advertising market k is given by the marginal

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distribution, integrating over all the products x :

$$s_k \triangleq \int_0^\infty f(x,k)\, dx. \tag{2}$$

Each buyer is only interested in one specific product x. A sale of product x occurs if and only if the buyer is interested in the product *and* she receives at least one message from firm x. A message by firm x is hence only effective if it is received by a buyer in segment x. In other words, we adopt the complementary view of advertising (see Bagwell (2007)), in which both the message and the right receiver are necessary to generate a purchase. Each sale generates a gross revenue of \$1, constant across all product markets.

The advertising policy of firm x determines the number of messages $m_{x,k}$ it distributes in advertising market k. Each message of advertiser x reaches a random consumer in advertising market k with uniform probability. Given the size of the advertising market s_k and the message volume $m_{x,k}$, the probability that a given consumer in market k is aware of product x is then given by:

$$\alpha\left(m_{x,k}, s_k\right) \triangleq 1 - e^{-m_{x,k}/s_k}. \tag{3}$$

We refer to $\alpha\left(m_{x,k}, s_k\right)$ as the awareness level for product x in advertising market k. The exponential form of the matching probability (3) is a result of the uniform random matching process. Suppose a large number of messages, denoted by m, is distributed with uniform probability across a large number of agents, denoted by s.[3] We then ask what is the probability that a representative agent has received *none* out of the m messages. This probability is given by:

$$(1 - 1/s)^m.$$

Now, by the definition of the exponential function, we have that:

$$\lim_{m,s \to \infty} (1 - 1/s)^m = e^{-m/s},$$

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[3] An early version of this paper considered a model with an infinite, but countable number, of advertisers and consumers. The current model with a continuum of advertisers can be viewed as the limit of the countable model. We present here the continuum model as it has the advantage that all marginal conditions are exact, rather than subject to integer constraints. The results in the countable and continuum models are identical up to the integer constraints.

and the complementary probability is then given (3).

The allocation of buyers across product and advertising markets is assumed to be governed by an exponential distribution. In particular, the market share of product x is given by:

$$s_x \triangleq \lambda e^{-\lambda x}. \tag{4}$$

The parameter $\lambda \geq 0$ measures the concentration in the product market, and a large value of λ represents a more concentrated product market. In turn, the conditional distribution of consumers in product segment x over advertising markets k is given by a (truncated) exponential distribution:

$$\frac{s_{x,k}}{s_x} \triangleq \begin{cases} e^{-\gamma x}, & \text{if} \quad k = 0, \\ \gamma e^{-\gamma(x-k)}, & \text{if} \quad k \leq x, \\ 0, & \text{if} \quad k > x. \end{cases} \tag{5}$$

The parameter $\gamma \geq 0$ measures the concentration of the consumers in the advertising markets. A larger value of γ represents a heavier concentration of more consumers in fewer advertising markets. The distributions of consumers across markets and products are conditionally independent. The corresponding unconditional market shares are given by:

$$s_{x,k} \triangleq \begin{cases} \lambda e^{-(\lambda+\gamma)x}, & \text{if} \quad k = 0, \\ \lambda \gamma e^{-(\lambda+\gamma)x} e^{\gamma k}, & \text{if} \quad k \leq x, \\ 0, & \text{if} \quad k > x. \end{cases}$$

For $\gamma > 0$, the distribution of consumers over product and advertising markets has a triangular structure. The consumers who are interested in product x are present in all advertising markets $k \leq x$, but are not present in the advertising markets $k > x$.

The distribution of consumers across a one-dimensional product space and a one-dimensional advertising space has a natural interpretation in terms of specialization of preferences and audiences. In this interpretation, a product with a larger index x represents a more specialized product with a smaller market. Correspondingly, an advertising market with a larger index k represents an advertising medium with a narrower audience. To give a precise example, consider the market for bicycles. Here, products naturally range from mass-produced comfort bikes, to quality-produced fitness bikes, to high-end racing bikes with successively

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smaller market shares. Similarly, there is a natural range of advertising markets, from daily newspapers with a large audience, to monthly magazine with well-defined audience such as "Sports Illustrated," to narrowly focused publications, such as "Velonews". Now, the triangular structure of the joint distribution implies that the consumer with an interest in racing bikes may read either one of the publications, but that a consumer with interest in fitness bikes does *not* read "Velonews," and by extension that a consumer with an interest in comfort bikes does *not* read "Velonews" nor "Sports Illustrated". In other words, the triangular structure represents a positive but less than perfect correlation of the preference and the audience characteristics of a consumer. The specific feature of the triangular structure, namely the unidirectional diffusion of the consumer x across advertising markets $k \leq x$, is not essential for the qualitative character of our results, but allow us to represent the strength of the targeting in a singe variable, namely the parameter γ of the exponential distribution.
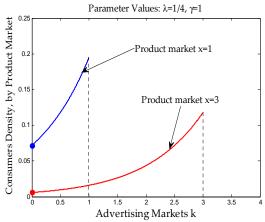
As we vary the targeting measure γ from 0 to ∞, we change the distribution and the concentration in each advertising market. With $\gamma = 0$, all consumers are located in the single large advertising market $k = 0$. As we increase γ, an increasing fraction of consumers of type x move from the large market to the smaller markets and as $\gamma \to \infty$, all consumers of type x are exclusively present in advertising market $k = x$. The limit values of γ, namely $\gamma = 0$ and $\gamma = \infty$, represent two special market structures. If $\gamma = 0$, then all consumers are present in advertising market 0 and hence there is a single advertising market. If, on the other hand, $\gamma \to \infty$, then all consumers of product x are present in advertising market x, and hence we have advertising markets with perfect targeting. The left panel of Figure 2 illustrates the cross section of how the customers of a firm are distributed across the advertising market (for several firms). The mass points indicate the number of consumers of each firm that are present in market 0. The right panel shows how an advertising market hosts consumers of different firms (for several advertising markets).
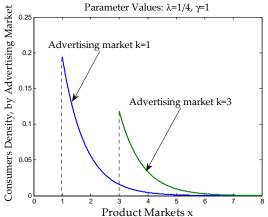
Finally, the supply of messages M_k in every advertising market k is proportional to the size s_k of the advertising market and given by

$$M_k \triangleq s_k \cdot M,$$

for some constant $M > 0$. The constant M can be interpreted as the attention or time

Figure 2: Distribution of Consumers across Product Markets and Advertising Markets

that each consumer allocates to receiving messages on the advertising market where he is located. We think of each advertising market k as the market for attention by a given set of consumers. For example, we may interpret each advertising market as a local, regional or national market for newspaper readers, radio listeners or TV audience. The equilibrium price p_k for messages placed in advertising market k is then determined by the market clearing condition in market k. We only depart from the competitive equilibrium paradigm in Section 7. There we investigate the equilibrium allocation where a single (or multiple) publisher controls the supply of messages. In particular, we allow the publisher to offer a nonlinear pricing schedule of the message volume purchased.

3 Single Advertising Market

We begin the equilibrium analysis with the benchmark of a single advertising market. In other words, consumers of all product market segments are present in a single advertising market $k = 0$. In terms of the distribution of the consumers over the advertising markets, it corresponds to setting $\gamma = 0$. Each firm x can now reach its consumers by placing messages in the single advertising market $k = 0$. Consequently, in this section we drop the subscript k in the notation without loss of generality. The objective of each firm x is to maximize the profit given the unit price for advertising p. The profit π_x is given by:

$$\pi_x = \max_{m_x} \; [s_x \alpha \left(m_x \right) - pm_x].$$

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An advertising policy m_x generates a gross revenue $s_x \cdot \alpha(m_x)$. The information technology $\alpha(m_x)$, given by (3), determines the probability that a representative consumer is aware of product x, and s_x is the proportion of buyers who are in market segment x. The cost of an advertising policy m_x is given by $p \cdot m_x$. The optimal demand of messages by firm x is determined by the first order conditions and yields a demand function:

$$m_x = \ln(s_x / p).$$

It is an implication of the above optimality conditions that firms with a larger market share s_x choose to send more messages to the consumers. In consequence, at the equilibrium price, the firms with the largest market share choose to advertise. Let $[0, X]$ be the set of participating firms, where X is the marginal firm, and let M be the total supply of messages. The market clearing condition is:

$$\int_0^X m_x dx = M.$$

Using the optimal demand of firm x and the formula for product market shares (4), we obtain

$$\int_0^X (\ln(\lambda / p) - \lambda x) \, dx = M. \tag{6}$$

The equilibrium price and participation are now determined by imposing $m_X = 0$ and the market clearing condition in (6). The competitive equilibrium is then characterized by (p^*, X^*) with:

$$p^* = \lambda e^{-\sqrt{2\lambda M}}, \tag{7}$$
$$X^* = \sqrt{2M / \lambda}. \tag{8}$$

By using these formulas in the equilibrium expressions, we obtain the competitive equilibrium allocation of messages for a single advertising market with a given capacity M,

$$m_x^* = \begin{cases} \sqrt{2\lambda M} - \lambda x, & \text{if } x \leq X^*, \\ 0, & \text{if } x > X^*. \end{cases} \tag{9}$$

To summarize, in the competitive equilibrium, the X^* largest firms enter the advertising market and the remaining smaller firms stay out of the advertising market. With the exponential distribution of consumers across products, the number of messages sent by an active firm is linear in its rank x in the market. The set of participating firms, the number of messages and the equilibrium price change continuously in λ and M. We determine how the equilibrium allocation depends on the primitives of the advertising market, namely λ and M, in the following comparative statics results.

Proposition 1 (Single Market, Comparative Statics)

1. *The equilibrium demand of messages m_x^* is increasing in λ for all $x \leq X^*/2$.*

2. *The number of advertising firms X^* is increasing in M and decreasing in λ.*

3. *The equilibrium price p^* is decreasing in M for all λ.*

4. *The equilibrium price p^* is increasing in λ iff $\lambda < 2/M$.*

5. *The price per consumer reached is increasing in x. It is decreasing in λ for $x \leq X^*/2$.*

6. *The social value of advertising is increasing in λ.*

As the message volume M of the advertising market increases, the number of participating firms X^* also increases. The population of consumers is segmented in many categories. As the market becomes more concentrated in fewer categories, the number of actively advertising firms is decreasing as well. The equilibrium price responds in a more subtle way to the concentration measure λ in the product market. If the product market is diffuse, then an increase in the concentration measure essentially increases the returns from advertising for most of the participating firms. In other words, the demand of the inframarginal firms has a larger effect on the price than the demand of the smaller, the marginal, firms. If on the other hand, the concentration in the product market is already large, then a further increase in the concentration weakens the marginal firm's demand for advertising. At the same time, as the market share of the large firms is already substantial, their increase in demand for advertising is not sufficient to pick up the decrease in demand of the marginal firm. The additional demand of the large firm is weak because an increase in the already large advertising volume leads to many more redundant messages, which do not generate

additional sales. Figure 3 shows the market demand and supply for different values of the concentration measure λ.
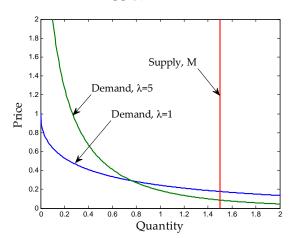
Figure 3: Demand and Supply, Different Concentration Measures

The dichotomy in the comparative static is thus driven by the determination of the marginal demand for advertising. If the source of the marginal demand is the marginal firm, then the price goes down with an increase in λ, and likewise if the marginal demand is driven by the inframarginal firms, then the advertising price is increasing with λ. In this sense, the non monotonic behavior of prices is not specific to the exponential distribution of firms' market shares. On the contrary, it is a consequence of the natural tension between competition and concentration.

Finally, notice that the competitive equilibrium implements the socially efficient allocation of advertisement messages (given λ). An easy way to see this is that with a uniform unit price of messages, the marginal returns to ads bought by different firms are equalized. A natural question is how does the social value of advertising depend on the product market concentration. Consider holding the allocation m_x^* fixed, and increasing λ. Now the total market share of the participating firms has increased, and fewer messages "get lost." At the new equilibrium, welfare will be even higher, as the allocation is adjusted for the new relative market shares of different products. In consequence, social welfare is increasing in the concentration measure.

One may wonder how relaxing the assumption of perfectly inelastic supply affects the comparative statics result in Proposition 1. For the case of constant supply elasticity $q =$

Mp^{ε}, we can show that the equilibrium price retains the same comparative static properties: it is first increasing, then decreasing in λ. Moreover, as M becomes larger, the equilibrium price will be increasing in λ over a larger range. In particular, when the product market is very concentrated (so that the marginal demand is low), a more elastic supply reduces the number of active firms in the market. A further increase in concentration may then increase the demand of the active firms, and therefore also the price. But for high values of λ, it continues to hold that the demand "falls off" fast enough that the equilibrium price decreases. In particular, as λ goes to infinity, both the price and the quantity traded go to zero. However, since an increase in λ causes a drop in the quantity sold, the welfare result with respect to an increase in the concentration measure λ now becomes ambiguous.

We observe that we assumed that the value of a match is constant across product markets. The introduction of product specific profit margins – which may be thought of as the value of a match – would affects the equilibrium price and the distribution of messages. In the case of exponentially declining profit levels, the rate of decrease of profits plays a role similar to that of the concentration parameter λ. Intuitively, faster declining profits imply a more skewed equilibrium allocation of messages. As the profit margins are declining faster, the competitive equilibrium displays a decline in the number of participating firms.

4   Many Advertising Markets

We are now in a position to analyze the general model with a continuum of advertising markets. The model with a single advertising market was described by $\gamma = 0$ and we now allow the targeting parameter γ to be strictly positive. The case of perfect targeting is described by $\gamma = \infty$. We described the distribution of consumers over different advertising markets by a (truncated) exponential distribution. We recall that the share of consumers who are active in product category x, and located in advertising market k was given by (5):

$$\frac{s_{x,k}}{s_x} \triangleq \begin{cases} e^{-\gamma x}, & \text{if} \quad k = 0, \\ \gamma e^{-\gamma(x-k)}, & \text{if} \quad k \leq x, \\ 0, & \text{if} \quad k > x. \end{cases}$$

The share of consumers active in product market x and located in advertising market $k = 0$ is given by the residual probability of the product market segment x. As a result, the

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population size in advertising market $k > 0$ is given by the integral over the population shares,

$$s_{k>0} \triangleq \int_k^\infty \lambda \gamma e^{-(\lambda+\gamma)x} e^{\gamma k} dx = \frac{\gamma\lambda}{\gamma+\lambda} e^{-\lambda k}. \tag{10}$$

For advertising market $k = 0$, it is given by

$$s_{k=0} \triangleq \int_0^\infty \lambda e^{-(\lambda+\gamma)x} dx = \frac{\lambda}{\gamma+\lambda}. \tag{11}$$

The volume of advertising messages is assumed to be proportional to the population size of advertising market k, hence $M_k = s_k M$. The common factor M again expresses the attention devoted to advertising messages by the consumers.

An important implication of the exponential distribution across advertising and product markets is a certain stationarity in the composition over the consumers across the advertising markets. In particular, the relative shares of the product markets are constant across advertising markets:

$$\frac{s_{x,k}}{s_k} = (\lambda+\gamma) e^{-(\lambda+\gamma)(x-k)} = \frac{s_{x+n,k+n}}{s_{k+n}},$$

for all $x \geq k$ and all $n \geq 0$. Thus, while the exact composition of each advertising market is different, the size distribution of the competing advertisers are constant across advertising markets. The stationarity property allows us to transfer many of the insights of the single advertising market to the world with many advertising markets.

Now we consider the demand function of firm x in market k,

$$m_{x,k} = \arg\max_m [s_{x,y}(1 - e^{-m/s_k}) - p_k m].$$

The first order condition for the firm's problem is given by

$$m_{x,k} = s_k \ln \frac{s_{x,k}}{p_k s_k}. \tag{12}$$

We can use (12), the market clearing condition, and the definition of the marginal firm in market k, given by $m_{X_k^*,k} = 0$. We then obtain the following equilibrium conditions:

$$\int_k^{X_k^*} m_{x,k} dx = s_k M,$$

and

$$s_{X_k^*, k} / s_k = p_k.$$

We can now characterize the equilibrium prices p_k^*, the number active firms $X_k^* - k$, and the allocation $m_{x,k}^*$ of messages. In particular, the price and the number of active firms are stationary in the index k of the advertising market, that is:

$$p_k^* = (\gamma + \lambda) e^{-\sqrt{2M(\gamma + \lambda)}}, \tag{13}$$

$$X_k^* - k = \sqrt{2M/(\gamma + \lambda)}, \tag{14}$$

for all $k \geq 0$. Finally, the allocation of messages is given by

$$m_{x,k}^* = \begin{cases} \gamma \lambda e^{-\lambda k}(\sqrt{2M/(\gamma + \lambda)} - (x - k)), & \text{if} \quad k > 0, \\ \lambda(\sqrt{2M/(\gamma + \lambda)} - x), & \text{if} \quad k = 0. \end{cases} \tag{15}$$

Clearly, the larger firms $x \geq k$ receive a higher fraction of the message supply. If in particular we consider firm $x = k$, then the number of messages it receives is also increasing in the targeting ability.

The stationarity of the equilibrium prices implies that the marginal utility of an additional message is equalized across markets. We therefore have the following result.

Proposition 2 (Efficiency)

1. *The efficient allocation of a fixed advertising space M is proportional to the size of the advertising market: $M_k = s_k \cdot M$.*

2. *The competitive equilibrium is efficient.*

3. *The social value of advertising is strictly increasing in the targeting ability γ.*

To understand the implications of targeting on social welfare, consider the relative size of consumer segment x in advertising market $k = x$:

$$\frac{s_{x,x}}{s_{k=x}} = \gamma + \lambda.$$

We observe that better targeting increases the value that firm x assigns to a message in the advertising market $k = x$. Now let us consider holding the allocation of messages $m_{x,k}$

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constant, and increasing the degree of targeting γ. The volume of matched consumers and firms is increasing because of the shift in the relative sizes of advertising markets. Since we know that the competitive allocation of messages is Pareto efficient, the equilibrium (for the new γ) has unambiguously improved the social value of advertising.

The comparative statics results (with respect to λ and M) do not differ qualitatively from the case of a single competitive market. More importantly, the effect of targeting ability γ and product market concentration λ on the equilibrium allocation is remarkably similar. In particular, the response of prices to changes in the concentration measure λ may be generalized as follows:

$$\operatorname{sign}\left(\partial p_k^* / \partial \lambda\right) = \operatorname{sign}\left(2 / M - \lambda - \gamma\right).$$

In particular, prices are increasing in λ if both the concentration and targeting parameters are low enough. We now focus on the comparative statics with respect to γ, where a higher γ means more precise targeting. We define the equilibrium advertising revenues on each advertising market k as $R_k^* \triangleq s_k p_k^*$.

Proposition 3 (Role of Targeting)

1. *The number of messages per capita $m_{x,k}^*/s_k$ is increasing in γ iff $x < (k + X_k^*)/2$.*

2. *The number of participating firms $X_k^* - k$ is decreasing in γ.*

3. *The equilibrium price p_k^* is increasing in γ iff $\lambda + \gamma < 2/M$.*

4. *The equilibrium revenue R_0^* is decreasing in γ. The revenues $R_{k>0}^*$ are increasing in γ iff $\gamma < (1 + \sqrt{1 + 2M\lambda})/M$.*

The equilibrium number of messages $m_{x,k}^*$ is increasing in γ for the participating firms larger than the median firm active on each market k. Furthermore, more precise targeting implies a lower number of active firms. The relationship between targeting ability and equilibrium price is generally inverse-U shaped. However, if either M or λ are large, then p_k^* is decreasing in γ for all values of γ. In other words, despite the increased social value of advertising, the equilibrium price of advertising is decreasing in the targeting ability over a large range of parameter values. In terms of revenues, it is immediate to see from equations

(10) and (11) that an increase in γ leads to an increase in the size of markets $k > 0$ and to a decrease in the size of market 0. Since prices are constant, revenues in market 0 are decreasing in γ. Finally, targeting has the same qualitative effect on the equilibrium revenues in all markets $k > 0$.

We now come back to the similar effects of concentration and targeting. In particular, as with product market concentration, an increase in targeting γ reduces the demand of the marginal firm on each advertising market k. At the same time, better targeting increases the demand of the inframarginal firms. The underlying tension is the one between identifying a consumer segment precisely, and finding several (competing) advertisers who are interested in it. The resulting trade-off between competition and inframarginal willingness to pay applies to a number of contexts, such as generic vs. specific keyword searches, and more or less precise attributes targeting on social networks.

This trade-off may be ameliorated if the media can offer a menu of advertising policies rather than sell the advertising messages at a single, competitive unit price. In the context of our model, menu pricing is equivalent to block sales of messages. This additional instrument may allow publishers to extract (a fraction of) the inframarginal rents, and therefore to serve a limited number of advertisers without suffering from decreasing marginal returns. We shall discuss the impact and limits of menu pricing in Section 7.

To conclude this section, we should point out that the exponential distributions over advertising and product markets provide particularly tractable expressions. But for robustness, we have also derived the main results under the alternative assumption of Pareto-distributed consumers over product and advertising markets. The key difference with the exponential distribution lies in the fat tails (and hence decreasing hazard rate) of the Pareto distribution. In the product markets, this means two niche (high x) products have more similar relative market shares, compared to two mass (low x) products. Analogously, consumers in smaller advertising markets are relatively more dispersed than in larger advertising markets. It follows that, in small advertising markets, the marginal and inframarginal firms have more similar message demands under the Pareto than under the exponential distribution. The number of active firms in each advertising market is then no longer a constant, but rather it is increasing in k. In consequence, the message demand of the marginal firm in each market k is decreasing in k, and therefore so are the equilibrium prices p_k.

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5 Media Competition

In this section, we deploy our model of general and targeted advertising markets as a framework to provide insights into the effects of competition between new and established media. For this reason, we shall weaken the single-homing assumption to allow each consumer to be present in multiple media. A first effect of competition is then to multiply the opportunities for matching an advertiser with a customer. At the same time, we maintain all the assumptions of the previous sections, namely that each buyer is only interested in one product, and that one message is sufficient to generate a sale.

We initially consider competition between traditional media, i.e. sellers of non-targeted messages, where each advertising medium is described by a single advertising market. For example, this may represent the competition between nation-wide TV broadcasting and nation-wide newspaper publishers. We initially abstract away from the role of targeting, in order to trace out the implications of (a) the number of consumers present on each market, and (b) the distribution of consumer characteristics in each market. The analysis of competition between traditional advertising markets can shed light on the interaction of new and established (offline and online) media along at least two dimensions. First, a new market is likely to have an initially smaller user base. As a consequence, advertisement messages have a more narrow reach, though a smaller market makes it easier to reach a large fraction of the audience. Second, the main feature of a targeted, online advertising market is a higher concentration of consumers of a particular product, compared to a traditional market. Therefore, the degree of product market concentration, which we focus on here, plays a similar role to the degree of advertising market targeting of Section 4. In particular, differences in market concentration lead firms to sort into those markets where their messages have a higher probability of forming a match with the desired customer population. Furthermore, our results show that the availability of a new market with a smaller user base induces the largest advertisers to buy messages on both markets. These firms purchase a constant number of advertising messages in the (new) smaller market. As a result, some "medium-sized" firms have a relatively larger presence on the new market, compared to the single market case.

5.1 Competition by Symmetric Offline Media

We begin the analysis with a model of competition between two traditional media. The two media have the same distribution of consumer characteristics in their respective advertising markets. This model provides a useful benchmark to understand the effects of different user bases and consumer distributions. Therefore, we consider two media, A and B, competing for advertisers. Let $m_{x,j}$ denote the number of messages bought by firm x on advertising market $j \in \{A, B\}$, and denote by M_j the exogenous supply of advertising space on each market. We can also interpret the supply of messages as the time the representative consumer spends on each media (market).

As in our baseline model, the fraction of consumers reached by firm x on advertising market j is given by

$$\alpha_{x,j} \triangleq 1 - e^{-m_{x,j}}.$$

The main novel feature of media competition is that each firm x views messages displayed in advertising markets A and B as (perfect) substitutes. We can therefore define the total awareness level generated by firm x as

$$\alpha_x \left(m_{x,A}, m_{x,B} \right) \triangleq \alpha_{x,A} + \alpha_{x,B} - \alpha_{x,A}\alpha_{x,B} = 1 - e^{-m_{x,A} - m_{x,B}}.$$

As each consumer is dual-homing, there is a loss in the frequency of productive matches generated by messages in market A because the consumer may have received a duplicate message in market B (and conversely). The distribution of consumers in product markets (x) is common to the two media (j), and it is given by $s_x = \lambda \exp{(-\lambda x)}$. Each firm then maximizes the following profit function:

$$\pi_x \triangleq s_x \alpha_x \left(m_{x,A}, m_{x,B} \right) - \sum_{j \in \{A,B\}} p_j m_{x,j}.$$

It follows that the demand function of firm x in market j is given by

$$m_{x,j} = \ln{(\lambda/p_j)} - m_{x,-j} - \lambda x.$$

This expression differs from the demand function in a single advertising market only because of the perfect substitutability of messages across markets. We denote by $m_x \triangleq \sum_j m_{x,j}$ the

total number of messages demanded by firm x, and we describe the equilibrium allocation in the following proposition.

Proposition 4 (Offline Media)

The equilibrium with two competing offline media is described by:

$$
\begin{aligned}
p_A^* &= p_B^* = \lambda e^{-\sqrt{2\lambda(M_A+M_B)}}, \\
m_x^* &= \sqrt{2\lambda\left(M_A + M_B\right)} - \lambda x, \text{ for } x \leq X^*, \\
X^* &= \sqrt{2\left(M_A + M_B\right)/\lambda}.
\end{aligned}
$$

Since the messages on the two markets are perfect substitutes, it is intuitive that the equilibrium prices must also be identical. The number of active firms X^* in equilibrium reflects the increase in the total supply of messages $(M_A + M_B)$, but it is not directly affected by competition between the two markets, and it is otherwise analogous to the case of a single advertising market.

In this symmetric model, the equilibrium allocation of messages is not characterized in terms of each $m_{x,j}$. This is because perfect substitutability of messages across the two media leads to an indeterminacy in the division of message purchased across the two media. In particular, both media specialization – in which each firm $x \leq X^*$ purchases messages exclusively on one market – and proportional representation of advertisers on each market, may occur in equilibrium.

Finally, notice that the equilibrium revenues of market j are non monotonic in the supply level M_j and decreasing in M_{-j}. Therefore, if we considered M_j as a strategic variable – such as a capacity choice – then market interaction would be analogous to a game of quantity competition between the two media. We address the role of endogenous supply decisions in Section 7.

5.2 Media Markets of Different Size

We now turn to the effects of introducing a new advertising medium with a smaller user base, which is visited only by a subset of the consumers. To capture this asymmetry between the new and the established medium in a simple way, let the number of consumers present on (new) market B be given by $\delta \leq 1$. Furthermore, all consumers who visit the new medium

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B also visit the established medium A. For example, one may think of the early days of online advertising, or more recently about new online advertising channels (such as social networks).

We normalize the supply of messages *per capita* to M_j in each market j. Since each firm x can reach a subset of its customers on the new market B, the profit function is given by

$$\pi_x = \lambda e^{-\lambda x}((1 - e^{-m_{x,A}}) + e^{-m_{x,A}}\delta(1 - e^{-m_{x,B}/\delta})) - \sum_{j \in \{A,B\}} p_j m_{x,j}.$$

Whenever firm x buys a positive number of messages on both media, the first order conditions imply the following demand functions:

$$m_{x,A} = \ln \frac{\lambda (1 - \delta)}{p_A - \delta p_B} - \lambda x, \qquad m_{x,B} = \delta \ln \frac{p_A - \delta p_B}{p_B (1 - \delta)}.$$

Notice that $m_{x,A}$ is decreasing as usual, while $m_{x,B}$ is a constant. In equilibrium, we can identify two thresholds, X and Z, such that firms $x \in [0, X]$ buy messages on both markets, while firms $x \in [X, Z]$ only buy on market B. We describe the equilibrium allocation in the following proposition, and then discuss its properties.

Proposition 5 (New Advertising Medium)

1. *The equilibrium allocation of messages in the established market A is:*

$$m_{x,A}^* = \sqrt{2\lambda M_A} - \lambda x, \quad \text{for } x \leq \sqrt{2M_A/\lambda}.$$

2. *The equilibrium allocation of messages in the new market B is:*

$$m_{x,B}^* = \begin{cases} \delta(\sqrt{2(M_A + M_B)\lambda} - \sqrt{2M_A\lambda}), & \text{for} & x \leq \sqrt{2M_A/\lambda}, \\ \delta(\sqrt{2(M_A + M_B)\lambda} - \lambda x), & \text{for} & \sqrt{2M_A/\lambda} < x \leq \sqrt{2(M_A + M_B)/\lambda}. \end{cases}$$

3. *The equilibrium prices are given by:*

$$\begin{aligned} p_A^* &= \delta \lambda e^{-\sqrt{2(M_A+M_B)\lambda}} + (1 - \delta) \lambda e^{-\sqrt{2M_A\lambda}}, \\ p_B^* &= \lambda e^{-\sqrt{2(M_A+M_B)\lambda}}. \end{aligned}$$

Figure 4 illustrates the allocation for $M_A = M_B = 1$, $\lambda = 2$, and several values of δ. When $\delta = 1$, we return to the case of symmetric advertising markets, and the specific

allocation displayed below is just one of the possible equilibrium allocations. The displayed allocation for $\delta = 1$ is however the unique limit for the equilibrium allocations as $\delta \to 1$.
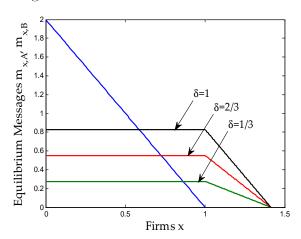
Figure 4: Allocation with Different Diffusion

Surprisingly, the largest firms enter the new market with a constant number of messages. Intuitively, larger firms stand more to lose by shifting messages to market B and reaching fewer potential customers. More formally, suppose (as is the case) that larger firms buy a larger number of messages on the established market (A). Given the substitutability of messages across markets, this increases the demand by smaller firms in the new market. In equilibrium, this effect exactly cancels the differences in firm size, and the allocation of messages on market B is flat for all dual-homing firms. Compared to the single market case, the new advertising market is then characterized by a strong presence of "medium-size" firms, and by a longer tail of relatively smaller firms.

Proposition 5 also shows that the number of active firms in market A is determined by the single market threshold, when supply is equal to M_A (that is, it is as if market B were not present). The total number of active firms is instead determined by the symmetric competition threshold (when supply is equal to $M_A + M_B$). Finally, the equilibrium price on the larger market p_A is decreasing in the size of the smaller market δ, while the price on the smaller market p_B is independent of δ.

5.3 Media Markets with Different Distributions

As we saw in Section 4, the key advantage of more targeted advertising markets is to allow fewer firms to deliver messages to a more concentrated consumer population. We now shift our attention to the role of the distribution of consumer characteristics for the competition between different media markets.

We consider two advertising markets, $j \in \{A, B\}$ and let the distribution of consumers in market j be given by $s_{x,j} \triangleq \lambda_j \exp(-\lambda_j x)$. We assume that the advertising market A has a more concentrated distribution over consumer characteristics than advertising market B, or $\lambda_A > \lambda_B$. As the distribution of consumers across advertising markets is now assumed to be different, it follows that not all consumer will be dual-homing. In particular, if a firm x has a larger presence in market A, then all its potential customers are present in market A, but only a subset of them is present in market B. Given that $\lambda_A > \lambda_B$, this is the case for the larger firms, for which $s_{x,A} > s_{x,B}$. The converse holds for the smaller firms, which have more consumers in market B. If we denote the matching technology by $\alpha(m)$, we can write the objective function of a large firm x (for which $s_{x,A} > s_{x,B}$) as:

$$\pi_x = s_{x,A}\alpha(m_{x,A}) + (1 - \alpha(m_{x,A}))\, s_{x,B}\alpha(m_{x,B}) - \sum_{j \in \{A,B\}} p_j m_{x,j}.$$

In other words, firm x perceives market B as a lower-quality substitute, analogous to a market with a smaller user base. Market A plays a similar role for smaller firms, for which $s_{x,A} < s_{x,B}$. It follows that larger firms have an incentive to focus on medium A and to disregard medium B.

The equilibrium allocation is now characterized by three threshold firms, $X < Y < Z$, and in particular:

1. The largest firms $x \in [0, X]$ only buy on market A.

2. A set of "medium-sized" firms $x \in [X, Y]$ buy on both markets. These firms divide their purchases in varying proportions. In particular, the demand for messages in market A is decreasing in x, while the demand on market B is increasing in x. The total demands are decreasing in x.

3. The relatively smaller firms $x \in [Y, Z]$ only buy on market B.

In equilibrium, the more concentrated market attracts the largest, most valuable, firms. In particular, large firms advertise exclusively on the more concentrated market, while a subset of medium-sized firms advertise on both, and relatively smaller firms only advertise on the more diffuse market, where they can reach a larger fraction of their consumers.

The cutoff values X, Y and Z solve the market clearing conditions given the demand functions. The equilibrium market shares do not allow for an explicit expression in the case of different concentration levels, and the details of the equilibrium construction are presented in Appendix B. In Figure 5, we show the allocations of messages $m_{x,A}$ and $m_{x,B}$ as a function of λ_A.

Figure 5: Allocation with Different Concentrations, $\lambda_B = 1$, $M_A = 1$, $M_B = 1$.

$$\lambda_A = 2 \qquad\qquad \lambda_A = \lambda_A^* \approx 4.73 \qquad\qquad \lambda_A = 8$$

For large differences in the concentration levels λ_j, all dual homing firms $x \in [X, Y]$ satisfy $s_{x,A} < s_{x,B}$, which means they are located to the right of the crossing point of the two density functions. For small differences in the concentration levels, all $x \in [X, Y]$ satisfy $s_{x,A} > s_{x,B}$. For a given choice of the parameters (λ_B, M_A, M_B), the number of dual-homing firms $(Y - X)$ is non monotonic in λ_A, and it is equal to zero for a single value $\lambda_A = \lambda_A^*$. When this is the case, the marginal firm $X = Y$ has an identical share of consumers in each of the two distributions.

The results in this section provide two kinds of insights into the interaction of online and offline advertising markets. Indeed, we can view each online advertising market as a separate medium with a higher concentration of consumers. With this interpretation, the prediction of the model is that Internet advertising induces the largest, most profitable advertisers to switch away from the offline medium, and to advertise only on the more

concentrated online markets.

In this sense, competition by a more concentrated (targeted) market is very different from an (identical) emerging market with a smaller user base. In the former case, the established media lose the most valuable firms, as these firms find a more profitable market where to reach their customers. In the latter case, the established media share the largest buyers with the new media, and actually hold a relatively favorable position (in terms of the allocation of messages purchased by the largest firms).

In an alternative interpretation, we can view market B as the newer medium, such as the Internet, with a relatively larger presence of consumers of small (long tail) firms. Competition with a more concentrated (established) market would then cause the demand for messages by smaller firms to completely crowd out the demand of large firms, and partially offset the demand of medium-size firms. In this sense, online advertising increases the number of firms that have access to messages in equilibrium, and allows for a more significant participation of smaller firms.

6   Offline vs. Online Media

The internet has introduced at least two technological innovations in advertising, namely (a) the ability to relate payments and performance (e.g. pay per click), and (b) an improved ability to target advertisement messages to users. We now focus on the latter aspect, and in particular on the equilibrium allocation of advertising when both traditional and targeted media are present.

The targeted markets represent specialized websites, and messages can be thought of as display advertisements. We therefore refer to the traditional medium as "offline," and to the many targeted markets as "online." We then consider a population of dual-homing consumers, who spend a total time of M_A on the offline medium and M_B on the online, targeted, medium. More specifically, $s_k M_B$ denotes the supply on each targeted market $k \geq 0$.

Because of the risk of duplication, messages sent online and offline are strategic substitutes for each firm. This is not the case for messages sent on two different online markets, since each consumer only visits one website (in addition to the offline market). Therefore, if firm x sends a total of m_x non targeted messages and $m_{x,k}$ messages on each online market

k, its profit function is given by

$$\pi_x = \int_0^x (s_{x,k}(1 - e^{-m_x - m_{x,k}/s_k}) - p_k m_{x,k})dk - pm_x.$$

The analysis of firms' advertising choices between offline and online media is intricate. In general, each firm x will want to advertise on a subset of the online markets $k \leq x$ where its consumers are located (see Figure 2), and some firms will also advertise offline. Both for tractability concerns, and to focus on the revenue implications of competition and targeting, we assume that the online medium allows to perfectly target messages to consumers. We then ask what is the equilibrium unit price of advertisement messages, and how it is affected by each firm's demands offline.

6.1 Perfect Targeting

With perfect targeting, each advertising market k is only visited by consumers of product k, or $x = k$. Since $s_{k=x} = s_x$, we immediately obtain the allocation and prices online from the individual firms' demands:

$$m_{x,x} = \lambda e^{-\lambda x} M_B, \tag{16}$$

$$p_{k=x} = e^{-M_B} e^{-m_x}. \tag{17}$$

Equation (16) implies that in equilibrium, given the supply of messages on each market, each firm reaches a constant fraction $1 - \exp(-M_B)$ of its customers.[4] Equation (17) shows that the more firm x advertises offline, the lower the price on the corresponding online market $k = x$. This is again a consequence of the substitutability of messages across media.

We now turn to the message demands offline. Since each firm reaches a constant fraction $1 - \exp(-M_B)$ of its customers online, the supply of messages online simply acts as a scaling factor for each firm's demand function offline. Intuitively, each firm now has $s_x \exp(-M_B)$

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[4]Strictly speaking, we should interpret this as the limit of a model with a discrete number of product and advertising markets. In the discrete model, all consumers of product x are located in the advertising market $k = x$. Each firm x only advertises in the online market $k = x$, supply is proportional to the number of consumers in the market, and as a consequence, the probability of a match is constant across firms. These results hold independently of the number of products and markets, and carry over to our continuous model.

potential customers offline. The equilibrium allocation is then given by

$$X^* = \sqrt{2M_A/\lambda}, \tag{18}$$

$$m_x^* = \sqrt{2\lambda M_A} - \lambda x. \tag{19}$$

Equations (18) and (19) show that the equilibrium distribution of offline messages across the participating firms, as well as the number of active firms, are both identical to the single market case. However, competition has a clear effect on the equilibrium prices and revenues, as we show in the next proposition.

Proposition 6 (Equilibrium Prices)

1. *The equilibrium price on the offline medium is given by*

$$p^* = \lambda \exp(-M_B - \sqrt{2\lambda M_A}).$$

2. *The equilibrium prices on the online markets are given by*

$$p_k^* = \begin{cases} \exp(\lambda k - M_B - \sqrt{2\lambda M_A}), & for \quad k \leq X^*, \\ \exp(-M_B), & for \quad k > X^*. \end{cases}$$

Consistent with intuition, the offline price p^* is decreasing in M_B, reflecting the drop in each firm's willingness to pay for regular advertisements in the presence of an alternative, better targeted market. In other words, the presence of a targeted online market does not modify the composition of the offline market but lowers the equilibrium profits. The prices in the online markets are initially increasing in k, and then constant. This reflects the allocation of messages offline, where relatively smaller firms buy a lower number of messages, and are willing to pay more for M_B messages per capita online. Furthermore, the prices online are constant for all those markets (firms) who do not participate in the offline market. In other words, "niche" online markets, where customers of long tail firms are likely to be present, are not affected at all by media competition. In this sense, as emphasized by Anderson (2006), online advertising allows to reach new segments of the consumer population, which are distinct from the intended audience of the firms that actively advertise offline.

Finally, we seek to compare the interaction between online and offline media, with the competition between offline media. We take the point of view of a offline market, and focus

on the effect of competition on the equilibrium price. For the purpose of this comparison, we consider two cases: (i) competition between an offline market and a continuum of perfectly targeted online markets; (ii) competition between two symmetric non-targeted markets. We interpret the supply as the outcome of the consumers' time allocation decisions. In particular, we assume each consumer spends a fraction β of her time M in the offline market. We then have $M_A = \beta M$ and $M_B = (1 - \beta) M$, with the understanding that, in the case of offline vs. online media competition, the offline medium is denoted by A and the online medium by B. On the basis of the previous results, the equilibrium price in the offline market (i.e. market A) is given by:

$$p^* = \begin{cases} \lambda \exp(-(\sqrt{2\lambda\beta M} + (1 - \beta) M)), & \text{with an} \quad \text{online competitor,} \\ \lambda \exp(-\sqrt{2\lambda M}), & \text{with an} \quad \text{offline competitor.} \end{cases} \tag{20}$$

We can now draw conclusions on how different kinds of competition affect the equilibrium revenues of offline media.

Proposition 7 (Price Comparison)

The equilibrium price of messages in the offline market is lower under competition by the online markets, compared to competition by an offline medium if and only if

$$\lambda < (M/2)(1 - \beta)^2 (1 - \sqrt{\beta})^{-2}.$$

Proposition 7 shows that if product markets are diffuse (λ is low), the equilibrium price of advertising on the offline market falls more when competing against perfectly targeted markets. Conversely, for highly concentrated product markets, an offline competitor is more detrimental to the profits of a offline market. Intuitively, consider the case of symmetric competition when λ is high: the two media are competing for a few valuable buyers, and the outcome is closer to Bertrand competition. Finally, as the consumer spends more time on the competing market (the lower β), competition by a targeted medium becomes even less attractive, relatively to competition by a offline market with the same capacity. In other words, the progressive growth of online advertising markets is shown to be more detrimental for the revenues of offline media than, say, the entry of new competitors in the television or newspaper industries.

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6.2 Imperfect Targeting

When we consider imperfect targeting levels, our predictions are similar to those of the model with different degrees of concentration. In particular, the online market $k = 0$ is a close substitute for the offline medium, as all consumer types are present (though with different intensities). Consider the hierarchical structure of the advertising markets on a targeted medium. If firm x advertises on all markets $k \leq x$, then it can reach all of its customers both offline and online. This means the two media are perfect substitutes, and the prices would have to make the firm indifferent, in order to justify dual homing. Therefore, the price offline must be to equal a weighted average of the prices on the online markets firm x is active in:

$$p = \int_0^x p_k s_k dk.$$

Clearly, this condition cannot hold for more than one firm. In equilibrium, it must then be the case that firms $[x_L, x_H]$ advertise offline, while no firm $x > x_L$ advertises on online market $k = 0$. The message is similar to the model with different concentrations. Indeed, the two models are very close, as the concentration parameter of the distribution of consumers on market 0 is equal to $\gamma + \lambda$. As a result, the largest firms leave the offline medium and advertise exclusively online, in the largest markets k, leading to a decrease in the price of the offline medium.

This effect is somewhat mitigated if the online market has a smaller user base. As in the case of competition between offline media of different sizes, we can show that all firms larger than a critical x^* advertise both offline and on all the available online markets (*i.e.*, each firm x buys messages on markets $k \in [0, x]$). In terms of comparative statics, better targeting reduces the demand for online messages by "long tail" firms, and induces a higher concentration in the offline medium. At the same time, larger firms are also able to reach a larger fraction of their customers online, and this reduces the overall profitability of the offline medium. As a consequence, the offline price p^* and the number of firms participating offline x^* are decreasing in γ, while the distribution of offline messages is more concentrated as γ increases.

7 Revenue Maximization

The analysis so far has considered competitive advertising markets with uniform, market-clearing, unit prices for advertising messages. In this section we explore the possibility that each advertising market is owned by a single or a few publishers who seek to maximize revenues by a menu of quantity and price pairs. We first focus on the problem of a monopolist with the ability to price discriminate across different advertising firms, while keeping the total number of available messages fixed. We then return to a model with unit prices, and consider the effects of endogenizing the total supply of messages.

7.1 Menu Pricing

We focus on a single advertising market, in which a single publisher sells M messages to a continuum of advertisers via nonlinear pricing schedules. The type of each advertiser is his rank in the product market, x. A strategy for the single publisher can be represented by a direct mechanism as a pair of functions, $m(x)$ and $p(x)$, assigning a number of messages and a total payment to each firm x. The indirect utility of firm x is then given by

$$U(x) = \max_{\hat{x}}[\lambda e^{-\lambda x}(1 - e^{-m(\hat{x})}) - p(\hat{x})].$$

The publisher then solves the following problem:

$$\max_{m(\cdot), U(\cdot)} \int_0^\infty (\lambda e^{-\lambda x}(1 - e^{-m(x)}) - U(x))dx.$$

The publisher's choice of $m(x)$ and $U(x)$ is constrained by the local incentive compatibility requirement

$$U'(x) = -\lambda^2 e^{-\lambda x}(1 - e^{-m(x)}),$$

by the monotonicity requirement,

$$m'(x) \leq 0,$$

and by the total capacity M, so that

$$\int_0^\infty m(x)\,dx \leq M.$$

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In the absence of production costs, the monopolist wants to sell as many units as possible, hence the capacity constraint is binding. The X largest firms participate, and the optimal message allocation is given by

$$m(x) = \lambda \left(X - x + \ln \frac{1 - \lambda x}{1 - \lambda X} \right),$$

where the logarithmic term differentiates the price discriminating monopolist's from the competitive solution. Notice that, given X, each firm $x \leq X$ receives more messages in the current specification. The equilibrium set of active firms is given by $[0, X^*]$, where X^* is the solution to

$$\int_0^{X^*} \ln \frac{(1 - \lambda x) e^{-\lambda x}}{(1 - \lambda X^*) e^{-\lambda X^*}} dx = M. \tag{21}$$

The marginal prices charged by the publisher can be easily characterized in terms of the nonlinear tariff $p(m)$ through the advertisers' first order condition

$$\lambda e^{-\lambda x} e^{-m} = p'(m). \tag{22}$$

We therefore establish the following properties of the revenue maximizing allocation.

Proposition 8 (Revenue Maximization)

1. *The revenue maximizing allocation is given by*

$$m^*(x) = \begin{cases} \lambda \left(X^* - x + \ln \frac{1 - \lambda x}{1 - \lambda X^*} \right), & \text{if } x \leq X^*, \\ 0, & \text{if } x > X^*, \end{cases}$$

   where X^* is given by the solution of (21).

2. *The number of active firms X^* is strictly increasing in M and strictly decreasing in λ. For all λ and M, we have $X^* \in \left[0, \lambda^{-1} \right]$.*

3. *The average unit price $p^*(m)/m$ is decreasing in m.*

4. *The publisher's revenue is increasing in λ and in M, and converges to e^{-1} as either $\lambda \to \infty$ or $M \to \infty$.*

The revenue maximizing allocation differs from the competitive one along several dimensions. First, compared to the competitive allocation, the equilibrium quantities are

distorted downwards. For a given X, the revenue maximizing allocation assigns more messages to larger firms, since the term $\ln \frac{1-\lambda x}{1-\lambda X}$ is decreasing. Second, fewer firms participate in equilibrium, and the largest firms receives more messages than in the competitive allocation. This is possible because the publisher can price differentially and extract more surplus from the advertisers. Third, the number of active firms is bounded from above by $1/\lambda$, and the publisher's equilibrium revenue converges to a positive number as $M \to \infty$. This is in sharp contrast with the competitive case, where the number of active firms increases without bound as $M \to \infty$, and revenues vanish. Finally, unit prices are decreasing in the volume m of messages purchases. As expected, the publisher offers quantity discounts.

7.2 Endogenous Capacity and Competition

What happens if the publisher can choose the capacity M instead? In the single firm case, there exists an optimal M, while more capacity is always more profitable in the price discrimination model. However, the analysis of competition among publishers is a natural question to address at this stage.

Suppose that more than one publisher is present in the advertising market. As in the analysis of market interaction, suppose that each consumer divides her time equally across two or more publishers. We are therefore motivated to analyze competition among n identical publishers as a Cournot game, with advertising volume as the strategic variable. Each consumer spends a total time of M/n on each publication. Each publisher chooses what level of capacity $q_j \in [0, M/n]$ to place on the market. Clearly, each publisher could fill the entire time with messages, or withhold some capacity and keep prices high. Following our analysis in Section 5, we know the equilibrium price of messages is given by

$$p\left(q\right) = \lambda \exp(-\sqrt{2\lambda \Sigma_j q_j}).$$

The resulting symmetric equilibrium quantities and price are given by

$$
\begin{aligned}
q^* &= \min\left\{M/n, 2n/\lambda\right\}, \\
p^* &= \max\{\lambda e^{-2n}, \lambda e^{-\sqrt{2\lambda M}}\}.
\end{aligned}
$$

In the next proposition, we relate the imperfect competition prices and quantities with the

competitive equilibrium benchmark.

Proposition 9 (Imperfect Competition)

For any M, there exists $n^*(M) \triangleq \sqrt{\lambda M / 2}$ such that, for all $n \geq n^*(M)$, imperfect competition yields the competitive benchmark outcome.

In particular, notice that the capacity constraints imply that imperfect competition will have no impact on the equilibrium outcomes if the number of publishers is high enough. Intuitively, each publisher's incentives to withhold capacity are highest in the monopoly case. Thus, our benchmark model may be viewed as describing a framework in which publishers have market power, but the number of competitors is high relatively to the time allocated to the medium.

Finally, consider the case of competitive menu pricing. Suppose that publishers can adopt menu pricing. Remember that messages bought on different media are perfect substitutes for advertisers. If publishers offer incentive compatible tariffs as the one described above, advertisers will buy their entire supply from a single publisher, so as to exploit declining unit prices. Given the menus offered by competitors, each publisher will then want to cut prices for the most lucrative market segments, in this case the small advertisers who are paying high unit prices. It follows that in equilibrium all sales must take place at a constant unit price. Therefore, given the capacity constraints, the nonlinear pricing problem reduces to a capacity choice for each publisher.

8 Concluding Remarks

In this paper, we developed a novel model to understand and evaluate the implications of targeting in advertising markets. The model provided a framework for the systematic analysis of the trade-offs that arise due to changes in the targeting technology. We adopted a hierarchical framework to rank products and advertising markets of different sizes. We explored in particular the tension between competition and value extraction. This tension appears as general issue as the targeting ability of the various media improve. In this sense, our model can provide insight into the effects of detailed users information in the hands of social networks and on the profitability of IP address tracking.

The analysis we have presented is the outcome of a number of modeling choices which

constrain the scope of our results in some directions. We now conclude by discussing two directions for future research. The price of advertising was determined in a competitive equilibrium model. While the competitive equilibrium is the natural benchmark, it would be of interest to consider the implications of other pricing rules for advertising messages. Clearly, the auctions for keywords in the sponsored search environment or the emerging ad exchange model might offer valuable additional insights in this respect.

In our model, the advertisers were competing for messages but they were not competing for consumers. The competition among firms for advertising messages therefore did not interact with their competition in the product market. A natural next step therefore might enrich the current model with advertisers which are directly competing in the product markets. The equilibrium price for advertising, in particular in highly targeted markets, may then become more responsive to the intensity of competition on the product market.

Appendix

Proof of Proposition 1. (1.)–(4.) The comparative statics results can be derived directly by differentiating expressions (7), (8), and (9) in the text.

(5.) The total expenditure of firm $x \leq X^*$ is given by

$$p^* m_x^* = \lambda e^{-\sqrt{2\lambda M}}(\sqrt{2\lambda M} - \lambda x),$$

and the total number of consumers reached is

$$s_x(1 - e^{-m_x^*}) = \lambda e^{-\lambda x}(1 - e^{\lambda x - \sqrt{2\lambda M}}).$$

Therefore, the price paid by firm x per consumer reached is given by

$$\frac{p^* m_x^*}{s_x (1 - e^{-m_x^*})} = \frac{\sqrt{2\lambda M} - \lambda x}{e^{\sqrt{2\lambda M} - \lambda x} - 1} = \frac{z}{e^z - 1},$$

which is decreasing in z (with $z = \sqrt{2\lambda M} - \lambda x$), and therefore increasing in x. It is also decreasing in λ if $x < \sqrt{M/2\lambda}$ (which represents the median active firm).

(6.) The average probability of a match, which is equal to the total fraction of consumers reached, is given by

$$W(\lambda, M) = \int_0^{X^*} s_x(1 - e^{-m_x^*})dx = 1 - \frac{1 + \sqrt{2M\lambda}}{e^{\sqrt{2M\lambda}}},$$

which is increasing in λ. ∎

Proof of Proposition 2. (1.)–(2.) The competitive equilibrium described in the text leads to uniform prices p_k^* across advertising markets. Therefore, for each firm and for each market, the marginal returns to messages are equalized. Since the match production function is concave in m, this allocation maximizes the probability of a match given the available supply of messages.

(3.) The average probability of a match now takes into account the fraction of consumers reached in the exterior market as well as in the interior markets. It is given by,

$$W(\lambda, \gamma, M) = \int_0^\infty \int_k^{X_k^*} s_{x,k}(1 - e^{-m_{x,k}/s_k})dxdk + \int_0^{X_0^*} s_{x,0}(1 - e^{-m_{x,0}/s_0})dx,$$

where $m^*_{x,k}$ is given by (15) in the text. Therefore, we obtain

$$W\left(\lambda, \gamma, M\right) = 1 - \frac{1 + \sqrt{2M\left(\lambda + \gamma\right)}}{e^{\sqrt{2M(\lambda+\gamma)}}},$$

which is increasing in λ and γ. ∎

Proof of Proposition 3. (1.)–(4.) These statements follow from differentiation of expressions (13), (14), and (15) in the text. ∎

Proof of Proposition 4. From the first order conditions for firm x, we obtain

$$
\begin{aligned}
1 - r_{x,A} &= e^{-m_x^A} = e^{\lambda x}\frac{p_A}{\lambda\left(1 - r_{x,B}\right)}, \\
1 - r_{x,B} &= e^{-m_x^B} = e^{\lambda x}\frac{p_B}{\lambda\left(1 - r_{x,A}\right)}.
\end{aligned}
$$

It follows that in equilibrium we must have $p_A = p_B = p$, and that the sum of the demands is given by

$$m_x^A + m_x^B = \ln\frac{\lambda}{p} - \lambda x.$$

Consider the market clearing condition for A and B combined,

$$\int_0^X \left(m_x^A + m_x^B\right)dx = M_A + M_B,$$

and the results follow as in the single-homing case. ∎

Proof of Proposition 5. The first order conditions may be written as

$$
\begin{aligned}
\lambda_A e^{-\lambda_A x}\left(1 - \delta\left(1 - e^{-m_B/\delta}\right)\right)e^{-m_A} - p_A &= 0, \\
\lambda_A e^{-\lambda_A x}e^{-m_A}\left(1 - e^{-m_B/\delta}\right) - p_B &= 0.
\end{aligned}
$$

Solving for $m_{x,A}$ and $m_{x,B}$, and simplifying, we obtain

$$
\begin{aligned}
m_A &= \ln\frac{\lambda\left(1 - \delta\right)}{p_A - \delta p_B} - \lambda x, \\
m_B &= m_B = \delta\ln\frac{p_A - \delta p_B}{p_B\left(1 - \delta\right)}, \text{ for } x \in [0, X].
\end{aligned}
$$

For all firms $x \in [X, Z]$, we have $m_A = 0$ and $m_B = \delta (\ln \lambda / p_B - \lambda x)$ as in the single-homing case. Since by construction, the marginal firm X satisfies $m_{X,A} = 0$, we have $(1 - \delta) \lambda \exp(-\lambda X) = p_A - \delta p_B$. Similarly, we have $m_{Z,B} = 0$, and so $\lambda \exp(-\lambda Z) = p_B$. We can now write the market clearing conditions as follows:

$$\int_0^X m_A dx = \int_0^X \lambda (X - x) \, dx = M_A,$$

$$Xm_B + \int_X^Z m_B dx = X\delta\lambda(Z - X) + \int_X^Z \delta\lambda(Z - x) \, dx = \delta M_B.$$

Therefore

$$X = \sqrt{\frac{2M_A}{\lambda}}, \quad Z = \sqrt{\frac{2(M_A + M_B)}{\lambda}},$$

which means

$$p_A = \delta\lambda e^{-\sqrt{2(M_A + M_B)\lambda}} + (1 - \delta)\lambda e^{-\sqrt{2M_A\lambda}}$$

$$p_B = \lambda e^{-\sqrt{2(M_A + M_B)\lambda}},$$

which completes the proof. ∎

Proof of Proposition 6. The price offline is equal to $\lambda \exp(-\lambda X^*)$, where X^* is the marginal firm characterized in (18). The prices offline follow from substitution of (19) into (16) and (17). ∎

Proof of Proposition 7. The inequality in the text follows from directly comparing the two expressions for the p^* in (20), and solving for λ. ∎

Proof of Proposition 8. (1.) The modified Hamiltonian is

$$H(x, m, U, a) = \lambda e^{-\lambda x}(1 - e^{-m}) - U - a\lambda^2 e^{-\lambda x}(1 - e^{-m}) + b(M - m)$$

where $a(x)$ and b are the multipliers on the incentive constraint and on the capacity constraint. The first order conditions are given by:

$$\lambda e^{-\lambda x}e^{-m} - a\lambda^2 e^{-\lambda x}e^{-m} = b, \quad a'(x) = 1, \quad a(0) = 0.$$

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Therefore, we have

$$
\begin{aligned}
a\left(x\right) &= x, \\
m\left(x\right) &= \ln \frac{\left(1 - \lambda x\right)\lambda e^{-\lambda x}}{b}, \\
b &= \lambda e^{-\lambda X}\left(1 - \lambda X\right).
\end{aligned}
$$

From the market clearing condition, we have

$$
\int_0^X \ln \frac{\left(1 - \lambda x\right) e^{-\lambda x}}{\left(1 - \lambda X\right) e^{-\lambda X}} dx = M,
$$

and by defining $z := \lambda X$, we obtain

$$
\frac{1}{2} z^2 - z - \ln \left(1 - z\right) = \lambda M. \tag{23}
$$

(2.) The left-hand side of (23) is increasing in z, and therefore z is increasing in M, and $z\left(M\right) \in \left[0, 1\right]$. Furthermore, $z \to 1$ (so $X \to \lambda^{-1}$) as $M \to \infty$. We can derive the last comparative statics result from

$$
\begin{aligned}
\lambda X'\left(\lambda\right) &= \left(M \frac{z - 1}{z\left(z - 2\right)} - \frac{z}{\lambda}\right) \\
&= -\left(1 - z\right)\ln\left(1 - z\right) + \frac{1}{2} z\left(z + 1\right)\left(z - 2\right) < 0.
\end{aligned}
$$

(3.) From the buyer's first order condition, we know that $\lambda e^{-\lambda x\left(m\right)} e^{-m} = p'\left(m\right)$, where $x\left(m\right)$ is determined by:
$$
\ln \frac{\left(1 - \lambda x\right) e^{-\lambda x}}{\left(1 - \lambda X\right) e^{-\lambda X}} - m = 0.
$$

Differentiating, we obtain

$$
p''\left(m\right) = -\lambda e^{-\lambda x\left(m\right)} e^{-m}\left(1 + \lambda x'\left(m\right)\right),
$$

and so

$$
p''\left(m\right) \propto -\left(1 - \frac{1 - \lambda x}{2 - \lambda x}\right) < 0.
$$

Finally, since $p\left(0\right) = 0$, the concavity of the function $p\left(m\right)$ implies that $p\left(m\right)/m$ is decreasing.

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(4.) By the IC constraint, advertisers' utility is given by

$$U(x) = \int_x^X \lambda^2 e^{-\lambda x}\left(1 - e^{-m(x)}\right) dx = \lambda\left(e^{-x\lambda} - e^{-\lambda X}\right) - (1 - \lambda X)\lambda e^{-\lambda X}\ln\frac{1 - \lambda x}{1 - \lambda X}.$$

Revenues are given by $\int_0^X p(x)\, dx$, which may be written as:

$$R = \int_0^X \left(\lambda e^{-\lambda x}\left(1 - e^{-m(x)}\right) - U(x)\right) dx = X^2\lambda^2 e^{-X\lambda} = z^2 e^{-z}.$$

Since $R(z)$ is increasing for $z \in [0,1]$, we conclude that R is increasing in λ and M. Furthermore, $z \to 1$ implies $R \to e^{-1}$. ∎

Appendix B

This Appendix contains the construction of the competitive equilibrium with competing media markets and different consumer concentrations across media markets.

The case of $s_{X,A} > s_{X,B}$ We begin with the case of $s_{X,A} > s_{X,B}$. Consider the objective function of any firm x such that $\lambda_A e^{-\lambda_A x} > \lambda_B e^{-\lambda_B x}$:

$$\pi_x = \lambda_A e^{-\lambda_A x} \left(1 - e^{-m_A}\right) + e^{-m_A} \lambda_B e^{-\lambda_B x} \left(1 - e^{-m_B}\right) - p_A m_A - p_B m_B.$$

The first order conditions are given by

$$e^{-m_A} \left(\lambda_A e^{-\lambda_A x} - \lambda_B e^{-\lambda_B x} \left(1 - e^{-m_B}\right)\right) - p_A = 0,$$
$$e^{-m_B} e^{-m_A} \lambda_B e^{-\lambda_B x} - p_B = 0.$$

By construction, the marginal firm satisfies $m_{X,A} > 0$ and $m_{X,B} = 0$. The indifference condition for the marginal firm requires that

$$e^{-m_A} \left(\lambda_A e^{-\lambda_A X} - \lambda_B e^{-\lambda_B X}\right) = p_A - p_B,$$

and therefore, from the first order conditions,

$$m_{X,A} = \ln\left(s_{X,A}/p_A\right),$$

or equivalently

$$X = \frac{1}{\lambda_A - \lambda_B} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B}.$$

Similarly, the marginal firm Y has $m_{Y,A} = 0$ and $m_{Y,B} > 0$. Therefore, indifference requires

$$\lambda_A e^{-\lambda_A Y} - \lambda_B e^{-\lambda_B Y} = p_A - p_B.$$

Since we know from the marginal firm X that $p_A > p_B$, we also know $\lambda_A e^{-\lambda_A Y} > \lambda_B e^{-\lambda_B Y}$, that is, all firms active on both media will lie before the crossing point of the two density functions. Furthermore, for all the dual-homing firms $x \in [X, Y]$, we know both first order

conditions hold with equality, and so

$$
e^{-m_{x,A}} = \frac{p_A - p_B}{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}
$$

$$
e^{m_{x,B}} = \frac{\lambda_B e^{-\lambda_B x}}{p_B} e^{-m_{x,A}}.
$$

Finally, exploiting the two conditions $m_{Y,A} = 0$ and $m_{X,B} = 0$, we obtain the equilibrium prices as a function of the cutoff firms:

$$
p_A = \lambda_A e^{-X\lambda_A} \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}
$$

$$
p_B = \lambda_B e^{-X\lambda_B} \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}.
$$

Now we consider the market clearing conditions:

$$
\int_0^X \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x \right) dx + \int_X^Y \ln \frac{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}{p_A - p_B} dx = M_A \tag{24}
$$

$$
-\int_X^Y \ln \frac{\lambda_A e^{-x\lambda_A} - \lambda_B e^{-x\lambda_B}}{p_A - p_B} dx + \int_X^Z \left(\ln \frac{\lambda_B}{p_B} - \lambda_B x \right) dx = M_B, \tag{25}
$$

where the marginal firm Z satisfies

$$
Z = \frac{1}{\lambda_B} \ln \frac{\lambda_B}{p_B} = X + \frac{1}{\lambda_B} \ln \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}.
$$

We can then substitute the expressions for p_A, p_B, and Z into (24) and (25), solve them numerically for the two unknowns X and Y, and verify that $X < Y < h(\lambda_A, \lambda_B)$, where

$$
h(\lambda_A, \lambda_B) = (\ln \lambda_A - \ln \lambda_B) / (\lambda_A - \lambda_B)
$$

represents the crossing point of the two distributions.

The case of $s_{X,A} < s_{X,B}$ Next we consider the case of $s_{X,A} < s_{X,B}$. Consider the objective function of any firm x such that $\lambda_A e^{-\lambda_A x} < \lambda_B e^{-\lambda_B x}$:

$$
\pi_x = \lambda_A e^{-\lambda_A x} e^{-m_B} \left(1 - e^{-m_A} \right) + \lambda_B e^{-\lambda_B x} \left(1 - e^{-m_B} \right) - p_A m_A - p_B m_B.
$$

The first order conditions are given by

$$\lambda_A e^{-\lambda_A x} e^{-m_B} e^{-m_A} - p_A = 0,$$

$$e^{-m_B}\left(\lambda_B e^{-\lambda_B x} - \lambda_A e^{-\lambda_A x}\left(1 - e^{-m_A}\right)\right) - p_B = 0.$$

By construction, $m_{X,A} > 0$ and $m_{X,B} = 0$. Again, indifference requires that

$$\lambda_A e^{-\lambda_A X} - \lambda_B e^{-\lambda_B X} = p_A - p_B.$$

Similarly, the marginal firm Y has $m_{Y,A} = 0$ and $m_{Y,B} > 0$. Therefore, indifference also requires

$$\frac{\lambda_A e^{-\lambda_A Y}}{\lambda_B e^{-\lambda_B Y}} = \frac{p_A}{p_B},$$

or equivalently

$$Y = \frac{1}{\lambda_B - \lambda_A} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B}.$$

For all the dual-homing firms $x \in [X, Y]$, we know both first order conditions must hold with equality, and so we have

$$e^{m_{x,A}} = \frac{\lambda_A e^{-\lambda_A x}}{p_A} e^{-m_{x,B}}$$

$$e^{-m_{x,B}} = \frac{p_B - p_A}{\lambda_B e^{-x\lambda_B} - \lambda_A e^{-x\lambda_A}}.$$

We can then solve for the prices,

$$p_A = \lambda_A e^{-Y\lambda_A} \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}$$

$$p_B = \lambda_B e^{-Y\lambda_B} \frac{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}.$$

Now we have the following market clearing conditions:

$$\int_0^X \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x\right) dx + \int_X^Y \left(\ln \frac{\lambda_A}{p_A} - \lambda_A x - m_B\right) dx = M_A$$

$$\int_X^Y \ln \frac{\lambda_B e^{-x\lambda_B} - \lambda_A e^{-x\lambda_A}}{p_B - p_A} dx + \int_Y^Z \left(\ln \frac{\lambda_B}{p_B} - \lambda_B x\right) dx = M_B$$

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where the marginal firm Z satisfies

$$\lambda_B e^{-\lambda_B Z} = p_B,$$

or

$$Z = Y + \frac{1}{\lambda_B} \ln \frac{\lambda_A e^{-Y\lambda_A} - \lambda_B e^{-Y\lambda_B}}{\lambda_A e^{-X\lambda_A} - \lambda_B e^{-X\lambda_B}}.$$

Again, we can use the expressions for p_A, p_B, and Z, solve the market clearing conditions numerically, and verify that $h(\lambda_A, \lambda_B) < X < Y$.

The case of $s_{X,A} = s_{X,B}$ We conclude the construction of the equilibrium by considering $s_{X,A} = s_{X,B}$ For the marginal firms X and Y to be both equal to $h(\lambda_A, \lambda_B) = (\ln \lambda_A - \ln \lambda_B) / (\lambda_A - \lambda_B)$, it must be that $p_A = p_B = p$ and that $X = Y$. Thus

$$X = \frac{1}{\lambda_A - \lambda_B} \ln \frac{\lambda_A/\lambda_B}{p_A/p_B} = \frac{1}{\lambda_A - \lambda_B} \ln (\lambda_A/\lambda_B). \tag{26}$$

Market clearing on $j = A$,

$$\int_0^X \left(\ln \frac{\lambda_A}{p} - \lambda_A x \right) dx =$$

$$X \ln \frac{\lambda_A}{p} - \lambda_A X^2/2 = M_A$$

$$\lambda_A \exp \left(-\frac{M_A}{X} - \frac{\lambda_A X}{2} \right) = p,$$

and then on $j = B$:

$$\int_X^{\frac{1}{\lambda_B} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} \right)} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} - \lambda_B x \right) dx = M_B.$$

Substituting the definition of X from (26), we obtain the condition that λ_A must satisfy:

$$\int_X^{\frac{1}{\lambda_B} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} \right)} \left(\ln \frac{\lambda_B}{\lambda_A} + \frac{M_A}{X} + \frac{\lambda_A X}{2} - \lambda_B x \right) dx = M_B$$

$$-\ln (\lambda_A/\lambda_B) \frac{1}{2} \frac{\lambda_A}{\lambda_A - \lambda_B} + \frac{(\lambda_A - \lambda_B) M_A}{\ln (\lambda_A/\lambda_B)} = \sqrt{2\lambda_B M_B}.$$

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