AFFECTIVE DECISION MAKING AND THE ELLSBERG PARADOX

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Abstract

We characterize, in the framework for variational preferences, the affective decision making model of choice under risk and uncertainty introduced by Bracha and Brown(2007). This characterization (i) provides a rigorus decision-theoretic foundation for affective decision making, (ii) offers an axiomatic explanation for ambiguity-seeking in the Ellsberg Paradox and (iii) suggests a dual representation of ADM games in terms of the Legendre-Fenchel conjugate.

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Keywords Ellsberg paradox, Schmeidler's axiom, Affective decision making, Variational preferences, Legendre-Fenchel conjugate.

1 Introduction

As is well known, the subjective expected utility (SEU) models of Savage (1954) and Anscombe and Aumann(1963) are refuted by the Ellsberg paradox (1961). In the Ellsberg experiment, individuals are asked to bet on a draw from an urn with 100 balls, some red and the rest black, where the distribution is unknown or bet on a draw from an urn with 50 black balls and 50 red balls. This experiment partitions the subjects into three disjoint groups: A, B, and C. Individuals in Group A preferred to bet on a black draw from the urn with the known distribution, rather than bet on a black draw from the urn with the unknown distribution and similarly for bets on drawing a red ball. Individuals in Group B, were indifferent between betting on draws from either urn. Individuals in Group C preferred to bet on the ambiguous urn.

In his thought experiment, Ellsberg (1961) on pg 651 suggests that the majority of people are in group A, but a small minority are in group C and he ignores the people in group B. As he points out, both Group A and C violate Savage's axioms for the SEU model. Subjects in Group A are said to be ambiguity-averse and subjects in Group C are said to be ambiguity-seeking.

A number of alternative models of choice under risk and uncertainty have been proposed as resolutions of the Ellsberg paradox, such as the maximin expected utility model of Gilboa and Schmeidler (1989) or more recently the multiplier preferences of Hansen and Sargent(2000). Recently, Maccheroni, Marinacci and Rustichini[MMR] (2006) proposed variational preferences as a general class of preferences that rationalize ambiguity-averse choices. MMR (2006) show that variational preferences subsume both maximin preferences and multiplier preferences and are characterized by six axioms, where axiom 5, due to Schmeidler (1989), is the axiom for ambiguity aversion. This axiom has the simple geometric interpretation that the preference relation over acts is quasi-concave. Moreover, if axiom 5 is replaced by axiom $5'$ where the preference relation over acts is quasi-linear, then axioms :$1-4$, $5'$ and 6 characterize the SEU model. Both of these results are proven in MMR (2006).

The remaining possibility is that the preference relation over acts is quasi-convex– a possibility anticipated by Ellsberg's thought experiment (1961), where the decision-makers in Group C were ambiguity-seeking. If so, then what is the behavioral interpretation of this axiom and do these preferences share with variational preferences a penalized SEU representation?

These are the questions we address in this paper.

In the variational preferences models the decision maker is playing a sequential game against a malevolent nature, where nature moves last. Hence the solution concept is maximin. In the affective decision making (ADM) model proposed by Bracha and Brown (2007) the rational and the emotional process of the decision-maker are engaged in a simultaneous move, potential game, where the solution concept is Nash equilibrium. Both classes of models are penalized SEU models. In the variational preferences models the penalty reflects the decision maker's uncertainty that her "subjective" beliefs about the states of the world are the correct state probabilities. In the ADM model, the penality reflects the mental cost of her "optimistic" beliefs about preferred outcomes.

We suggest that the outcomes of Ellsberg's thought experiment are not paradoxical, but allow for three mutually exclusive formulations of Schmeidler's axiom. That is, preferences over acts can be quasi-concave, quasi-linear or quasi-convex. If in addition preferences satisfy axioms $1-4$ and axiom 6 in MMR (2006), then the corresponding classes of preferences over acts are: variational preferences, SEU preferences and ADM preferences.We show that if axiom $5: f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succcurlyeq f$, the axiom that the preference relation over acts is quasi-concave, is replaced with axiom $\widehat{5}$: $f \sim g \Longrightarrow \alpha f + (1-\alpha)g \preccurlyeq f$,the axiom that the preference relation over acts is quasi-convex, then the preference relation has an ADM representation if and only if it satisfies axiom $\widehat{5}$ and axioms $1-4$ and 6 for variational preferences.

In the next section, we use the proof in MMR (2006) of their representation theorem for variational preferences, with axiom $\widehat{5}$ in lieu of axiom 5, to prove the representation theorem for ADM preferences.

2 A Decision-Theoretic Foundation for ADM

We follow the SETUP in MMR(2006), where: S is the set of states of the world; Σ is an algebra of subsets of S, the set of events; and X, the set of consequences, is a convex subset of some vector space. F is the set of (simple) acts, i.e., finite-valued Σ-measureable functions $f : S \longrightarrow X$. $B(\Sigma)$ is the set of all bounded Σ -measureable functions. $B(\Sigma)$ with the sup-norm is an AM -space with unit, the constant function 1. $B_o(\Sigma)$ the set of Σ -measureable simple functions is norm dense in $B(\Sigma)$.The norm dual of $B(\Sigma)$ is $ba(\Sigma)$, finitely additive signed measures of bounded variation on Σ– for further discussion see chapter 13 in Aliprantis and Border(1999). If $u : X \longrightarrow R$, then $u(f) \in B_o(\Sigma)$ for every $f \in F$.

Niveloids are functionals on function spaces that are monotone: $\varphi \leq \eta \Longrightarrow I(\varphi) \leq I(\eta)$and vertically invariant: $I(\varphi + r) = I(\varphi) + r$ for all φ and $r \in R$–see Dolecki and Greco (1995) for additional discussion. Epstein, Marinacci and Seo [EMS] (2007) show in lemma $A.5$ that niveloids are Lipschitz continuous on any convex cone of an AM-space with unit and concave(convex) if and only if they are quasi-concave(convex).

The general representation for variational preferences is:

$V(f) = \min_{p\in\Delta} \left\{ \int u(f)dp + c(p) \right\}$. $c : \Delta \longrightarrow [0, \infty]$ is a convex functional on the simplex Δ, the family of positive, finitely additive measures of bounded variation in $ba(\Sigma)$, and u is an affine function on X. If we rewrite $V(f) = \min_{p\in\Delta} \left\{ \int u(f)dp - q(p) \right\}$, where $q : \Delta \to [0, \infty]$ is a concave functional on the simplex Δ, then $V(f)$ is the Legendre-Fenchel conjugate of the concave function $q(p)$– see Rockafellar (1970), pg 308, for finite state spaces. If the decision-maker maximizes $V(f)$ over her choice set K, then $\max_{f\in K} V(f) = \max_{f\in K} \min_{p\in\Delta} \left\{ \int u(f)dp + c(p) \right\}$. Hence her optimal decision is a maximin equilibrium of the game against nature.

The potential function, $\Pi(f, p)$, for an ADM intrapersonal game is: $\Pi(f, p) = \int u(f)dp - c(p)$, where $c : \Delta \to [0, \infty]$ is a convex functional on the simplex Δ. If $W(f) = \max_{p\in\Delta} \left\{ \int u(f)dp - c(p) \right\}$, then $W(f)$ is the Legendre-Fenchel conjugate of the convex function $c(p)$–see Rockafellar(1970), pg 104 for finite state spaces and Zălinescu (2002), pg 75 for infinite state spaces. If the decision-maker maximizes $W(f)$ over her choice set K, then $\max_{f\in K} W(f) = \max_{f\in K} \max_{p\in\Delta} \left\{ \int u(f)dp - c(p) \right\} = \max_{f\in K, p\in\Delta} \Pi(f, p)$. Hence $\arg\max_{f\in K, p\in\Delta} \Pi(f, p)$ is a pure strategy Nash equilibrium of the ADM intrapersonal game. That is, maximax rather than maximin. Here are our axioms, again we follow MMR(2006):

$A.1$(Weak Order): If f, g, $h \in F$, (a) either $g \succsim f$ or $f \succsim g$, and (b) $f \succsim g$ and $g \succsim h \Longrightarrow f \succsim h$.

$A.2$(Weak Certainty Independence): If f, $g \in F$, x, $y \in X$ and $\alpha \in (0, 1)$,then
$\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x \Longrightarrow \alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y$.

$A.3$(Continuity): If f, g, $h \in F$, the sets$\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h\}$ and $\{\alpha \in [0, 1] : h \succsim \alpha f + (1 - \alpha)g\}$are closed

$A.4$(Monoticity): If f, $g \in F$ and $f(s) \succsim g(s)$ for all $s \in S$,the set of states, then $f \succsim g$.

$A.\widehat{5}$(Quasi-Convexity): If f, $g \in F$ and $\alpha \in (0,1)$, then
$f \sim g \Longrightarrow \alpha f + (1-\alpha)g \precsim f$
$A.6$(Nondegeneracy): $f \succ g$ for some f, $g \in F$

Theorem 1 *Let $\succsim$ be a binary order on F. The following conditions are equivalent:*

(1)The relation $\succsim$ satisfies axioms $A.1 - A.6$
(2)There exists a nonconstant affine function $u : X \longrightarrow R$ and a continuous, convex function $c : \Delta \longrightarrow [0, \infty]$ where for all f, $g \in F$, $f \succsim g \iff$
$W(f) \geq W(g)$ and $W(h) = \max_{p\in\Delta} \left\{ \int u(h)dp - c(p) \right\}$ for all $h \in F$

Proof. Axioms $1 - 4$ are used in MMR(2006) to derive a nonconstant affine utility function, u, over the space of consequences, X. u is extended to the space of simple acts, F, using certainty equivalents. That is, $U(f) = u(x_f)$ for each $f \in F$, where x_f.is the certainty equivalent of f. This is lemma 28 in MMR(2006), where $I(f) = U(f)$ is a niveloid on $\Phi = \{\varphi : \varphi = u(f)$ for some $f \in F\}$. Φ is a convex subset of $B(M)$ and by Schmeidlers's axiom 5, I is quasi-concave on Φ. We also assume axioms1-4, so lemma 28 in MRR (2006) holds for the niveloid J in the ADM representation theorem. By axiom $\widehat{5}$, J is quasi-convex on Φ.

MMR(2006) show in lemma 25 that I is concave if and only if I is quasi- concave. Hence J is convex if and only if J is quasi-convex, since J is convex(quasi-convex) if and only if $-J$ is concave(quasi-concave). MMR(2004) extend I to a concave niveloid $\widehat{I}$ on all of $B(\Sigma)$–see lemma 25 in MMR (2004). Since $B(\Sigma)$ is a convex cone in an AM-space with unit, $\widehat{I}$ is Lipshitz continuous. It follows from the theorem of the bi-conjugate for continuous, concave functionals that $I(\varphi) = \inf_{p\in ba(\Sigma)} \left\{ \int \varphi dp - \widehat{I}^*(p) \right\}$, where $\widehat{I}^*(p) = \inf_{\varphi \in B_o(\Sigma)} \left\{ \int \varphi dp - \widehat{I}(\varphi) \right\}$ is the concave, conjugate of $\widehat{I}(\varphi)$– see Rockafellar (1970), pg 308.for finite state spaces. MMR(2006) show on pg. 1476 that we can restrict attention to Δ, the family of positive, finitely additive measures of bounded variation in $ba(\Sigma)$. Hence $I(\varphi) = \min_{p\in\Delta} \left\{ \int \varphi dp - \widehat{I}^*(p) \right\} = \min_{p\in\Delta} \left\{ \int u(f)dp + c(p) \right\}$, where $\varphi = u(f)$ and
$c(p) = -\widehat{I}^*(p)$. $c(p)$ is convex since $\widehat{I}^*(p)$ is concave.

Extending -J to -$\widehat{J}$ on $B(\Sigma)$, using lemma 25 in MMR (2004), it follows from the theorem of the biconjugate for continuous, convex functionals that
$J(\varphi) = \max_{p\in ba(\Sigma)} \left\{ \int \varphi dp - \widehat{J}^*(p) \right\}$ where $\widehat{J}^*(p) = \max_{\varphi\in B_o(\Sigma)} \left\{ \int \varphi dp - \widehat{J}(\varphi) \right\}$ is the convex, conjugate of $\widehat{J}(\varphi)$ –see Rockafellar (1970), pg 104 for finite state spaces and Zălinescu (2002), pg 77 for infinite state spaces.

Again it follows from MMR (2006) that $J(\varphi) = \max_{p\in\Delta} \left\{ \int \varphi dp - \widehat{J}^*(p) \right\} = \max_{p\in\Delta} \left\{ \int u(f)dp - c(p) \right\} = W(f)$, where $\varphi = u(f)$ and $c(p) = \widehat{J}^*(p)$. $c(p)$ is convex since $\widehat{J}^*(p)$ is convex.

$f \succsim g \iff J(u(f)) \geq J(u(g)) \iff W(f) \geq W(g)$.

Remark 2 *The u in our representation theorem is affine. To obtain a concave u as assumed in Bracha and Brown (2007), it may suffice to consider some combination of axioms in Savage (1954) and axioms in MMR (2006), – see Strzalecki (2007)– and replace $A.5$ by $\widehat{A5}$.*

■

3 A Dual Representation of ADM Games

Bracha and Brown (2007) define the potential function, $\Pi(x,p)$ for an ADM game as $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - c(p)$. $\overrightarrow{u}(x) = (u(x_1), u(x_2), ..., u(x_K))$ is the state -utility vector for a smooth, concave Bernoulli utility function $u : R_{++} \to R_{++}$, $c : (R^K_{++})^* \to R$ is the smooth, convex cost function of the emotional process and $(R^K)^*$ is the dual space of R^K. The proof of the axiomatic characterization of ADM preferences and variational preferences presented in section 2 suggests a dual characterization of ADM games in terms of $\overrightarrow{u}(x)$ and $c^*(y)$, the Legendre-Fenchel conjugate of $c(p)$.

This dual representation has important, empirical implications for rationalizing market data with ambiguity-seeking or ambiguity- averse preferences. Here is the intutition for the dual representation. We restrict attention to finite state spaces for ease of exposition.

Given the cost function $c(p)$ defined on the simplex $\Delta \subset (R^K_+)^*$, extend c to $\widehat{c}$ on all of $(R^K)^*$ by defining $\widehat{c}(p)$ to be $+\infty$ for all $p \notin \Delta$ and $\widehat{c}(p) = c(p)$ for all $p \in \Delta$. The Legendre-Fenchel conjugate of $\widehat{c}(p)$ denoted $\widehat{c}^*(y)$ is a convex function on R^K with values in the extended reals $\overline{R}$. $\widehat{c}^*(\overrightarrow{u}(x))$ can be interepted as the optimal value function defined by the convex, upper envelope of the family of value functions of the concave maximization problems of the rational process, parameterized by the best responses of the emotional process: – see the discussion of the envelope theorem in Dixit (1990) and his figure 5.1.

Following section 26 in Rockafellar (1970), a smooth, monotone, stictly convex cost function, $c(p)$, on $(R^K_{++})^*$ is said to be of Legendre type if $\nabla c(p)$ is a diffeomorphism from $(R^K_{++})^*$ onto R^K_{++}. It follows from Hadamard's theorem (1932) – see Theorem 6.2.8 in Krantz and Parks (2002) – that this condition is equivalent to the "essentially smooth" condition given on pg.258 in Rockafellar. If $c(p)$ is of Legendre type on $(R^K_{++})^*$, then the Legendre conjugate of $c(p)$, $c^*(y) = \langle y, (\nabla c)^{-1}(y)\rangle - c((\nabla c)^{-1}(y))$ is of Legendre type on R^K_{++}. Moreover, $\nabla c^* = (\nabla c)^{-1}$ and the Legendre conjugate of $c^*(y)$, $c^{**}(p) = c(p)$. An ADM game with potential function $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - c(p)$ is said to be of Legendre type if $c(p)$ is of Legendre type.

The market data $D = \{(q^i, x^i)\}_{i=1}^N$, where q^i are the market prices and x^i are the choices of the decision maker.

Theorem 3 *If $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - c(p)$ is the potential function for an ADM game of Legendre type, then $c^*(\overrightarrow{u(x)}) = \Pi(x, p(x))$ is the optimal value function for* *the concave maximization problems of the rational process, parameterized by the best responses of the emotional process, where* $p(x) = (\nabla c)^{-1}(\overrightarrow{u}(x))$ *is the best response of the emotional process, i.e.,* $\nabla c^*(\overrightarrow{u(x)}) = p(x)$.

Proof. $c^*(\overrightarrow{u}(x)) = \sup_{p \in (R^K_{++})^*} \{\langle \overrightarrow{u}(x), p\rangle - c(p)\} = \sup_{p \in \Delta} \Pi(x,p)$. The first order condition for this maximization problem is $\nabla c(p) = \overrightarrow{u}(x)$. If $p(x) = (\nabla c)^{-1}(\overrightarrow{u}(x))$, then it follows from the envelope theorem that the optimal value function $\Pi(x, p(x)) = c^*(\overrightarrow{u}(x)) = \langle \overrightarrow{u}(x), p(x)\rangle - c(p(x))$ is the upper envelope of the family of value functions for the concave maximization problems : $\sup_{\{x \in R^K_{++} :\, q \cdot x = I\}} \{\langle \overrightarrow{u}(x), p(x)\rangle - c(p(x))\}$, parameterized by $p(x)$ where $\nabla c^*(\overrightarrow{u(x)}) = p(x)$. $c^*(\overrightarrow{u}(x))$ is the sup of a family of functions affine in $\overrightarrow{u}(x)$, hence convex in $\overrightarrow{u}(x)$. ■

Theorem 4 *If D is rationalized by an ADM game of Legendre type with potential function* $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - c(p)$, *then there exists a function* $J(y)$ *of Legendre type on* R^K_{++}, *where* $J(\overrightarrow{u}(x)) = c^*(\overrightarrow{u}(x))$ *and* $\nabla_x J(u(x^k)) = \lambda_k q^k$ *for* $k = 1, 2, .., N$.

Proof. Let $J(y) = c^*(y) = \sup_{p \in (R^K_{++})^*} \{\langle y, p\rangle - c(p)\}$, where $y \in R^K_{++}$. $\nabla_x J(u(x^k)) = \nabla c^*(\overrightarrow{u}(x^k)) \circ \nabla \overrightarrow{u}(x^k) = p(x^k) \circ \nabla \overrightarrow{u}(x^k)$, where $z \circ w$ is the Hadamard product of $z \in (R^K)^*_{++}$and $w \in R^K_{++}$.Hence $\nabla_x J(u(x^k)) = \lambda_k q^k$ for $k = 1, 2, .., N$ is a pure strategy Nash equilibrium for each k, since x^k is the best response of the rational process to $p(x^k)$, the best response of the emotional process. ■

Theorem 5 *If there exists a function* $J(y)$ *of Legendre type on* R^K_{++} *and a smooth, monotone, concave Bernoulli utility* $u : R_{++} \to R_{++}$, *where* $\nabla_x J(\overrightarrow{u}(x^k)) = \lambda_k q^k$ *for* $k = 1, 2, .., N$, *then D is rationalized by the ADM game of Legendre type with potential function* $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - J^*(p)$.

Proof. This is the converse of Theorem 4, and follows from the theorem of the biconjugate, i.e., $J(y)$ is the Legendre conjugate of $J^*(p)$. ■

Theorem 6 *If an ADM game is of Legendre type with potential function* $\Pi(x,p) = \langle \overrightarrow{u}(x), p\rangle - c(p)$, *then* $(\overline{x}, \overline{p})$ *is a pure strategy Nash equilibrium of the ADM game if and only if* $c^*(\overrightarrow{u}(\overline{x})) + c(\overline{p}) = \langle \overrightarrow{u}(\overline{x}), \overline{p}\rangle$ *and* $\nabla_x c^*(\overrightarrow{u}(\overline{x})) = \lambda \overline{q}$, *where* $\overline{q}$ *defines the decision maker's budget set*

Proof. $c^*(\overrightarrow{u}(\overline{x})) + c(\overline{p}) = \langle \overrightarrow{u}(\overline{x}), \overline{p}\rangle$ if and only if $\nabla c(\overline{p}) = \overrightarrow{u}(\overline{x})$ and $\nabla c^*(\overrightarrow{u}(\overline{x})) = \overline{p}$ – see Proposition 11.3 in Rockafellar and Wets (1998). $\nabla c^*(\overrightarrow{u}(\overline{x})) = p$ is the best response of the emotional process to $\overrightarrow{u}(\overline{x})$, the decision of the rational process. $\nabla c(\overline{p}) = \overrightarrow{u}(\overline{x})$ is the best response of the rational process to $\overline{p}$, the decision of the emotional process, if $\nabla_x c^*(\overrightarrow{u}(\overline{x})) = \lambda \overline{q}$. ■

A similar analysis holds for variational preferences and concave cost functions of Legendre type.

Remark 7 *The notion of risk aversion is problematic in the case of ambiguity- seeking decision makers, since* $J(\overrightarrow{u}(x))$ *need not be concave in* x*, if* J *is convex in* $\overrightarrow{u}(x)$*.Consider the following examples of the composition of a convex function* J *and a concave function* u*, where* $J : R_{++} \to R_{++}$ *and* $u : R_{++} \to R_{++}$ *:(i) If* $J(y) = \exp((1/2)y)$ *and* $u(x) = \ln(x)$*, then* $\Phi(x) = J(u(x)) = x^{1/2}$ *(risk-averse) (ii)If* $J(y) = \exp(y)$ *and* $u(x) = \ln(x)$*, then* $\Phi(x) = J(u(x)) = x$ *(risk-neutral) (iii)If* $J(y) = \exp(2y)$ *and* $u(x) = \ln(x)$*, then* $\Phi(x) = J(u(x)) = x^2$ *(risk-seeking). Of course, if J is a monotone concave funtion as is the case with SEU and variational preferences, then* $\overrightarrow{u}(x)$ *risk-averse in* x *implies* $J(\overrightarrow{u}(x))$ *is risk-averse in* x*.*

References

[1]Aliprantis, C.D., and Border, K.C., (1999):**Infinite Dimensional Analysis**, Springer-Verlag, New York

[2]Anscombe, F., and Aumann, R. (1963):"A Definition of Subjective Probability," The Annals of Mathematical Statistics, 34, 199 – 205.

[3]Bracha, A., and Brown, D.J., (2007):"Affective Decision Making:A Behavioral Theory of Choice," CFDP 1633.

[4]Dixit, A.K., (1990):**Optimization in Economic Theory**, Oxford University Press, Oxford

[5]Dolecki, S., and Greco, G.H., (1995):"Niveloids," Topological Methods in Nonlinear Analysis, 6, 1 – 22

[6]Ellsberg, D., (1961):"Risk, Ambiguity, and The Savage Axioms," Quarterly Journal of Economics, 75, 643 – 669

[7]Epstein, L.G., Marinacci, M., and Seo, K., (2007):"Course Contingencies and Ambiguity,"Theoretical Economics, 2, 355 – 394

[8]Gilboa, I., and Schmeidler, D., (1989):"Maximin Expected Utilty Theory With a Non-unique Prior," Journal of Mathematical Economics, 18, 141 – 153.

[9]Hadamard, J., (1932):"Sur les Problems aux Derivees Partielles et Leur Signification Physique," Princeton University Bulletin, 49 – 52

[10]Hansen, L., and Sargent, T., (2000)"Wanting Robustness in Macroeconomics," Mimeo, University of Chicago and Stanford University

[11]Krantz , S.G., and Parks, H.R., (2002):**The Implicit Function Theorem:History, Theory and Applications**, Birkhauser, Basel Switzerland

[12]Maccheroni, F., Marinacci, M., and Rustichini, A., (2004):"Varionational Representation of Preferences Under Ambiguity," Working Paper

[13]Maccheroni, F., Marinacci, M., and Rustichini, A., (2006):"Ambiguity Aversion, Robustness,and The Variational Representation of Preferences," Econometrica, $74, 1447 - 1498$

[14]Rockafellar, R.,T., (1970):**Convex Analysis**, Princeton Univ.Press, Princeton

[15]Rockafellar, R.,T., and Wets, R., (1998):**Variational Analysis**, Springer-Verlag, New York

[16]Savage, L., J.,(1954):**Foundations of Statistics**, John Wiley and Sons, New York

[17]Schmeidler, D.,(1989):"Subjective Probability and Expected Utility Without Additivity," Econometrica, $57, 571 - 587$

[18]Strzalecki, T., (2007):"Axiomatic Foundations of Multiplier Preferences," Northwestern Working Paper

[19]Zălinescu, C., (2002):**Convex Analysis in General Vector Spaces**, World Scientific, New Jersey