

Global Games: Theory and Applications*

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Abstract

Global games are games of incomplete information whose type space is determined by the players each observing a noisy signal of the underlying state. With strategic complementarities, global games often have a unique, dominance solvable equilibrium, allowing analysis of a number of economic models of coordination failure. For symmetric binary action global games, equilibrium strategies in the limit (as noise becomes negligible) are simple to characterize in terms of ‘diffuse’ beliefs over the actions of others. We describe a number of economic applications that fall in this category. We also explore the distinctive roles of public and private information in this setting, review results for general global games, discuss the relationship between global games and a literature on higher order beliefs in game theory and describe the relationship to local interaction and dynamic games with payoff shocks.

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1. Introduction

Many economic problems are naturally modelled as a game of incomplete information, where a player's payoff depends on his own action, the actions of others, and some unknown economic fundamentals. For example, many accounts of currency attacks, bank runs and liquidity crises give a central role to players' uncertainty about other players' actions. Because other players' actions in such situations are motivated by their beliefs, the decision maker must take account of the beliefs held by other players. We know from the classic contribution of Harsanyi (1967-8) that rational behavior in such environments depends not only on economic agents' beliefs about economic fundamentals, but also depends on beliefs of higher order - that is, players' beliefs about other players' beliefs, players' beliefs about other players' beliefs about other players' beliefs, and so on. Indeed, Mertens and Zamir (1985) have shown how one can give a complete description of the "type" of a player in an incomplete information game in terms of a full hierarchy of beliefs at all levels.

In principle, optimal strategic behavior should be analyzed in the space of all possible infinite hierarchies of beliefs; however, such analysis is highly complex for players and analysts alike and is likely to prove intractable in general. It is therefore useful to identify strategic environments with incomplete information that are rich enough to capture the important role of higher order beliefs in economic settings, but simple enough to allow tractable analysis. Global games, first studied by Carlsson and van Damme (1993a), represent one such environment. Uncertain economic fundamentals are summarized by a state θ and each player observes a different signal of the state with a small amount of noise. Assuming that the noise technology is common knowledge among the players, each player's signal generates beliefs about fundamentals, beliefs about other players' beliefs about fundamentals, and so on. Our purpose in this paper is to describe how such models work, how global game reasoning can be applied to economic problems and how this analysis relates to more general analysis of higher order beliefs in strategic settings.

One theme that emerges is that taking higher order beliefs seriously does not require extremely sophisticated reasoning on the part of players. In section 2, we present a benchmark result for binary action continuum player games with strategic complementarities where each player has the same payoff function. In a global games setting, there is a unique equilibrium where each player chooses the action that is a best response to a uniform belief over the proportion of his opponents choosing each action. Thus, when faced with some information concerning the underlying state of the world, the prescription for each player is to hypothesize that the proportion of other players who will opt for a particular

action is a random variable that is uniformly distributed over the unit interval and choose the best action under these circumstances. We dub such beliefs (and the actions that they elicit) as being *Laplacian*, following Laplace's (1824) suggestion that one should apply a uniform prior to unknown events from the "principle of insufficient reason".

A striking feature of this conclusion is that it reconciles Harsanyi's fully rational view of optimal behavior in incomplete information settings with the dissenting view of Kadane and Larkey (1982) and others that rational behavior in games should imply only that each player chooses an optimal action in the light of his subjective beliefs about others' behavior, without deducing his subjective beliefs as part of the theory. If we let those subjective beliefs be the agnostic Laplacian prior, then there is no contradiction with Harsanyi's view that players should *deduce* rational beliefs about others' behavior in incomplete information settings.

The importance of such analysis is not that we have an adequate account of the subtle reasoning undertaken by the players in the game - it clearly does not do justice to the reasoning inherent in the Harsanyi program. Rather, its importance lies in the fact that we have access to a form of short-cut, or heuristic device that allows the economist to identify the actual *outcomes* in such games, and thereby open up the possibility of systematic analysis of economic questions which may otherwise appear to be intractable.

One instance of this can be found in the debate concerning self-fulfilling beliefs and multiple equilibria. If one set of beliefs motivate actions which bring about the state of affairs envisaged in those beliefs, while another set of self-fulfilling beliefs bring about quite different outcomes, then there is an apparent indeterminacy in the theory. In both cases, the beliefs are logically coherent, consistent with the known features of the economy, and are borne out by subsequent events. However, we do not have any guidance on which outcome will transpire without an account of how the initial beliefs are determined. We have argued elsewhere (Morris and Shin (2000)) that the apparent indeterminacy of beliefs in many models with multiple equilibria can be seen as the consequence of two modelling assumptions introduced to simplify the theory. First, the economic fundamentals are assumed to be common knowledge; and second, economic agents are assumed to be certain about others' behavior in equilibrium. Both assumptions are made for the sake of tractability, but they do much more besides. They allow agents' actions and beliefs to be perfectly co-ordinated in a way that invites multiplicity of equilibria. In contrast, global games allow theorists to model information in a more realistic way, and thereby escape this straightjacket. More importantly, through the heuristic device of Laplacian actions, global games allow modelers to pin down which set of self-fulfilling beliefs will prevail in equilibrium.

As well as any theoretical satisfaction at identifying a unique outcome in a game, there are more substantial issues at stake. Global games allow us to capture the idea that economic agents may be pushed into taking a particular action because of their belief that others are taking such actions. Thus, inefficient outcomes may be forced on the agents by the external circumstances even though they would all be better off if everyone refrained from such actions. Bank runs and financial crises are prime examples of such cases. We can draw the important distinction between whether there can be inefficient equilibrium outcomes and whether there is a unique outcome in equilibrium. Global games, therefore, are of more than purely theoretical interest. They allow more enlightened debate on substantial economic questions. In section 2.3, we discuss applications that model economic problems using global games.

Global games open up other interesting avenues of investigation. One of them is the importance of public information in contexts where there is an element of coordination between the players. There is plentiful anecdotal evidence from a variety of contexts that public information has an apparently disproportionate impact relative to private information. Financial markets apparently “overreact” to announcements from central bankers that merely state the obvious, or reaffirm widely known policy stances. But a closer look at this phenomenon with the benefit of the insights given by global games makes such instances less mysterious. If market participants are concerned about the reaction of other participants to the news, the public nature of the news conveys more information than simply the “face value” of the announcement. It conveys important strategic information on the likely beliefs of other market participants. In this case, the “overreaction” would be entirely rational and determined by the type of equilibrium logic inherent in a game of incomplete information. In section 3, these issues are developed more systematically.

Global games should be seen as a particular instance of equilibrium selection through perturbations. The set of perturbations is especially rich because it turns out that they allow for a rich structure of higher order beliefs. In section 4, we delve somewhat deeper into the properties of general global games - not merely those whose action sets are binary. We discuss how global games are related to other notions of equilibrium refinements and what is the nature of the perturbation implicit in global games. The general framework allows us to disentangle two properties of global games. The first property is that a unique outcome is selected in the game. A second, more subtle, question is how such a unique outcome depends on the underlying information structure and the noise in the players’ signals. Although in some cases the outcome is sensitive to the details of the information structure, there are cases where a particular outcome is selected and where this outcome turns out to be robust to the form of the noise

in the players' signals. The theory of “robustness to incomplete information” as developed by Kajii and Morris (1997) holds the key to this property. We also discuss a larger theoretical literature on higher order beliefs and the relation to global games.

In section 5, we show how recent work on local interaction games and dynamic games with payoff shocks employ a similar logic to global games in reaching unique predictions.

2. Symmetric Binary Action Global Games

2.1. Linear Example

Let us begin with the following example taken from Carlsson and van Damme (1993a). Two players are deciding whether to invest. There is a safe action (not invest); there is a risky action (invest) which gives a higher payoff if the other player invests. Payoffs are given by the following matrix.

	Invest	NotInvest	
Invest	θ, θ	$\theta - 1, 0$	(2.1)
NotInvest	$0, \theta - 1$	$0, 0$	

If there were complete information about θ , there would be three cases to consider.

- If $\theta > 1$, each player has a dominant strategy to invest.
- If $\theta \in [0, 1]$, there are two pure strategy Nash equilibria: both invest and both not invest.
- If $\theta < 0$, each player has a dominant strategy not to invest.

But there is incomplete information about θ . Player i observes a private signal $x_i = \theta + \varepsilon_i$. Each ε_i is independently normally distributed with mean 0 and standard deviation σ . We assume that θ is randomly drawn from the real line, with each realization equally likely. This implies that a player observing signal x considers θ to be distributed normally with mean x and standard deviation σ . This in turn implies that he thinks his opponent's signal x' is normally distributed with mean x and standard deviation $\sqrt{2}\sigma$. The assumption that θ is uniformly distributed on the real line is non-standard but presents no technical difficulties. Such “improper priors” (with an infinite mass) are well behaved as long as we are concerned only with conditional beliefs. See Hartigan (1983) for a discussion of improper priors. We will also see later that an improper prior can be seen

as a limiting case *either* as the prior distribution of θ becomes diffuse or as the standard deviation of the noise σ becomes small.

A strategy is a function specifying an action for each possible private signal; a natural kind of strategy we might consider is one where a player takes the risky action only if he observes a private signal above some cutoff point, k :

$$s(x) = \begin{cases} \text{Invest}, & \text{if } x > k \\ \text{NotInvest}, & \text{if } x \leq k \end{cases}$$

We will refer to this strategy as the switching strategy around k . Now suppose that a player observed signal x and thought that his opponent was following such a “switching” strategy with cutoff point k . His expectation of θ will be x . He will assign probability $\Phi\left(\frac{1}{\sqrt{2}\sigma}(k-x)\right)$ to his opponent observing a signal less than k (where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution). In particular, if he has observed a signal equal to the cutoff point of his opponent ($x = k$), he will assign probability $\frac{1}{2}$ to his opponent investing. Thus there will be an equilibrium where both players follow switching strategies with cutoff $\frac{1}{2}$.

In fact, a switching strategy with cutoff $\frac{1}{2}$ is the unique strategy surviving iterated deletion of strictly interim dominated strategies. To see why¹, first define $b(k)$ to be the unique value of x solving the equation

$$x - \Phi\left(\frac{k-x}{\sqrt{2}\sigma}\right) = 0. \tag{2.2}$$

There is a unique such value because the left hand side is strictly increasing in x and strictly decreasing in k . These properties also imply that $b(\cdot)$ is strictly increasing. So if your opponent is following a switching strategy with cutoff k , your best response is to follow a switching strategy with cutoff $b(k)$. We will argue that if a strategy s survives n rounds of iterated deletion of strictly dominated strategies, then

$$s(x) = \begin{cases} \text{Invest}, & \text{if } x > b^{n-1}(1) \\ \text{NotInvest}, & \text{if } x < b^{n-1}(0) \end{cases} \tag{2.3}$$

We argue the second clause by induction (the argument for the first clause is symmetric). The claim is true for $n = 1$, since as we noted above, **NotInvest** is a dominant strategy if the expected value of θ is less than 0. Now suppose the claim is true for arbitrary n . If a player knew that his opponent would choose action **NotInvest** if he had observed a signal less than $b^{n-1}(1)$, his best response would

¹An alternative argument follows Milgrom and Roberts (1990): if a symmetric game with strategic complementarities has a unique symmetric Nash equilibrium, then the strategy played in that unique Nash equilibrium is also the unique strategy surviving iterated deletion of strictly dominated strategies.

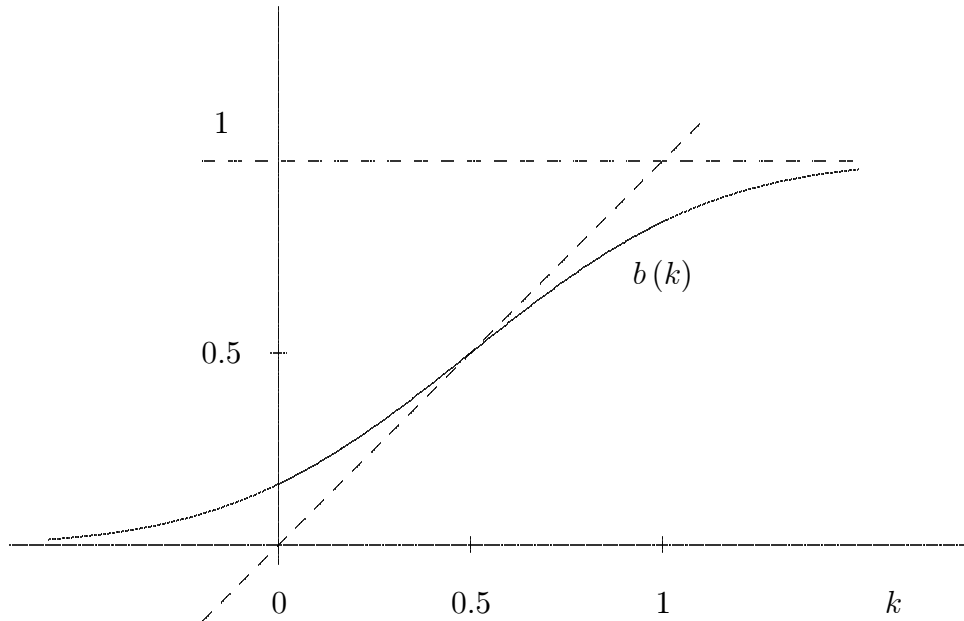


Figure 2.1: Function $b(k)$

always be to choose action **NotInvest** if his signal was less than $b(b^{n-1}(1))$. Since $b(\cdot)$ is strictly increasing and has a unique fixed point at $\frac{1}{2}$, $b^n(0)$ and $b^n(1)$ both tend to $\frac{1}{2}$ as $n \rightarrow \infty$.

The unique equilibrium has both players investing only if they observe a signal greater than $\frac{1}{2}$. In the underlying symmetric payoff complete information game, investing is a risk dominant action (Harsanyi and Selten (1988)) exactly if $\theta \geq \frac{1}{2}$; not investing is a risk dominant action exactly if $\theta \leq \frac{1}{2}$. The striking feature of this result is that no matter how small σ is, players' behavior is influenced by the existence of the ex ante possibility that their opponent has a dominant strategy to choose each action.² The probability that either individual invests is

$$\Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right);$$

Conditional on θ , their investment decisions are independent.

The above example and analysis are due to Carlsson and van Damme (1993a).

²Thus a "grain of doubt" concerning the opponent's behavior has large consequences. This element has been linked by van Damme (1997) to the classic analysis of surprise attacks of Schelling (1960), chapter 9.

There is a many-players analogue of this game, whose solution is no more difficult to arrive at. A continuum of players are deciding whether to invest. The payoff to not investing is 0. The payoff to investing is $\theta - 1 + l$, where l is the proportion of other players choosing to invest. The information structure is as before, with each player i observing a private signal $x_i = \theta + \varepsilon_i$, where each ε_i is independently normally distributed with mean 0 and standard deviation σ . In this case also, the unique strategy surviving iterated deletion of strictly dominated strategies has each player investing if they observe a signal above $\frac{1}{2}$, and not investing if they observe a signal below $\frac{1}{2}$. We will briefly sketch why this is the case.

Consider a player who has observed signal x and thinks that *all his opponents* are following the “switching” strategy with cutoff point k . As before, his expectation of θ will be x . As before, he will assign probability $\Phi\left(\frac{k-x}{\sqrt{2}\sigma}\right)$ to any given opponent observing a signal less than k . But since the realization of the signals are independent conditional on θ , his expectation of the proportion of players who observe a signal less than k will be exactly equal to the probability he assigns to any one opponent observing a signal less than k . Thus his expected payoff to investing will be $x - \Phi\left(\frac{k-x}{\sqrt{2}\sigma}\right)$, as before, and all the previous arguments go through.

The argument above shows the importance of keeping track of the layers of beliefs across players, and as such may seem rather daunting from the point of view of an individual player. However, the equilibrium outcome is also consistent with a procedure which places far less demands on the capacity of the players, and which seem to be far removed from equilibrium of any kind. This procedure has the following three steps.

- Estimate θ from the signal x .
- Postulate that l is distributed uniformly on the unit interval $[0, 1]$
- Take the optimal action.

Since the expectation of θ conditional on x is simply x itself, the expected payoff to investing if l is uniformly distributed is $x - \frac{1}{2}$ while the expected payoff to not investing is zero. Thus, a player following this procedure will choose to invest or not depending on whether x is greater or smaller than $\frac{1}{2}$, which is identical to the unique equilibrium strategy outlined above. The belief summarized in the second bullet point is *Laplacian* in the sense introduced in the introductory section. It represents a “diffuse” or “agnostic” view on the actions of other players in the game. We see that an apparently naive and simplistic strategy coincides with the equilibrium strategy. This is not an accident. There are good reasons why the Laplacian action is the correct one in this game, and why it turns out to be

an approximately optimal action in many binary action global games. The key to understanding this feature is to consider the following question asked by a player in this game.

“My signal has realization x . What is the probability that proportion less than z of my opponents have a signal higher than mine?”

The answer to this question would be especially important if everyone is using the switching strategy around x , since the proportion of players who invest is equal to the proportion whose signal is above x . If the true state is θ , the proportion of players who receive a signal higher than x is given by $1 - \Phi\left(\frac{x-\theta}{\sigma}\right)$. So, this proportion is less than z if the state θ is such that $1 - \Phi\left(\frac{x-\theta}{\sigma}\right) \leq z$. That is, when

$$\theta \leq x - \sigma\Phi^{-1}(1 - z) \tag{2.4}$$

The probability of this event conditional on x is

$$\Phi\left(\frac{x - \sigma\Phi^{-1}(1 - z) - x}{\sigma}\right) = z$$

In other words, the cumulative distribution function of z is the identity function, implying that the density of z is uniform over the unit interval. If x is to serve as the switching point of an equilibrium switching strategy, a player must be indifferent between choosing to invest and not to invest given that the proportion who invest is uniformly distributed on $[0, 1]$.

More importantly, even away from the switching point, the optimal action motivated by this belief coincides with the equilibrium action, even though the (Laplacian) belief may not be correct. Away from the switching point, the density of the random variable representing the proportion of players who invest will not be uniform. However, as long as the payoff advantage to investing is increasing in θ , the Laplacian action coincides with the equilibrium action. Thus, the apparently naive procedure outlined by the three bullet points gives the correct prediction as to what the equilibrium action will be. In the next section, we will show that the lessons drawn from this simple example extend to cover a wide class of binary action global games.

We will focus on the continuum player case in most of this paper. However, as suggested by this example, the qualitative analysis is very similar independent of the number of players. In particular, the analysis of the continuum player game with linear payoffs applies equally well to any finite number of players (where each player observes a signal with an independent normal noise term). Independent of the number of players, the cutoff signal in the unique equilibrium is $\frac{1}{2}$. However,

a distinctive implication of the infinite player case is that the outcome is a deterministic function of the realized state. In particular, once we know the realization of θ , we can calculate exactly the proportion of players who will invest. It is

$$\widehat{\xi}(\theta) = 1 - \Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right).$$

With a finite number of players (I), we write $\xi_{\lambda,I}(\theta)$ for the probability that at least proportion λ out of the I players invest when the realized state is θ :

$$\xi_{\lambda,I}(\theta) = \sum_{n \geq \lambda I} \binom{I}{n} \left[\Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right) \right]^{I-n} \left[1 - \Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right) \right]^n.$$

Observe, however, that the many finite player case converges naturally to the continuum model: by the law of large numbers, as $I \rightarrow \infty$,

$$\begin{aligned} \xi_{\lambda,I}(\theta) &\rightarrow 1 \text{ if } \lambda < \widehat{\xi}(\theta) \\ \text{and } \xi_{\lambda,I}(\theta) &\rightarrow 0 \text{ if } \lambda > \widehat{\xi}(\theta). \end{aligned}$$

2.2. Symmetric Binary Action Global Games: A General Approach

Let us now take one step in making the argument more general. We deal first with the case where there is a uniform prior on the initial state and each player's signal is a sufficient statistic for how much they care about the state (we call this the private values case). In this case, the analysis is especially clean and it is possible to prove a uniqueness result and characterize the unique equilibrium independent of both the structure and size of the noise in players' signals. We then show that the analysis can be extended to deal with general priors and payoffs that depend on the realized state.

2.2.1. Continuum Players, Uniform Prior and Private Values

There is a continuum of players. Each player has to choose an action $a \in \{0, 1\}$. All players have the same payoff function, $u : \{0, 1\} \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, where $u(a, l, x)$ is a player's payoff if he chooses action a , proportion l of his opponents choose action 1, and his "private signal" is x . Thus we assume that his payoff is independent of which of his opponents choose action 1. To analyze best responses, it is enough to know the payoff gain from choosing one action rather than the other. Thus the utility function is parameterized by a function $\pi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ with

$$\pi(l, x) \equiv u(1, l, x) - u(0, l, x).$$

Formally, we say that an action is the *Laplacian* action if it is a best response to a uniform prior over the opponents' choice of action. Thus action 1 is the *Laplacian* action at x if

$$\int_{l=0}^1 u(1, l, x) dl > \int_{l=0}^1 u(0, l, x) dl,$$

or, equivalently,

$$\int_{l=0}^1 \pi(l, x) dl > 0;$$

action 0 is the *Laplacian* action at x if

$$\int_{l=0}^1 \pi(l, x) dl < 0.$$

Generically, a continuum player, symmetric payoff, two action game will have exactly one Laplacian action.

A state $\theta \in \mathbb{R}$ is drawn according to the (improper) uniform density on the real line. Player i observes a private signal $x_i = \theta + \sigma \varepsilon_i$, where $\sigma > 0$. The noise terms ε_i are i.i.d. with continuous density $f(\cdot)$, with support on the real line.³ We note that this density need not be symmetric around the mean, nor even have zero mean. The uniform prior on the real line is “improper,” i.e., has infinite probability mass, but the conditional probabilities are well defined: a player observing signal x_i puts density $\frac{1}{\sigma} f\left(\frac{x_i - \theta}{\sigma}\right)$ on state θ (see Hartigan (1983)). The example of the previous section fits this setting where $f(\cdot)$ is the standard normal distribution and $\pi(l, x) = x + l - 1$.

We will initially impose five properties on the payoffs:

A1: **Strategic Complementarities:** $\pi(l, \theta)$ is non-decreasing in l .

A2: **State Monotonicity:** $\pi(l, \theta)$ is non-decreasing in θ .

A3: **Continuum Player Single Crossing:** There exists a unique θ^* solving $\int_{l=0}^1 \pi(l, \theta^*) dl = 0$.

A4: **Limit Dominance:** There exist $\underline{\theta} \in \mathbb{R}$ and $\bar{\theta} \in \mathbb{R}$ such that [1] $\pi(l, x) < 0$ for all $l \in [0, 1]$ and $x \leq \underline{\theta}$; and [2] $\pi(l, x) > 0$ for all $l \in [0, 1]$ and $x \geq \bar{\theta}$.

³With small changes in terminology, the argument will extend to the case where $f(\cdot)$ has support on some bounded interval of the real line.

A5: **Continuity:** $\int_{l=0}^1 g(l) \pi(l, x) dl$ is continuous with respect to signal x and density g .

Condition A1 implies that the incentive to choose action 1 is increasing in the proportion of other players' actions who use action 1. Thus there are *strategic complementarities* between players' actions (Bulow, Geanakoplos and Klemperer (1985)). Condition A2 implies monotonicity of the Laplacian action with respect to the private signal: as the private signal increases, the Laplacian action is (weakly) higher. Condition A3 represents a slight strengthening of A2 to ensure that there is at most one crossing. Condition A4 requires that action 0 is a dominant strategy for sufficiently low signals and action 1 is a dominant strategy for sufficiently high signals. In condition A5, continuity in g is with respect to the weak topology. Note that this condition allows for some discontinuities in payoffs. For example,

$$\pi(l, x) = \begin{cases} 0, & \text{if } l \leq x \\ 1, & \text{if } l > x \end{cases}$$

satisfies A5 as for any given x , it is discontinuous at only one value of l .

We denote by $G^*(\sigma)$ this incomplete information game - with the uniform prior and satisfying A1 through A5. A strategy for a player in the incomplete information game is a function $s : \mathbb{R} \rightarrow \{0, 1\}$, where $s(x)$ is the action chosen if a player observes signal x . We will be interested in strategy profiles, $\mathbf{s} = (s_i)_{i \in [0,1]}$, that form a Bayesian Nash equilibrium of $G^*(\sigma)$. We will show not merely that there is a unique Bayesian Nash equilibrium of the game, but that a unique strategy profile survives iterated deletion of strictly (interim) dominated strategies.

Proposition 2.1. *Let θ^* be defined as in (A3). The essentially unique strategy surviving iterated deletion of strictly dominated strategies in $G^*(\sigma)$ satisfies $s(x) = 0$ for all $x < \theta^*$ and $s(x) = 1$ for all $x > \theta^*$.*

The “essential” qualification arises because either action may be played if the private signal is exactly equal to θ^* . The key idea of the proof is that with a uniform prior on θ , observing x_i gives no information to a player on his ranking within the population of signals. Thus he will have a uniform prior belief over the proportion of players who will observe higher signals.

PROOF. Write $\pi_\sigma^*(x, k)$ for the expected payoff gain to choosing action 1 for a player who has observed a signal x and knows that all other players will choose

action 0 if they observe signals less than k :

$$\pi_\sigma^*(x, k) \equiv \int_{\theta=-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \pi\left(1 - F\left(\frac{k-\theta}{\sigma}\right), x\right) d\theta. \quad (2.5)$$

First observe that $\pi_\sigma^*(x, k)$ is continuous in x and k , increasing in x and decreasing in k , $\pi_\sigma^*(x, k) < 0$ if $x \leq \underline{\theta}$ and $\pi_\sigma^*(x, k) > 0$ if $x \geq \bar{\theta}$. We will argue by induction that a strategy survives n rounds of iterated deletion of strictly interim dominated strategies if and only if

$$s(x) = \begin{cases} 0, & \text{if } x < \underline{\xi}_n \\ 1, & \text{if } x > \bar{\xi}_n \end{cases}$$

where $\underline{\xi}_0 = -\infty$ and $\bar{\xi}_0 = +\infty$, and $\underline{\xi}_n$ and $\bar{\xi}_n$ are defined inductively by

$$\begin{aligned} \underline{\xi}_{n+1} &= \min \left\{ x : \pi_\sigma^*(x, \underline{\xi}_n) = 0 \right\} \\ \text{and } \bar{\xi}_{n+1} &= \max \left\{ x : \pi_\sigma^*(x, \bar{\xi}_n) = 0 \right\}. \end{aligned}$$

Suppose the claim were true for n . By strategic complementarities, if action 1 were ever to be a best response to a strategy surviving n rounds, it must be a best response to the switching strategy with cutoff $\underline{\xi}_n$; $\underline{\xi}_{n+1}$ is defined to be the lowest signal where this occurs. Similarly, if action 0 were ever to be a best response to a strategy surviving n rounds, it must be a best response to the switching strategy with cutoff $\bar{\xi}_n$; $\bar{\xi}_{n+1}$ is defined to be the highest signal where this occurs.

Now note that $\underline{\xi}_n$ and $\bar{\xi}_n$ are increasing and decreasing sequences, respectively, since $\underline{\xi}_0 = -\infty < \underline{\theta} < \underline{\xi}_1$, $\bar{\xi}_0 = \infty > \bar{\theta} > \bar{\xi}_1$, and $\pi_\sigma^*(x, k)$ is increasing in x and decreasing in k . Thus $\underline{\xi}_n \rightarrow \underline{\xi}$ and $\bar{\xi}_n \rightarrow \bar{\xi}$ as $n \rightarrow \infty$. The continuity of π_σ^* and the construction of $\underline{\xi}$ and $\bar{\xi}$ imply that we must have $\pi_\sigma^*(\underline{\xi}, \underline{\xi}) = 0$ and $\pi_\sigma^*(\bar{\xi}, \bar{\xi}) = 0$. Thus the last step of our proof is to show that θ^* is the unique solution to the equation $\pi_\sigma^*(x, x) = 0$.

To see this last step, write $\Psi_\sigma^*(l; x, k)$ for the probability that a player assigns to proportion less than l of the other players observing a signal greater than k , if he has observed signal x . Observe that if the true state is θ , the proportion of players observing a signal greater than k is $1 - F\left(\frac{k-\theta}{\sigma}\right)$. This proportion is less than l if $\theta \leq k - \sigma F^{-1}(1-l)$. So

$$\begin{aligned} \Psi_\sigma^*(l; x, k) &= \int_{\theta=-\infty}^{k - \sigma F^{-1}(1-l)} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) d\theta \\ &= \int_{z=\frac{x-k}{\sigma} + F^{-1}(1-l)}^{\infty} f(z) dz, \text{ changing variables to } z = \frac{x-\theta}{\sigma} \end{aligned}$$

$$= 1 - F\left(\frac{x - k}{\sigma} + F^{-1}(1 - l)\right) \quad (2.6)$$

Also observe that if $x = k$, then $\Psi_\sigma^*(\cdot; x, k)$ is the identity function (i.e., $\Psi_\sigma^*(l; x, k) = l$), so it is the cumulative distribution function of the uniform density. Thus

$$\pi_\sigma^*(x, x) = \int_{l=0}^1 \pi(l, x) dl.$$

Now by A3, $\pi_\sigma^*(x, x) = 0$ implies $x = \theta^*$. ■

2.2.2. Continuum Players, General Prior and Common Values

Now suppose instead that θ is drawn from a continuously differentiable strictly positive density $p(\cdot)$ on the real line and that a player's utility depend on the realized state θ , not his signal of θ . Thus $u(a, l, \theta)$ is his payoff if he chooses action a , proportion l of his opponents choose action 1, *and the state is θ* , and as before $\pi(l, \theta) \equiv u(1, l, \theta) - u(0, l, \theta)$. We must also impose two extra technical assumptions.

A4*: **Uniform Limit Dominance:** There exist $\underline{\theta} \in \mathbb{R}$, $\bar{\theta} \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_{++}$ such that [1] $\pi(l, \theta) \leq -\varepsilon$ for all $l \in [0, 1]$ and $\theta \leq \underline{\theta}$; and [2] there exists $\bar{\theta}$ such that $\pi(l, \theta) > \varepsilon$ for all $l \in [0, 1]$ and $\theta \geq \bar{\theta}$.

Property A4* strengthens property A4 by requiring that the payoff gain to choosing action 0 is *uniformly* positive for sufficiently low values of θ , and the payoff gain to choosing action 1 is *uniformly* positive for sufficiently high values of θ .

A6: **Finite Expectations of Signals:** $\int_{z=-\infty}^{\infty} z f(z) dz$ is well defined.

Property A6 requires that the distribution of noise is integrable.

We will denote by $G(\sigma)$ this incomplete information game - with prior $p(\cdot)$ and satisfying A1, A2, A3, A4*, A5 and A6.

Proposition 2.2. *Let θ^* be defined as in (A3). For any $\delta > 0$, there exists $\bar{\sigma} > 0$ such that for all $\sigma \leq \bar{\sigma}$, if strategy s survives iterated deletion of strictly dominated strategies in the game $G(\sigma)$, then $s(x) = 0$ for all $x \leq \theta^* - \delta$ and $s(x) = 1$ for all $x \geq \theta^* + \delta$.*

We will sketch here why this general prior, common values, game $G(\sigma)$ becomes like the uniform prior, private values, game $G^*(\sigma)$ as σ becomes small. A

more formal proof is relegated to the appendix. Consider $\Psi_\sigma(l; x, k)$, the probability that a player assigns to proportion less than or equal to l of the other players observing a signal greater than or equal to k , if he has observed signal x :

$$\begin{aligned} \Psi_\sigma(l; x, k) &= \frac{\int_{\theta=-\infty}^{k-\sigma F^{-1}(1-l)} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta}{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta} \\ &= \frac{\int_{z=\frac{x-k}{\sigma}+F^{-1}(1-l)}^{\infty} p(x-\sigma z) f(z) dz}{\int_{z=-\infty}^{\infty} p(x-\sigma z) f(z) dz}, \text{ changing variables to } z = \frac{x-\theta}{\sigma}. \end{aligned}$$

For small σ , the shape of the prior will not matter and the posterior beliefs over l will depend only on $\frac{x-k}{\sigma}$, the normalized difference between the x and k . Formally, setting $\kappa = -\frac{x-k}{\sigma}$, we have

$$\Psi_\sigma^*(l; x, x + \sigma\kappa) = \frac{\int_{z=\kappa+F^{-1}(1-l)}^{\infty} p(x-\sigma z) f(z) dz}{\int_{z=-\infty}^{\infty} p(x-\sigma z) f(z) dz},$$

so that as $\sigma \rightarrow 0$,

$$\begin{aligned} \Psi_\sigma^*(l; x, x + \sigma\kappa) &\rightarrow \int_{z=\kappa+F^{-1}(1-l)}^{\infty} f(z) dz \\ &= 1 - F(\kappa + F^{-1}(1-l)). \end{aligned} \tag{2.7}$$

In other words, for small σ , posterior beliefs concerning the proportion of opponents choosing each action are almost the same as under a uniform prior. The formal proof of proposition 2.2 presented in the appendix consists of showing, first, that convergence of posterior beliefs described above is uniform; and, second, that the small amount of uncertainty about payoffs in the common value case does not affect the analysis sufficiently to matter.

The proofs of propositions 2.1 and 2.2 generalize arguments presented in Morris and Shin (1998). The technique of analyzing the uniform prior private values game, and then showing continuity with respect to the general prior, common values game, follows Frankel, Morris and Pauzner (2000) (this paper is discussed further in section 4.1 below). Carlsson and van Damme (1993b) showed a version

of the uniform prior result (proposition 2.1) in the finite player case (see also Kim (1996)). We briefly discuss the relation to the finite player case in appendix B.

How does proposition 2.2 make use of the underlying assumptions? We used strategic complementarities to show the existence of a largest and smallest equilibrium, each of which was itself an equilibrium. The strategies played in these largest and smallest equilibria are monotonic (i.e., switching strategies). This step of the argument follows Milgrom and Roberts (1990) and Vives (1990). Once we know that we are interested in monotonic strategies, the very weak single crossing property A3 is sufficient to ensure the equivalence of the largest and smallest equilibria and thus uniqueness of equilibrium.

It is interesting to know whether one can dispense with the full force of the strategic complementarities assumption, especially if one was interested in showing uniqueness within the class of monotonic equilibria (without ruling out the existence of non-monotonic equilibria). Clearly, the very weak property A3 is sufficient to show that there is at most one unique symmetric switching (monotonic) equilibrium. But in the absence of strategic complementarities, some additional restrictions are required to ensure the existence of monotonic equilibria. In particular, one can show that if strategic complementarities are not satisfied at the cutoff point θ^* (i.e., $\pi(l, \theta^*)$ is decreasing in l over some range), then one can find a problematic noise distribution $f(\cdot)$ such that the symmetric switching strategy profile with cutoff point θ^* is *not* an equilibrium.

However, there are two ways one could hope to weaken the strategic complementarities assumption. First, one could restrict the class of noise distributions (recall that our results so far have placed no restrictions on $f(\cdot)$). In the next section, we will discuss an application of Goldstein and Pauzner (2000a) where there is a unique symmetric switching equilibrium if the noise distribution is uniform, but where monotonic equilibria will fail to exist for sufficiently eccentric noise distributions. Athey (2000) has proved existence of monotonic pure strategy equilibria in a general class of incomplete information games, without using strategic complementarities. It might be possible to use sufficient conditions from that paper to find joint restrictions on payoffs and noise distributions that are sufficient for the existence of monotonic equilibria. Uniqueness of monotonic equilibria then easily follows from A3 and the uniform prior / small noise properties of global games.

Second, even allowing all noise distributions, the full force of the strategic complementarities assumption is not required away from θ^* . We will report one weakening that is sufficient. Let $g(\cdot)$ and $h(\cdot)$ be densities on the interval $[0, 1]$. We say that g *stochastically dominates* h ($g \succeq h$) if $\int_{z=0}^l g(z) dz \leq \int_{z=0}^l h(z) dz$ for all $l \in [0, 1]$. We write $\bar{g}(\cdot)$ for the uniform density on $[0, 1]$, i.e., $\bar{g}(l) = 1$ for all

$l \in [0, 1]$. Now consider

A7: There exists θ^* which solves $\int_{l=0}^1 \pi(l, \theta^*) dl = 0$ such that [1] $\int_{l=0}^1 g(l) \pi(l, x) dl \geq 0$ for all $x \geq \theta^*$ and $g \succeq \bar{g}$, with strict inequality if $x > \theta^*$; and [2] $\int_{l=0}^1 g(l) \pi(l, x) dl \leq 0$ for all $x \leq \theta^*$ and $g \preceq \bar{g}$, with strict inequality if $x < \theta^*$.

We can replace A1-A3 with A7 in all our arguments and results above, with no change. Observe that A1-A3 straightforwardly imply A7. Also, observe that A7 implies that $\pi(l, \theta^*)$ be non-decreasing in l (suppose that $l > l'$ and $\pi(l, \theta^*) < \pi(l', \theta^*)$; now start with the uniform distribution \bar{g} and shift mass from l' to l). But A7 allows some failure of A1 (strategic complementarities) away from θ^* .

Propositions 2.1 and 2.2 deliver strong negative conclusions about the efficiency of non-cooperative outcomes in global games. In the limit, all players will be choosing action 1 when the state is θ if $\int_{l=0}^1 \pi(l, \theta) dl > 0$. However, it is *efficient* to choose action 1 at state θ if $u(1, 1, \theta) > u(0, 0, \theta)$. These conditions will not coincide in general. For example, in the investment example we had $u(1, l, \theta) = \theta + l - 1$, $u(0, l, \theta) = 0$ and thus $\pi(l, \theta) = \theta + l - 1$. So in the limiting equilibrium, both players will be investing if the state θ is at least $\frac{1}{2}$, while it is efficient for them to be investing if the state is at least 0.

The analysis of the unique non-cooperative equilibrium serves as a benchmark describing what will happen in the absence of other considerations. In practise, repeated play or other institutions will often allow players to do better. We will briefly consider what happens in the game if players were allowed to make cheap talk statements about the signals that they have observed in the investment example (for this exercise, it is most natural to consider a finite player case, so we consider the two player case). Our arguments here follow Baliga and Morris (2000). The investment example as formulated has a non-generic feature, which is that if a player plans not to invest, he is exactly indifferent about which action his opponent will take. To make the problem more interesting, let us perturb the payoffs to remove this tie:

	Invest	NotInvest
Invest	$\theta + \delta, \theta + \delta$	$\theta - 1, \delta$
NotInvest	$\delta, \theta - 1$	$0, 0$

Thus each player receives a small payoff δ (which may be positive or negative) if the other player invests, independent of his own action. This change does not influence each player's best responses, and the analysis of this game in the

absence of cheap talk is unchanged by the payoff change. But observe that if $\delta \leq 0$, there is an equilibrium of the game with cheap talk, where each player truthfully announces his signal, and invests if the (common) expectation of θ conditional on both announcements is greater than $-\delta$ (this gives the efficient outcome). On the other hand, if $\delta > 0$, then each player would like to convince the other to invest even he does not plan to do so. In this case, there cannot be a truth-telling equilibrium where the efficient equilibrium is achieved, although it is possible that there are equilibria with some partially revealing cheap talk that improve on the no cheap talk outcome.

2.3. Applications

We now turn to applications of these results and describe models of pricing debt (Morris and Shin (1999b)), currency crises (Morris and Shin (1998)) and bank runs (Goldstein and Pauzner (2000a)).⁴ Each of these papers makes specific assumptions about the distribution of payoffs and signals. But if one is only interested in analyzing the limiting behavior as noise about θ becomes small, the results of the previous section imply that we can identify the limiting behavior independently of the prior beliefs and the shape of the noise.⁵ In each example, we describe one comparative static exercise changing the payoffs of the game, illustrating how changing payoffs has a direct effect on outcomes and an indirect, strategic, effect via the impact on the cutoff point of the unique equilibrium. We emphasize that it is also interesting in the applications to study behavior away from the limit; indeed, the focus of the analysis in Morris and Shin (1999b) is on comparative statics away from the limit. More assumptions on the shape of the prior and noise are required in this case. We study behavior away from the limit in section 3.

2.3.1. Pricing Debt

In Morris and Shin (1999b), we consider a simple model of debt pricing. In period 1, a continuum of investors hold collateralized debt that will pay 1 in period 2 if it is rolled over and if an underlying investment project is successful; the debt will pay 0 in period 2 if the project is not successful. If an investor does not roll

⁴Other applications include Karp's (2000) noisy version of Krugman's (1991) multiple equilibrium model of sectoral shifts; Scaramozzino and Vulkan's (1999) noisy model of Schliefer's (1986) multiple equilibrium model of implementation cycles; and Dönges and Heinemann's (2000) model of competition between dealer markets and crossing networks in financial markets.

⁵The model in Goldstein and Pauzner (2000) fails the strategic complementarity property (A1) of the previous section, but they are nonetheless able to prove the uniqueness of a symmetric switching equilibrium, exploiting their assumption that noise terms are distributed uniformly. However, as we noted in the previous section, whenever there is a unique equilibrium, it must satisfy the Laplacian characterization with the cutoff point θ^* defined as in A3.

over his debt, he receives the value of the collateral, $\kappa \in (0, 1)$. The success of the project depends on the proportion of investors who do not rollover and the state of the economy, θ . Specifically, the project is successful if the proportion of investors not rolling over is less than $\frac{\theta}{z}$. Writing 1 for the action “rollover” and 0 for the action “do not rollover,” payoffs can be described as follows:

$$\begin{aligned} u(1, l, \theta) &= \begin{cases} 1, & \text{if } z(1-l) \leq \theta \\ 0, & \text{if } z(1-l) > \theta \end{cases} \\ u(0, l, \theta) &= \kappa \end{aligned}$$

So

$$\begin{aligned} \pi(l, \theta) &\equiv u(1, l, \theta) - u(0, l, \theta) \\ &= \begin{cases} 1 - \kappa, & \text{if } z(1-l) \leq \theta \\ -\kappa, & \text{if } z(1-l) > \theta \end{cases} \end{aligned}$$

Now

$$\int_{l=0}^1 \pi(l, \theta) dl = \begin{cases} -\kappa, & \text{if } \theta \leq 0 \\ \frac{\theta}{z} - \kappa, & \text{if } 0 \leq \theta \leq z \\ 1 - \kappa, & \text{if } z \leq \theta \end{cases}$$

Thus $\theta^* = z\kappa$. In other words, if private information about θ among the investors is sufficiently accurate, the project will collapse exactly if $\theta \leq z\kappa$. We can now ask how debt would be priced ex ante in this model (before anyone observed private signals about θ). Recalling that $p(\cdot)$ is the density of the prior on θ , and writing $P(\cdot)$ for the corresponding c.d.f., the value of the collateralized debt will be

$$\begin{aligned} V(\kappa) &\equiv \kappa P(z\kappa) + 1 - P(z\kappa) \\ &= 1 - (1 - \kappa) P(z\kappa) \end{aligned}$$

and

$$\frac{dV}{d\kappa} = P(z\kappa) - z(1 - \kappa)p(z\kappa)$$

Thus increasing the value of collateral has two effects: first, it increases the value of debt in the event of default (the direct effect). But second, it increases the range of θ at which default occurs (the strategic effect). For small κ , the strategic effect outweighs the direct effect, while for large κ , the direct effect outweighs the strategic effect. Figure 2.2 plots $V(\cdot)$ for the case where $z = 10$ and $p(\cdot)$ is the standard normal density.

Morris and Shin (1999b) study the model away from the limit and argue that taking the strategic, or liquidity, effect into account in debt pricing can

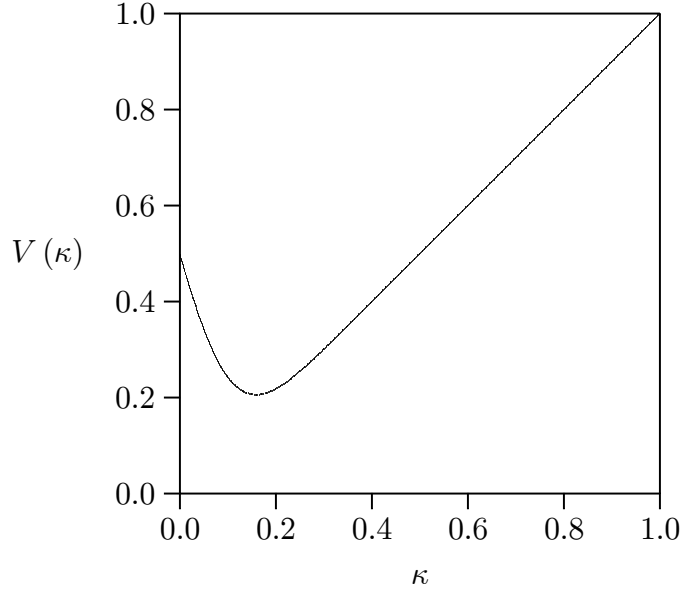


Figure 2.2: Function $V(\kappa)$

help explain anomalies in empirical implementation of the standard debt pricing theory of Merton (1974). Brunner and Krahen (2000) present evidence of the importance of debtor coordination in distressed lending relationships in Germany. See also Chui, Gai and Haldane (2000) and Hubert and Schäfer (2000).

2.3.2. Currency Crises

In Morris and Shin (1998), a continuum of speculators must decide whether to attack a fixed exchange rate regime by selling the currency short. Each speculator may only short a unit amount. The current value of the currency is e^* ; if the monetary authority does not defend the currency, the currency will float to the shadow rate $\zeta(\theta)$, where θ is the state of fundamentals. There is a fixed transaction cost t of attacking. This can be interpreted as an actual transaction cost or as the interest rate differential between currencies. The monetary authority defends the currency if the cost of doing so is not too large. Assuming that the costs of defending the currency are increasing in the proportion of speculators who attack and decreasing in the state of fundamentals, there will be some critical proportion of speculators, $a(\theta)$, increasing in θ , who must attack in order for a devaluation to occur. Thus writing 1 for the action “not attack” and 0 for the action “attack,” payoffs can be described as follows:

$$u(1, l, \theta) = 0$$

$$u(0, l, \theta) = \begin{cases} e^* - \zeta(\theta) - t, & \text{if } l \leq 1 - a(\theta) \\ -t, & \text{if } l > 1 - a(\theta) \end{cases}$$

where $\zeta(\cdot)$ and $a(\cdot)$ are increasing functions, with $\zeta(\theta) \leq e^* - t$ for all θ . Now

$$\pi(l, \theta) = \begin{cases} \zeta(\theta) + t - e^*, & \text{if } l \leq 1 - a(\theta) \\ t, & \text{if } l > 1 - a(\theta) \end{cases}$$

If θ were common knowledge, there would be three ranges of parameters. If $\theta < a^{-1}(0)$, each player has a dominant strategy to attack. If $a^{-1}(0) \leq \theta \leq a^{-1}(1)$, then there is an equilibrium where all speculators attack and another equilibrium where all speculators do not attack. If $\theta > a^{-1}(1)$, each player has a dominant strategy to attack. This tripartite division of fundamentals arises in a range of models in the literature on currency crises (see Obstfeld (1996)).

However, if θ is observed with noise, we can apply the results of the previous section, since $\pi(l, \theta)$ is weakly increasing in l , and weakly increasing in θ .

$$\begin{aligned} \int_{l=0}^1 \pi(l, \theta) dl &= (1 - a(\theta)) (\zeta(\theta) + t - e^*) + a(\theta) t \\ &= t - (1 - a(\theta)) (e^* - \zeta(\theta)) \end{aligned}$$

Thus θ^* is implicitly defined by

$$(1 - a(\theta)) (e^* - \zeta(\theta)) = t.$$

Theorem 2 in Morris and Shin (1998) gave an incorrect statement of this condition. We are grateful to Heinemann (2000) for pointing out the error and giving a correct characterization.

Again, we will describe one simple comparative statics exercise. Consider a costly ex ante action R for the monetary authority that lowered their costs of defending the currency. For example, R might represent the value of foreign currency reserves or (as in the recent case of Argentina) a line of credit with foreign banks to provide credit in the event of a crisis. Thus the critical proportion of speculators for which an attack occurs becomes $a(\theta, R)$, where $a(\cdot)$ is increasing in R . Now write $\theta^*(R)$ for the unique value of θ solving

$$(1 - a(\theta, R)) (e^* - \zeta(\theta)) = t.$$

The ex ante probability that the currency will collapse is

$$P(\theta^*(R))$$

So the reduction in the probability of collapse resulting from a marginal increase in R is

$$-p(\theta^*(R)) \frac{d\theta^*}{dR} = p(\theta^*(R)) \frac{\frac{\partial a}{\partial R}}{\frac{\partial a}{\partial \theta} + \frac{1-a(\theta, R)}{e^* - \zeta(\theta)} \frac{d\zeta}{d\theta}}$$

This comparative static refers to the limit (as noise becomes very small) and the effect is entirely strategic, i.e., the increased value of R reduces the probability of attack *only* because it influences speculators' equilibrium strategies ("builds confidence") and not because the increase in R actually prevents an attack in any relevant contingency.

In section 4.1, we very briefly discuss Corsetti, Dasgupta, Morris and Shin (2000), an extension of this model of currency attacks where a large speculator is added to the continuum of small traders. See also Goldstein and Pauzner (2000b), Heinemann and Illing (2000), Hellwig (2000), Marx (2000) and Morris and Shin (1999a).

2.3.3. Bank Runs

We describe a model of Goldstein and Pauzner (2000a), who add noise to the classic bank runs model of Diamond and Dybvig (1983). A continuum of depositors (with total deposits normalized to 1) must decide whether to withdraw their money from a bank or not. If the depositors withdraw their money in period 1, they will receive $r > 1$ (if there are not enough resources to fund all those who try to withdraw, then the remaining cash is divided equally among early withdrawers). Any remaining money earns a total return $R(\theta) > 0$ in period 2, and is divided equally among those who chose to wait until period 2 to withdraw their money. Proportion λ of depositors will have consumption needs only in period 1, and will thus have a dominant strategy to withdraw. We will be concerned with the game among the proportion $1 - \lambda$ of depositors who have consumption needs in period 2. Consumers have utility $U(y)$ from consumption y , where the relative risk aversion coefficient of U is strictly greater than 1. They note that if $R(\theta)$ were greater than one and θ were common knowledge, the ex ante optimal choice of r maximizing

$$\lambda U(r) + (1 - \lambda) U\left(\frac{1 - \lambda r}{1 - \lambda} R(\theta)\right)$$

would be strictly greater than 1. But if θ is not common knowledge, we have a global game. Writing 1 for the action "withdraw in period 2" and 0 for the action "withdraw in period 1," and l for the proportion of late consumers who do not withdraw early, the money payoffs in this game can be summarized in the

following table:

		$l \leq \frac{r-1}{r(1-\lambda)}$	$l \geq \frac{r-1}{r(1-\lambda)}$
Early Withdrawal	0	$\frac{1-\lambda r}{(1-\lambda)(1-l)r}$	r
Late Withdrawal	1	0	$\left(r - \frac{r-1}{l(1-\lambda)}\right) R(\theta)$

Observe that if θ is sufficiently small (and so $R(\theta)$ is sufficiently small) all players have a dominant incentive to withdraw early. Goldstein and Pauzner assume that if θ is sufficiently large, all players have a dominant strategy to withdraw late (a number of natural economic stories could justify this variation in the payoffs).

Thus the payoffs in the game among late consumers are:

$$u(1, l, \theta) = \begin{cases} U(0), & \text{if } l \leq \frac{r-1}{r(1-\lambda)} \\ U\left(\left(r - \frac{r-1}{l(1-\lambda)}\right) R(\theta)\right), & \text{if } l \geq \frac{r-1}{r(1-\lambda)} \end{cases}$$

$$u(0, l, \theta) = \begin{cases} U\left(\frac{1}{1-l(1-\lambda)}\right), & \text{if } l \leq \frac{r-1}{r(1-\lambda)} \\ U(r), & \text{if } l \geq \frac{r-1}{r(1-\lambda)} \end{cases}$$

so that

$$\pi(l, \theta) = \begin{cases} U(0) - U\left(\frac{1}{1-l(1-\lambda)}\right), & \text{if } l \leq \frac{r-1}{r(1-\lambda)} \\ U\left(\left(r - \frac{r-1}{l(1-\lambda)}\right) R(\theta)\right) - U(r), & \text{if } l \geq \frac{r-1}{r(1-\lambda)} \end{cases}$$

The threshold state θ^* is implicitly defined by

$$\int_{l=0}^{\frac{r-1}{r(1-\lambda)}} U(0) - U\left(\frac{1}{1-l(1-\lambda)}\right) dl + \int_{l=\frac{r-1}{r(1-\lambda)}}^1 U\left(\left(r - \frac{r-1}{l(1-\lambda)}\right) R(\theta)\right) - U(r) dl = 0$$

The ex ante welfare of consumers as a function of r (as noise goes to zero) is

$$W(r) = P(\theta^*(r)) U(1) + \int_{\theta=\theta^*(r)}^{\infty} p(\theta) \left(\lambda U(r) + (1-\lambda) U\left(\frac{1-\lambda r}{1-\lambda} R(\theta)\right) \right)$$

There are two effects of increasing r : the direct effect on welfare is the increased value of insurance in the case where there is not a bank run. But there is also the strategic effect that an increase in r will lower $\theta^*(r)$.

Morris and Shin (2000) examine a stripped down version of this model where alternative assumptions on the investment technology and utility functions imply that payoffs reduce to those of the linear example in section 2.1. See also Boonprakaikawe and Ghosal (2000), Dasgupta (2000b), Goldstein (2000) and Rochet and Vives (2000).

3. Public versus Private Information

The analysis so far has all been concerned with behavior when either there is a uniform prior or the noise is very small. In this section, we look at the behavior of the model with large noise and non-uniform priors. There are three reasons for doing this. First, we want to understand how extreme the assumptions required for uniqueness are. We will provide sufficient conditions for uniqueness depending on the relative accuracy of private and public (or prior) signals. Second, away from the limit, prior beliefs play an important role in determining outcomes. In particular, we will see how even with a continuum of players and a unique equilibrium, public information contained in the prior beliefs plays a significant role in determining outcomes, *even controlling for beliefs concerning the fundamentals*. Finally, by seeing how and when the model jumps from having one equilibrium to multiple equilibria, it is possible to develop a better intuition for what is driving results.

We return the linear example of section 2.1: there is a continuum of players, the payoff to not investing is 0, the payoff to investing is $\theta + l - 1$, where θ is the state and l is the proportion of the population investing. It may help in following in the analysis to recall that with linear payoffs, the exact number of players is irrelevant in identifying symmetric equilibrium strategies (and we will see that symmetric equilibrium strategies will naturally arise). Thus the analysis below applies equally to a two player game.

Now assume that θ is normally distributed with mean y and standard deviation τ . The mean y is publicly observed. As before, each player observes a private signal $x_i = \theta + \varepsilon_i$, where the ε_i are distributed normally in the population with mean 0 and standard deviation σ . Thus each player i observes a public signal $y \in \mathbb{R}$ and a private signal $x_i \in \mathbb{R}$. To analyze the equilibria of this game, first fix the public signal y . Suppose that a player observed private signal x . His expectation of θ is

$$\bar{\theta} = \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2}.$$

It is useful to conduct analysis in terms of these posterior expectations of θ . In particular, we may consider a switching strategy of the following form:

$$s(\bar{\theta}) = \begin{cases} \text{Invest, if } \bar{\theta} > \kappa \\ \text{NotInvest, if } \bar{\theta} \leq \kappa \end{cases}$$

If the standard deviation of players' private signals is sufficiently small relative to the standard deviation of the public signal in the prior, then there is a strategy surviving iterated deletion of strictly dominated strategies. Specifically, let

$$\gamma \equiv \tilde{\gamma}(\sigma, \tau) \equiv \frac{\sigma^2}{\tau^4} \left(\frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2} \right)$$

Now we have

Proposition 3.1. *The game has a symmetric switching strategy equilibrium with cutoff κ if κ solves the equation*

$$\kappa = \Phi(\sqrt{\gamma}(\kappa - y)); \quad (3.1)$$

if $\tilde{\gamma}(\sigma, \tau) \leq 2\pi$, then there is a unique value of κ solving (3.1) and the strategy with that trigger is the essentially unique strategy surviving iterated deletion of strictly dominated strategies; if $\tilde{\gamma}(\sigma, \tau) > 2\pi$, then (for some values of y) there are multiple values of κ solving (3.1) and multiple symmetric switching strategy equilibria.

Figure 3.1 plots the regions in $\sigma^2 - \tau^2$ space where uniqueness holds.

In Morris and Shin (2000), we gave a detailed version of the uniqueness part of this result in appendix A. Here we sketch the idea. Consider a player who has observed private signal x . By standard properties of the normal distribution (see DeGroot (1970)), his posterior beliefs about θ would be normal with mean

$$\bar{\theta} = \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2}$$

and standard deviation

$$\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}.$$

He knows that any other player's signal, x' , is equal to θ plus a noise term with mean 0 and standard deviation σ . Thus he believes that x' is distributed normally with mean $\bar{\theta}$ and standard deviation

$$\sqrt{\frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}}.$$

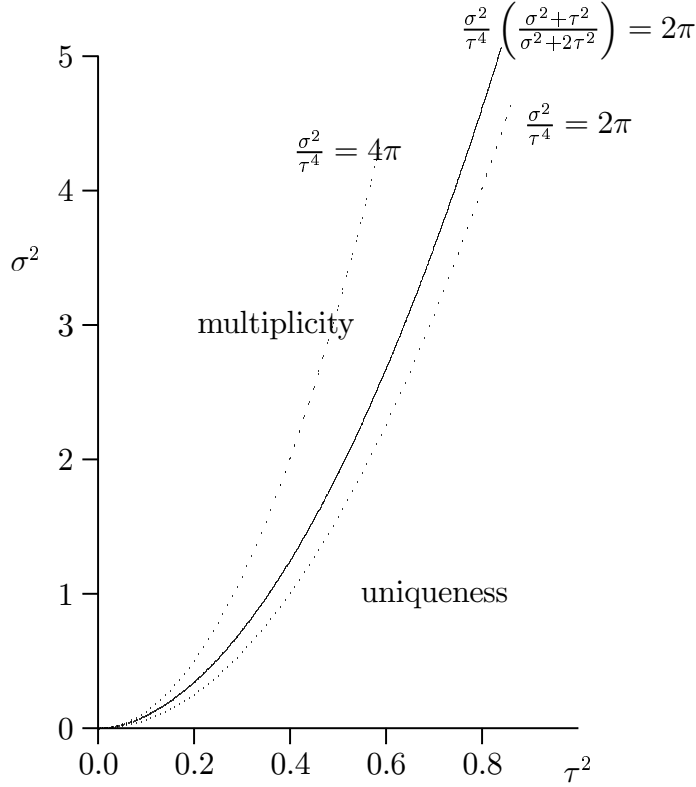


Figure 3.1: Parameter Range for Unique Equilibrium

Now suppose he believed that all other players will invest exactly if their expectation of θ is at least κ , i.e., if their private signals x' satisfy $\frac{\sigma^2 y + \tau^2 x'}{\sigma^2 + \tau^2} \geq \kappa$, or $x' \geq \kappa + \frac{\sigma^2}{\tau^2} (\kappa - y)$. Thus he assigns probability

$$1 - \Phi \left(\frac{\kappa - \bar{\theta} + \frac{\sigma^2}{\tau^2} (\kappa - y)}{\sqrt{\frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}}} \right) \quad (3.2)$$

to any particular opponent investing. But his expectation of the proportion of his opponents investing must equal to the probability he assigns to any one opponent investing. Thus (3.2) is also equal to his expectation of the proportion of his opponents investing. Since his payoff to investing is $\theta + l - 1$, his expected payoff to investing is $\bar{\theta}$ plus expression (3.2) minus one, i.e.,

$$v(\bar{\theta}, \kappa) \equiv \bar{\theta} - \Phi \left(\frac{\kappa - \bar{\theta} + \frac{\sigma^2}{\tau^2} (\kappa - y)}{\sqrt{\frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}}} \right)$$

His payoff to not investing is 0. Since $v(\bar{\theta}, \kappa)$ is increasing in $\bar{\theta}$, we have that there is a symmetric equilibrium with switching point κ exactly if $v^*(\kappa) \equiv v(\kappa, \kappa) = 0$. But

$$\begin{aligned} v^*(\kappa) &\equiv v(\kappa, \kappa) \\ &= \kappa - \Phi\left(\frac{\sigma^2(\kappa - y)}{\tau^2 \sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}}\right) \\ &= \kappa - \Phi(\sqrt{\gamma}(\kappa - y)) \end{aligned}$$

Figure 3.2 plots the function $v^*(\kappa)$ for $y = \frac{1}{2}$ and $\gamma = 1000, 10, 5$ and 0.1 , respectively.

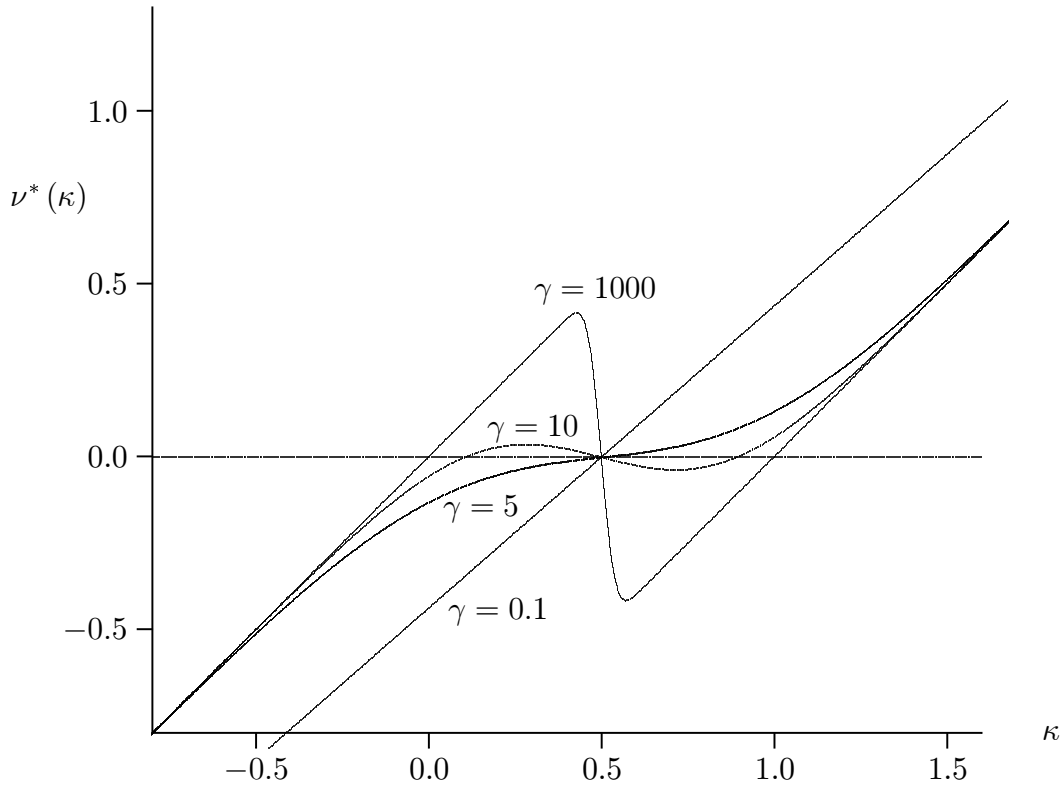


Figure 3.2: Function $v^*(\kappa)$

The intuition for these graphs is the following. If public information is relatively large (i.e., $\sigma \gg \tau$ and thus γ is large), then players with posterior expectation κ less than $y = \frac{1}{2}$ confidently expect that their opponent will have observed a

higher signal, and therefore will be investing. Thus his expected utility is (about) κ . But as κ moves above $y = \frac{1}{2}$, he rapidly becomes confident that his opponent has observed a lower signal and will not be investing. Thus his expected utility drops rapidly, around y , to (about) $\kappa - 1$. But if public information is relatively small (i.e., $\sigma \ll \tau$ and γ is small), then players with κ not too far above or below $y = \frac{1}{2}$ attach probability (about) $\frac{1}{2}$ to their opponent observing a higher signal. Thus his expected utility is (about) $\kappa - \frac{1}{2}$.

We can identify analytically when there is a unique solution: observe that

$$\frac{dv^*}{d\kappa} = 1 - \sqrt{\gamma}\phi(\sqrt{\gamma}(\kappa - y)).$$

Recall that $\phi(x)$, the density of the standard normal, attains its maximum of $\frac{1}{\sqrt{2\pi}}$ at $x = 0$. Thus if $\gamma \leq 2\pi$, $\frac{dv^*}{d\kappa}$ is greater than or equal to zero always, and strictly greater than zero except when $\kappa = y$. So (3.1) has a unique solution. But if $\gamma > 2\pi$ and $y = \frac{1}{2}$, then setting $\kappa = \frac{1}{2}$ solves (3.1) but $\frac{dv^*}{d\kappa}\big|_{\kappa=\frac{1}{2}} < 0$, so (3.1) has two other solutions.

Throughout the remainder of this section, we assume that there is a unique equilibrium, i.e., that $\tilde{\gamma}(\alpha, \beta) \leq 2\pi$. Under this assumption, we can invert the equilibrium condition (3.1) to show in (κ, y) space what the unique equilibrium looks like

$$y = h_\gamma(\kappa) = \kappa - \frac{1}{\sqrt{\gamma}}\Phi^{-1}(\kappa) \quad (3.3)$$

Figure 3.3 plots this for $\gamma = 5$ and $\gamma = \frac{1}{1000}$.

The picture has an elementary intuition. If $\kappa < 0$, it is optimal to not invest (independent of the public signal). If $\kappa > 1$, it is optimal to invest (independent of the public signal). But if $0 < \kappa < 1$, there is a trade-off. The higher y is (for a given κ), the more likely it is that the other player will invest. Thus if $0 < \kappa < 1$, the player will always invest for sufficiently high y , and not invest for sufficiently low y . This implies in particular that changing y has a larger impact on a player's action than changing his private signal (controlling for the informativeness of the signals). We next turn to examining this "publicity" effect.

3.1. The Publicity Multiplier

To explore the strategic impact of public information, we examine how much a player's private signal must adjust to compensate for a given change in the public signal. Equation (3.1) can be written as

$$\frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} - \Phi\left(\sqrt{\gamma}\left(\frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} - y\right)\right) = 0.$$

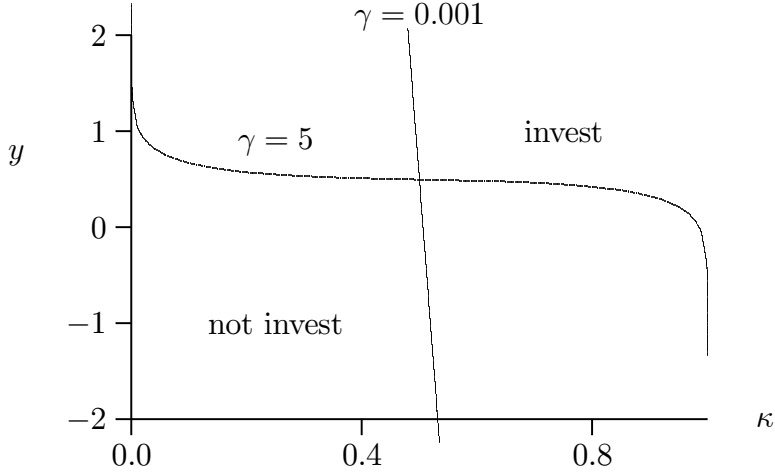


Figure 3.3: Investment takes place above and to the right of line

Totally differentiating with respect to y gives

$$\frac{dx}{dy} = -\frac{\frac{\sigma^2}{\tau^2} + \sqrt{\gamma}\phi(\cdot)}{1 - \sqrt{\gamma}\phi(\cdot)}.$$

This measures how much the private signal would have to change to compensate for a change in the public signal (and still leave the player indifferent between investing or not investing). We can similarly see how much the private signal would have to change to compensate for a change in the public signal, if there was no strategic effect. Totally differentiating

$$\bar{\theta} = \frac{\sigma^2 y + \tau^2 x}{\sigma^2 + \tau^2} = k,$$

we obtain

$$\frac{dx}{dy} = -\frac{\sigma^2}{\tau^2}.$$

Define the *publicity multiplier* as the ratio of these two:

$$\zeta = \frac{1 + \frac{\tau^2}{\sigma^2}\sqrt{\gamma}\phi(\cdot)}{1 - \sqrt{\gamma}\phi(\cdot)}.$$

Thus suppose a player's expectation of θ is $\bar{\theta}$ and he has observed the public signal that makes him indifferent between investing and not investing ($y = \bar{\theta} -$

$\frac{1}{\sqrt{\gamma}}\Phi^{-1}(\bar{\theta})$); the publicity multiplier evaluated at this point will be:

$$\zeta = \frac{1 + \frac{\tau^2}{\sigma^2}\sqrt{\gamma}\phi(\Phi^{-1}(\bar{\theta}))}{1 - \sqrt{\gamma}\phi(\Phi^{-1}(\bar{\theta}))}.$$

Notice that (for any given σ and τ) the publicity multiplier is maximized when $\bar{\theta} = \frac{1}{2}$, and thus the critical public signal $y = \frac{1}{2}$. Thus it is precisely when there is no conflict between private and public signals that the multiplier has its biggest effect. Here the publicity multiplier equals

$$\zeta^* = \frac{1 + \frac{\tau^2}{\sigma^2}\sqrt{\frac{\gamma}{2\pi}}}{1 - \sqrt{\frac{\gamma}{2\pi}}}.$$

Notice that when private information is very accurate relative to private information (i.e., $\sigma \rightarrow 0$ and $\gamma \rightarrow 0$), the publicity multiplier is very small. The multiplier is biggest just before we hit the multiplicity zone of the parameter space (i.e., when $\gamma \approx 2\pi$).

There is plentiful anecdotal evidence that in settings where co-ordination is important, public signals play a role in co-ordinating outcomes that exceeds the information content of those announcements. For example, financial markets apparently “overreact” to announcements from the Federal Reserve Board and public announcements in general. If market participants are concerned about the reaction of other participants to the news, the “overreaction” may be rational and determined by the type of equilibrium logic of our example. Further evidence for this is briefings on market conditions by key players in financial markets using conference calls with hundreds of participants. Such public briefings have a larger impact on the market than bilateral briefings with the same information, because they automatically convey to participants not only information about market conditions but also valuable information about the beliefs of the other participants.

Urban renewal also has a co-ordination aspect. Private firms’ incentives to invest in a run down neighborhood depend partly on exogenous characteristics of the neighborhood, but also depend to a great extent on whether other firms are investing. A well publicized investment in the neighborhood might be expected to have an apparently disproportionate effect on the probability of ending in the good equilibrium. The willingness of public authorities to subsidize football stadiums and conference centers is consistent with this view.

An indirect econometric test of the publicity effect is performed by Chwe (1998). Chwe observes that the per viewer price of advertising during the Super Bowl is exceptionally high (i.e., the price of advertising increases more than linearly in the number of viewers). The premium price is explained by the fact that

any information conveyed by those advertisements becomes not merely known to the wide audience, but also common knowledge among them. The value of this common knowledge to advertisers should depend on whether there is a significant co-ordination problem in consumers' decisions whether to purchase the product. Chwe makes some plausible ex ante guesses about when co-ordination is an important issue because of network externalities (e.g., the Apple Macintosh) or social consumption (e.g., beer); and when it is not (e.g., batteries). He then confirms econometrically that it is the advertisers of co-ordination goods who pay a premium for large audiences.

In Morris and Shin (1999b), we use the publicity effect to explain an anomaly in the pricing of debt. Empirically, the option pricing model of debt due to Merton (1974) underestimates the yield on debt (i.e., underestimates the empirical default rate). This deviation from theory is largest for low grade (high risk) bonds. A deterioration in public signals for low grade bonds generates a large publicity effect: the deterioration makes investors more pessimistic about default for any given strategies of the other players, but more importantly the deterioration makes investors more pessimistic about other players' strategies.

3.2. Limiting Behavior

If we increase the precision of public signals, while holding the precision of private signals fixed (i.e., let $\tau \rightarrow 0$ for fixed σ), then we clearly exit the unique equilibrium zone.⁶ If we increase the precision of private signals, while holding the precision of public signals fixed (i.e., let $\sigma \rightarrow 0$ for fixed τ), then we return to the uniform prior setting of section 2.1. But we can also examine what happens to the unique equilibrium as the precision of both signals increases in such a way that uniqueness is maintained. Specifically, let $\tau \rightarrow 0$ and let $\sigma^2 \rightarrow c\tau^4$, where $c < 4\pi$. In this case,

$$\begin{aligned} \tilde{\gamma}(\sigma, \tau) &= \frac{\sigma^2}{\tau^4} \left(\frac{\sigma^2 + \tau^2}{\sigma^2 + 2\tau^2} \right) \\ &\rightarrow \frac{c\tau^4}{\tau^4} \left(\frac{c\tau^4 + \tau^2}{c\tau^4 + 2\tau^2} \right) \\ &\rightarrow \frac{c}{2} \\ &< 2\pi. \end{aligned}$$

⁶For sufficiently small τ , either action is rationalizable as long as $y \in (0, 1)$ and $\bar{\theta} \in (0, 1)$. If either $\bar{\theta} \geq 1$ or $\bar{\theta} > 0$ and $y \geq 1$, then only investing is rationalizable. If either $\bar{\theta} \leq 0$ or $\bar{\theta} < 1$ and $y \leq 0$, then only not investing is rationalizable.

Thus

$$h_{\bar{\gamma}(\sigma, \tau)}(\bar{\theta}) \rightarrow \bar{\theta} - \left(\sqrt{\frac{2}{c}} \right) \Phi^{-1}(\bar{\theta}).$$

This result says that *even though the public signal becomes irrelevant to a player's expected value of θ in the limit*, it continues to have a large impact on the outcome. For example, suppose $c = 1$ and $y = \frac{1}{3}$ (i.e., public information looks bad). Each player will invest only if $\bar{\theta} \geq 0.7$, i.e., they will be very conservative. This is true even as they ignore y (i.e., $\bar{\theta} \rightarrow x$).

The intuition for this result is the following. Suppose public information looks bad ($y < \frac{1}{2}$). If each player's private information is much more accurate than the public signal, each player will mostly ignore the public signal in forming his own expectation of θ . But each will nonetheless expect the other to have observed a somewhat worse signal than themselves. This pessimism about the other's signal makes it very hard to support an investment equilibrium.

3.3. Sufficient Conditions for Uniqueness

We derived a very simple necessary and sufficient condition for uniqueness in the linear example, depending only of the precision of public and private signals. In this section, we briefly demonstrate that a similar sufficient condition works for general payoff functions. In particular, we will show that there is always a unique equilibrium if $\frac{\sigma^2}{\tau^4}$ is sufficiently small.⁷

We will show this in a simple setting although the argument can be extended. We maintain the normal distribution assumptions on the prior and signals, but let the payoffs be as in section 2.2, so that $\pi(l, \theta)$ is the payoff gain from choosing action 1 instead of action 0. Furthermore, we will focus on the continuum players case, where $\pi(l, \theta)$ is differentiable and strictly increasing in l and θ , with $\frac{d\pi}{dl}(l, \theta) \leq K$ and $\frac{d\pi}{d\theta}(l, \theta) \geq \varepsilon$ for all l and θ .

Under these assumptions, we may look at the expected gain to choosing action 1 rather than action 0 if your expectation of θ is $\bar{\theta}$ and you think that others follow a switching strategy at κ :

$$\begin{aligned} V(\bar{\theta}, \kappa) &= \int_{\theta=-\infty}^{\infty} \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}} \phi\left(\frac{\theta - \bar{\theta}}{\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}}\right) \pi\left(1 - \Phi\left(\frac{\kappa - \theta + \frac{\sigma^2}{\tau^2}(\kappa - y)}{\sigma}\right), \theta\right) d\theta \\ &= \int_{\theta'=-\infty}^{\infty} \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}} \phi\left(\frac{\theta'}{\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}}\right) \pi\left(1 - \Phi\left(\frac{-\theta' + \kappa - \bar{\theta} + \frac{\sigma^2}{\tau^2}(\kappa - y)}{\sigma}\right), \theta' + \bar{\theta}\right) d\theta'. \end{aligned}$$

⁷Hellwig (2000) performs a related exercise in a version of our currency attacks model (Morris and Shin (1998)).

Now to apply our earlier argument for uniqueness, it is enough to show that expression is increasing in $\bar{\theta}$ and $V(\kappa, \kappa) = 0$ has a unique solution. The former is clearly true; to show the latter, observe that

$$V(\kappa, \kappa) = \int_{\theta'=-\infty}^{\infty} \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}} \phi\left(\frac{\theta'}{\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}}\right) \pi\left(1 - \Phi\left(\frac{-\theta' + \frac{\sigma^2}{\tau^2}(\kappa - y)}{\sigma}\right), \theta' + \kappa\right) d\theta',$$

so

$$\begin{aligned} \frac{dV(\kappa, \kappa)}{d\kappa} &= \int_{\theta'=-\infty}^{\infty} \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}} \phi\left(\frac{\theta'}{\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}}\right) \left[\frac{d\pi(\cdot)}{d\theta} - \frac{d\pi(\cdot)}{dl} \phi(\cdot) \frac{\sigma}{\tau^2} \right] d\theta' \\ &= \int_{\theta'=-\infty}^{\infty} \sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}} \phi\left(\frac{\theta'}{\sqrt{\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}}}\right) \frac{d\pi(\cdot)}{d\theta} \left[1 - \frac{\frac{d\pi(\cdot)}{dl}}{\frac{d\pi(\cdot)}{d\theta}} \phi(\cdot) \frac{\sigma}{\tau^2} \right] d\theta' \end{aligned} \quad (3.4)$$

If this expression is always positive, then there is a unique value of κ solving $V(\kappa, \kappa) = 0$ and the unique strategy surviving iterated deletion of strictly dominated strategies is the switching strategy with that cutoff. Since $\phi(\cdot)$ is at most $\frac{1}{\sqrt{2\pi}}$, the expression in square brackets within equation (3.4) is positive as long as

$$\frac{K}{\varepsilon} \geq \frac{\frac{d\pi(\cdot)}{dl}}{\frac{d\pi(\cdot)}{d\theta}} > \frac{\tau^2 \sqrt{2\pi}}{\sigma},$$

i.e.,

$$\frac{\sigma^2}{\tau^4} < 2\pi \left(\frac{\varepsilon}{K}\right)^2.$$

4. Theoretical Underpinnings

4.1. General Global Games

All the analysis thus far has dealt with symmetric payoff games. The analysis of Carlsson and van Damme (1993a) in fact provided a remarkably general result for two player, two action games, even with asymmetric payoffs. Let the payoffs of a two player, two action game be given by the following matrix:

	1	0
1	θ_1, θ_2	θ_3, θ_4
0	θ_5, θ_6	θ_7, θ_8

Thus a vector $\theta \in \mathbb{R}^8$ describes the payoffs of the game. Each player i observes a signal $x_i = \theta + \sigma \varepsilon_i$, where the ε_i are 8-dimensional noise terms. This set up describes an incomplete information game parameterized by σ . Under mild technical assumptions,⁸ as $\sigma \rightarrow 0$, any sequence of strategy profiles surviving iterated deletion of strictly dominated strategies converges to a unique limit. Moreover that limit is independent of the distribution of the noise, and has the unique Nash equilibrium being played (if there is one) and has the risk dominant Nash equilibrium played (if there are two strict Nash equilibria).

To understand if and when this remarkable result might extend to many player, many action games, it is useful to first observe that there are two independent things being proved here. First, there is a *limit uniqueness* result. As the noise goes to zero, there is a unique strategy profile surviving iterated deletion of strictly dominated strategies. Given that with no noise, we know that there are multiple equilibria, this is a striking result by itself. Second, there is a *noise independent selection* result. We can characterize behavior in that unique limit as a function of the complete information payoffs in the limit, and thus independently of the shape of the prior beliefs on θ and the distribution of noise. Thus Carlsson and van Damme's two player, two action analysis combines separate limit uniqueness and noise independent selection results. Similarly, the results in section 2 for continuum player, symmetric binary action games simultaneously showed that there was a unique strategy surviving iterated deletion of strictly dominated strategies in the limit (a limit uniqueness result) and characterized behavior in the limit (the Laplacian action) independent of the structure of the noise (a noise independent selection result).

Frankel, Morris and Pauzner (2000) (hereafter, FMP) examine global games with many players, asymmetric payoffs and many actions. They show that a limit uniqueness result holds quite generally, as long as some monotonicity properties are satisfied. They consider the following environment. Each player has an ordered set of actions (finite or continuum); his payoff depends on the action profile played and a payoff parameter $\theta \in \mathbb{R}$; he observes a signal $x_i = \theta + \sigma \varepsilon_i$, where $\sigma > 0$ and ε_i is an independently distributed noise term. *Limit Dominance* requires that for sufficiently low values of θ , each player has a dominant strategy to choose his lowest action, and that for sufficiently high values of θ , each player has a dominant strategy to choose his highest action. *Strategic Complementarities* requires that each player's payoff gain in choosing a higher action is increasing in his opponents' actions (for any θ). A *Single Crossing* property requires that each

⁸The following technical conditions are sufficient (Carlsson and van Damme's actual set-up is a little more general): payoff vector θ is drawn according to a strictly positive, continuously differentiable, bounded density on \mathbb{R}^8 ; and the noise terms $(\varepsilon_1, \varepsilon_2)$ are drawn according to a continuous density with bounded support, independently of θ .

player's payoff gain in choosing a higher action is increasing in the payoff parameter θ (for any given actions of this opponents). The latter two assumptions assure that a player's best response is increasing in his opponents' actions and θ . Under these three substantive assumptions, and additional technical assumptions,⁹ FMP show a limit uniqueness result. The proof uses the technique, also followed in section 2.2, of first analyzing the uniform prior, private values game and showing a uniqueness result independent of the size of the noise; and then showing that if the noise is small, all equilibria of the game with a general prior and common values are close to the unique equilibrium of the uniform prior, private values game. The limit uniqueness result of FMP provides a natural many player, many action generalization of Carlsson and van Damme (1993a). It is true that Carlsson and van Damme required no strategic complementarity and other monotonicity properties. But when a two player, two action game has multiple Nash equilibria (the interesting case for Carlsson and van Damme's analysis), there are automatically strategic complementarities. FMP's limit uniqueness results could presumably be extended straightforwardly to many dimensional payoff parameters and signals, if the relevant monotonicity conditions were suitably adjusted.¹⁰

Within this class of monotonic global games where limit uniqueness holds, FMP also provide sufficient conditions for noise independent selection. They generalize the notion of a potential maximizing action, due to Monderer and Shapley (1996). We will discuss these generalized potential conditions in more detail below in section 4.4, as they are also sufficient for the (more demanding) property of being *robust to incomplete information*. The sufficient conditions for noise independent selection encompass two classes of games already discussed in this survey: many player, two action, symmetric payoff games (where the Laplacian action is played); and two player, two action games, with possibly asymmetric payoffs (where the risk dominant equilibrium is played). They also encompass two player, three action games with symmetric payoffs. And they encompass the minimum effort game of Bryant (1983).¹¹

FMP also provide an example of a two player, four action, symmetric payoff

⁹Payoffs are continuous with respect to actions and θ and there is a Lipschitz bound on the sensitivity of payoffs to changes in own and others actions. The state is drawn according to a continuous and positive density, and signals are drawn according to a continuous and positive density with bounded support.

¹⁰However, strategic complementarities are surely not a necessary condition for limit uniqueness and the FMP conditions could presumably be weakened in a number of directions. For example, in the two player bargaining model of Carlsson (1991), strategic complementarities hold locally but FMP's strategic complementarities condition fails. In an environment very like a global game, a limit uniqueness result holds (Carlsson's model has local action trembles rather than local payoff trembles but they appear to play a similar role).

¹¹Carlsson and Ganslandt (1998) show the potential maximizing action is selected in the minimum effort game when players' continuous actions are perturbed.

game where noise independent selection fails. Thus there is a unique limit as the noise goes to zero, but the nature of the limit depends on the exact distribution of the noise. Carlsson (1989) gave a three player, two action example in which noise independent selection failed. Corsetti, Dasgupta, Morris and Shin (2000) describe a global games model of currency crises, where there is a continuum of small traders and a single large trader. This is thus a many player, two action game with *asymmetric* payoffs. We show that the equilibrium selected as noise goes to zero depends on the relative informativeness of the large and small traders' signals. This is thus an application where noise independent selection fails.

We conclude this brief summary by noting one consequence of FMP for the earlier analysis in this paper. In section 2.2, it was shown that the Laplacian action was selected in symmetric binary action global games. The argument exploited the fact that players observed signals with i.i.d. noise in that class of games. But FMP show noise independent selection of the Laplacian action *independent of the distribution of noise*. If the distribution of noise is very different for different players, we surely cannot guarantee that each player has a uniform belief over the proportion of his opponents taking each action. Nonetheless, the Laplacian action must be played in the limit. We can illustrate this implication with a simple example. Consider a three player game, with binary action set $\{0, 1\}$. The payoff to action 1 is θ if both other players choose action 1, $\theta - z$ if one other player chooses action 1 and $\theta - 1$ if neither other player chooses action 1 (where $0 < z < 1$). The payoff to action 0 is zero. State θ is uniformly distributed on the real line. Observe that the Laplacian action is 1 if $\frac{1}{3}\theta + \frac{1}{3}(\theta - z) + \frac{1}{3}(\theta - 1) > 0$, i.e., $\theta > \frac{1}{3}(z + 1)$. Let ε_1 , ε_2 and ε_3 be i.i.d. with c.d.f. $F(\cdot)$, let δ be a very small positive number and let σ be a parameter describing the size of the noise. The players' signals x_1 , x_2 and x_3 are given by

$$\begin{aligned} x_1 &= \theta + \sigma\delta\varepsilon_1 \\ x_2 &= \theta + \sigma\delta\varepsilon_2 \\ x_3 &= \theta + \sigma\varepsilon_3 \end{aligned}$$

Thus 1 and 2 observe much more informative signals. We will look for a switching strategy equilibrium where players 1 and 2 use cutoff \hat{x}_σ and player 3 uses cutoff \tilde{x}_σ . Let

$$\lambda_\sigma = F\left(\frac{\tilde{x}_\sigma - \bar{x}_\sigma}{\sigma}\right)$$

We are interested in what happens in the limit as first we take $\delta \rightarrow 0$, and then take the limit as $\sigma \rightarrow 0$. As δ becomes very small, if player 1 or 2 observes signal \bar{x}_σ , he will assign probability (about) $\frac{1}{2}(\frac{1}{2} - \lambda_\sigma)$ to both players choosing action 1, probability (about) $\frac{1}{2}$ to one player choosing action 1, and probability (about)

$\frac{1}{2} (\frac{1}{2} + \lambda_\sigma)$ to neither player choosing action 1; while if player 3 observes signal \tilde{x}_σ , he will assign probability $\frac{1}{2} + \lambda_\sigma$ to both players choosing action 1, probability 0 to one player choosing action 1, and probability $\frac{1}{2} - \lambda_\sigma$ to neither player choosing action 1.

Thus we must have:

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{2} - \lambda_\sigma \right) \bar{x}_\sigma + \frac{1}{2} (\bar{x}_\sigma - z) + \frac{1}{2} \left(\frac{1}{2} + \lambda_\sigma \right) (\bar{x}_\sigma - 1) &= 0 \\ \left(\frac{1}{2} + \lambda_\sigma \right) \tilde{x}_\sigma + 0 (\tilde{x}_\sigma - z) + \left(\frac{1}{2} - \lambda_\sigma \right) (\tilde{x}_\sigma - 1) &= 0 \end{aligned}$$

Rearranging gives:

$$\begin{aligned} \bar{x}_\sigma &= \frac{1}{2} z + \frac{1}{2} \left(\frac{1}{2} + \lambda_\sigma \right) \\ \tilde{x}_\sigma &= \frac{1}{2} - \lambda_\sigma \end{aligned}$$

As $\sigma \rightarrow 0$, we must have $\bar{x}_\sigma \rightarrow \tilde{x}_\sigma$ and thus $\lambda_\sigma \rightarrow \frac{1}{3} (\frac{1}{2} - z)$ (so $\frac{\tilde{x}_\sigma - \bar{x}_\sigma}{\sigma} \rightarrow F^{-1}(\lambda_\sigma)$). So \bar{x}_σ and \tilde{x}_σ must both converge to $\frac{1}{3} (z + 1)$. But this gives the result that the Laplacian action is played by all players in the limit, independent of the shape of F .

4.2. Higher Order Beliefs

In global games, the importance of the noisy observation of the underlying state lies in the fact that it generates *strategic uncertainty*: that is, uncertainty about others' behavior in equilibrium. That strategic uncertainty is generated by players' uncertainty about other players' payoffs. Thus understanding global games involves understanding how equilibria depend on players' uncertainty about other players' payoffs. But clearly, it is not going to be enough to know each player's beliefs about other players' payoffs. We must also take into account each player's beliefs about other players' beliefs about his payoffs; and further such *higher order beliefs*. Players' payoffs and higher order beliefs about payoffs are the true primitives of a game of incomplete information, not the asymmetric information structure. In earlier sections we told an asymmetric information story about how there is a true state of fundamentals θ drawn from some prior and each player observes a signal of θ generated by some technology. But our analysis of the resulting game implicitly assumes that there is common knowledge of the prior distribution of θ and the signalling technologies. It is hard to defend this assumption literally when the original purpose was to get away from the unrealistic assumption that there is common knowledge of the realization of θ . The classic

arguments of Harsanyi (1967-1968) and Mertens and Zamir (1985) tell us that we can assume common knowledge of *some* state space without loss of generality. But such a common knowledge state space makes sense with an incomplete information interpretation (a player’s “type” is a description of his higher order beliefs about payoffs), but not with an asymmetric information interpretation (a player’s “type” is a signal drawn according to some ex ante fixed distribution); see Battigalli (1999) and Dekel and Gul (1996) for forceful defenses of this position. Thus we believe that the noise structures analyzed in global games are interesting because they represent a tractable way of generating a rich structure of higher order beliefs. The analysis of global games represents a natural vehicle to illustrate the power of higher order beliefs at work in applications.¹² But then the natural way to understand the “trick” to global games analysis is to go back and understand what is going on in terms of higher order beliefs.

Even if one is uninterested in the philosophical distinction between incomplete information and asymmetric information, there is a second reason why the higher order beliefs literature may contribute to our understanding of global games. Even keeping a pure asymmetric information interpretation, we can calculate (from the prior distribution over θ and the signal technologies) the players’ higher order beliefs about payoffs. Statements about higher order beliefs about payoffs turn out to represent a natural mathematical way of characterizing which properties of the prior distribution and signal technologies matter for the results.

The pedagogical risk of emphasizing higher order beliefs is that readers may conclude that playing in the uniquely rational way in a global game requires fancy powers of reasoning, some kind of hyper-rationality that allows them to reason to arbitrarily high number of levels. We emphasize that the fact that either the analyst or a player expresses information about the game in terms of higher order beliefs does not make standard equilibrium concepts any less compelling and does not suggest any particular view about how equilibrium behavior might be arrived at. In particular, recall that there is a very simple heuristic that will generate equilibrium behavior in symmetric binary action games. If there is not common knowledge of the environment you are in, you should hold diffuse beliefs about others’ behavior. In particular, if you are on the margin between your two actions, it seems reasonable to take the agnostic view that you are equally likely to hold any rank in the population concerning your evaluation of the desirability of the two actions. Thus if other people behave like you, you should make your decision on the assumption that the proportion of other players choosing each action is uniformly distributed. This reasoning sounds very suspicious game theoretically,

¹²For work on higher order beliefs not using the global games technology, see Townsend (1983), Allen, Morris and Postlewaite (1992), Shin (1996) and the discussion of section 4.1 of Allen and Morris (2000).

but actually generates a very simple heuristic for behavior that is consistent with the unique rational behavior.

In the remainder of this section, we first informally discuss the role of higher order beliefs in a global game example. Then we review briefly the theoretical literature on higher order beliefs in games.¹³ Finally, we show how results from that literature can be taken back to the analysis of global games.

Monderer and Samet (1989) introduced a natural language for characterizing players' higher order beliefs. Fix a probability $p \in (0, 1]$. Let Ω be a set of possible states and let E be any subset of Ω . The event E is *p-believed* at state ω among some fixed group of individuals if everyone believes that it is true with probability at least p (and we write $B^p E$ for the set of states where event E is *p-believed*). The event E is *common p-belief* at state ω if it is *p-believed*, it is *p-believed* that it is *p-believed*, and so on, up to an arbitrary number of levels (and we write $C^p(E)$ for the set of states where event E is common *p-belief*). The event E is *p-evident* if whenever it is true, it is *p-believed* (i.e., $E \subseteq B^p E$). Monderer and Samet proved the following result:

Proposition 4.1. *Event E is common p -belief at ω (i.e., $\omega \in C^p(E)$) if and only if there exists a p -evident event F such that $\omega \in F \subseteq B^p E$.*

This result provides a fixed point characterization (i.e., using the *p-evident* property) of an iterative definition of common *p-belief*. It thus generalizes Aumann's classic characterization of *common knowledge* (Aumann (1976)).

We will illustrate these properties of higher order beliefs in the global games setting.¹⁴ So consider again the two player example of section 2.1: θ is drawn uniformly from the real line and players $i = 1, 2$ each observe a signal $x_i = \theta + \varepsilon_i$, where ε_i is distributed normally with mean 0 and standard deviation σ . Thus the relevant state space is \mathbb{R}^3 , with typical element (θ, x_1, x_2) . Fix the payoff relevant event $E_k = \{(\theta, x_1, x_2) : \theta \geq k\}$; this is the set of states where the true θ is at least k . If player i observes signal x_i , he will assign probability $\Phi\left(\frac{x_i - k}{\sigma}\right)$ to the event E_k being true. Thus he will assign probability at least p to the event E_k exactly if $x_i \geq k + \sigma\Phi^{-1}(p) \geq k$. Thus

$$B^p E_k = \{(\theta, x_1, x_2) : x_i \geq k + \sigma\Phi^{-1}(p), \text{ for } i = 1, 2\}.$$

¹³Our review of this literature is much abbreviated and highly selective. See Fudenberg and Tirole (1991) chapter 14, Osborne and Rubinstein (1994) chapter 5, Geanakoplos (1994) and Dekel and Gul (1996) for more background on this material. Morris and Shin (1997) survey the higher order beliefs in game theory literature with a focus on the relationship to related literatures in philosophy and computer science. Kajii and Morris (1997c) survey this literature with a focus on the relation to the standard *refinements* literature in game theory.

¹⁴Monderer and Samet (1989) characterized common *p-belief* for discrete state spaces, but Kajii and Morris (1997b) show the straightforward extension to continuum state spaces.

Now if player i observes x_i , he assigns probability $\Phi\left(\frac{x_i - \kappa}{\sqrt{2}\sigma}\right)$ to player j observing a signal above κ , and he assigns probability at least p to that event exactly if $x_i \geq \kappa + \sqrt{2}\sigma\Phi^{-1}(p)$. In addition, player i knows for sure whether x_i is greater than κ . Thus

$$B^p B^p E_k = \left\{ (\theta, x_1, x_2) : x_i \geq k + \sigma\Phi^{-1}(p) + \max\left\{0, \sqrt{2}\sigma\Phi^{-1}(p)\right\}, \text{ for } i = 1, 2 \right\}$$

and, by induction,

$$[B^p]^n E_k = \left\{ (\theta, x_1, x_2) : x_i \geq k + \sigma\Phi^{-1}(p) + (n-1) \max\left\{0, \sqrt{2}\sigma\Phi^{-1}(p)\right\}, \text{ for } i = 1, 2 \right\}. \quad (4.1)$$

So

$$C^p E_k = \bigcap_{n \geq 1} [B^p]^n E_k = \begin{cases} \emptyset, & \text{if } p > \frac{1}{2} \\ \{(\theta, x_1, x_2) : x_i \geq k + \sigma\Phi^{-1}(p), \text{ for } i = 1, 2\}, & \text{if } p \leq \frac{1}{2} \end{cases}.$$

Thus a remarkable feature of this simple example is that for *any* $p > \frac{1}{2}$, there is never common p -belief that θ is greater than k , for *any* k . We could also have shown this using the characterization of common p -belief described in Proposition 4.1. For any k , event E_k is p -evident only if $p \leq \frac{1}{2}$. This is because a player observing signal k will always assign probability $\frac{1}{2}$ to his opponent observing a signal less than k . A key property of global games is that they fail to deliver non-trivial common p -belief and p -evident events (for high p). As we will see, the existence of such events is key to supporting multiple equilibria in incomplete information games.

Combining this information structure with the payoffs from the two player example of section 2.1, we can illustrate the extreme sensitivity of strategic outcomes to players' higher order beliefs. Recall that each player had to choose between not investing (with payoff 0) and investing (with payoff θ if the other player invests, and payoff $\theta - 1$ otherwise). The unique equilibrium involved each player i investing if his signal x_i were greater than $\frac{1}{2}$, and not otherwise. This result was independent of σ (the scale variable of the noise). Now observe that if

$$\sigma \leq \frac{1}{5(1 + (n-1)\sqrt{2})\Phi^{-1}(p)}$$

then (by equation (4.1)) for all θ ,

$$\left(\theta, \frac{2}{5}, \frac{2}{5}\right) \in [B^p]^n E_{\frac{1}{5}}$$

In words, suppose that each player observed signal $\frac{2}{5}$. If we fix *any* integer n and *any* $p < 1$, we may choose σ sufficiently small such that it is p -believed that it is

p -believed that (n times)... that θ is greater than $\frac{1}{5}$. If it were common knowledge that θ were greater than $\frac{1}{5}$, it would clearly be rational for both players to invest. But the unique rational behavior has each player not investing.

Rubinstein (1989) used his *electronic mail game* to illustrate this sensitivity of strategic outcomes to common knowledge. Monderer and Samet (1989) showed why n levels of p -belief or even knowledge was not enough to approximate common knowledge in strategic settings, and common p -belief, i.e., an infinite number of levels, is required. The idea behind this observation is illustrated in the next section. Morris, Rob and Shin (1995) showed why only some Nash equilibria (e.g. risk dominated equilibria) were sensitive to higher order beliefs and not others, and provided a characterization - related to common p -belief - of which (discrete state) information systems displayed an extreme sensitivity to higher order beliefs. Kajii and Morris (1997a) introduced a notion of robustness to incomplete information to characterize equilibria that are not sensitive to higher order beliefs. This work is reviewed and related back to global games in sections 4.4 and 4.5.

4.3. Common p -Belief and Game Theory

Fix a finite set of players $1, \dots, I$ and a finite action set A_i for each player i . A *complete information game* is then a vector of payoff functions, $g \equiv (g_1, \dots, g_I)$, where each $g_i : A \rightarrow \mathbb{R}$. A (discrete state) *incomplete information game* is then a collection $\left\{ \Omega, \pi, (\mathcal{P}_i)_{i=1}^I, (u_i)_{i=1}^I \right\}$, where Ω is a countable state space, $\pi \in \Delta(\Omega)$ is a prior probability on that state space, \mathcal{P}_i is the partition of the state space of player i ; and $u_i : A \times \Omega \rightarrow \mathbb{R}$ is the payoff function of player i .

For any given incomplete information game $\left\{ \Omega, \pi, (\mathcal{P}_i)_{i=1}^I, (u_i)_{i=1}^I \right\}$, we may write $|g|$ for the set of states in the incomplete information game where payoffs are given by g . Thus

$$|g| = \{ \omega \in \Omega \mid u_i(a, \omega) = g_i(a) \text{ for all } a \in A \text{ and } i = 1, \dots, I \}$$

Using this language, we can summarize some key observations from the theoretical literature on higher order beliefs in game theory. A pure strategy Nash equilibrium a^* of a complete information game, g , is said to be a *p -dominant* equilibrium (Morris, Rob and Shin (1995)) if each player's action is a best response whenever he assigns probability at least p to his opponents choosing according to a^* , i.e.,

$$\sum_{a_{-i} \in A_i} \lambda(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_i} \lambda(a_{-i}) g_i(a_i, a_{-i})$$

for all $i = 1, \dots, I$, $a_i \in A_i$ and $\lambda \in \Delta(A_{-i})$ such that $\lambda(a_{-i}^*) \geq p$.

Lemma 4.2. *If a^* is a p -dominant equilibrium of complete information game g , then every incomplete information game $\left\{ \Omega, \pi, (\mathcal{P}_i)_{i=1}^I, (u_i)_{i=1}^I \right\}$ has an equilibrium where a^* is played with probability 1 on the event $C^p(|g|)$.*

The proof of this result is straightforward. The event $C^p(|g|)$ is itself a p -evident event. Consider the modified incomplete information game where each player is constrained to choose according to a^* when he p -believes the event $C^p(|g|)$. Find an equilibrium of that modified game. By construction, a^* is played with probability 1 on the event $C^p(|g|)$. But the equilibrium of the modified game is also an equilibrium of the original game. If a player i p -believes the event $C^p(|g|)$, then he p -believes that other players are choosing a_{-i}^* . But since his payoffs are given by g and a^* is a p -dominant equilibrium, a_i^* must be a best response for player i .

Since every strict Nash equilibrium is a p -dominant equilibrium for some $p < 1$, we immediately have:

Corollary 4.3. *If a^* is a strict Nash equilibrium of complete information game g , then there exists $p < 1$ such that every incomplete information game $\left\{ \Omega, \pi, (\mathcal{P}_i)_{i=1}^I, (u_i)_{i=1}^I \right\}$ has an equilibrium where a^* is played on the event $C^p(|g|)$.*

Thus if we took a sequence of incomplete information games where in the limit payoffs are common knowledge, and close to the limit they are common p -belief (with p close to 1) with ex ante probability close to 1, then payoffs from equilibria of that sequence of incomplete information games must converge to payoffs in the limit game. Monderer and Samet (1989) proved such a lower hemicontinuity result. One can also ask a converse question: what is the relevant topology on information systems such that information systems close to common knowledge information systems deliver outcomes that are close to common knowledge outcomes. Monderer and Samet (1996) and Kajii and Morris (1998) characterize such topologies (for different kinds of information system).

4.4. Robustness to Incomplete Information

Let a^* be a pure strategy Nash equilibrium of complete information game g ; a^* is *robust to incomplete information* if every incomplete information game where payoffs are almost always given by g has an equilibrium where players almost always choose a^* (Kajii and Morris [KM] (1997a)).¹⁵ More precisely, a^* is robust

¹⁵KM define the property of robustness to incomplete information for mixed strategy equilibria also, but most of the sufficient conditions described below apply only to pure strategy profiles. For this reason, we focus on pure strategy profiles in the discussion that follows.

to incomplete information if, for all $\delta > 0$, there exists $\varepsilon > 0$, such that every incomplete information game where $\pi(|g|) \geq 1 - \varepsilon$ has an equilibrium where a^* is played by all players on an event with probability at least $1 - \delta$.

Robustness (to incomplete information) can be seen as a very strong refinement of Nash equilibrium. Kajii and Morris (1997b) provides a detailed account of the relation between robustness and the existing refinements literature, which we briefly summarize here. The refinements literature examines what happens to a given Nash equilibrium in perturbed versions of the complete information game. A weak class of refinements requires only that the Nash equilibrium continues to be equilibrium in *some* nearby perturbed game (Selten's (1975) notion of *perfect equilibria* is the leading example of this class); a stronger class requires that the Nash equilibrium continues to be played in *all* perturbed nearby games (Kohlberg and Mertens' (1986) notion of *stable equilibria* is the leading example of this class). Robustness belongs to the latter, stronger, class of refinements. Moreover, robustness to incomplete information allows an extremely rich set of "perturbed games". In particular, while Kohlberg and Mertens allow only independent action trembles across players, the definition of robustness leads to highly correlated trembles and thus an even stronger refinement. Indeed KM construct an example in the spirit of Rubinstein (1989) to show that even a game with a unique Nash equilibrium, which is strict, may fail to have any robust equilibrium.

Yet it turns out that a large set of games do have robust equilibria. KM provided two sufficient conditions. The first is that if a^* is the unique correlated equilibrium of g , then a^* is robust. The second sufficient condition comes from a generalization of the notion of p -dominance. Fix a vector of probabilities, $\mathbf{p} = (p_1, \dots, p_I)$, one for each player. Action profile a^* is a \mathbf{p} -dominant equilibrium if each player i 's action is a best response whenever he assigns probability at least p_i to his opponents choosing according to a^* , i.e.,

$$\sum_{a_{-i} \in A_i} \lambda(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i} \in A_i} \lambda(a_{-i}) g_i(a_i, a_{-i})$$

for all $i = 1, \dots, I$, $a_i \in A_i$ and $\lambda \in \Delta(A_{-i})$ such that $\lambda(a_{-i}^*) \geq p_i$. If a^* is a \mathbf{p} -dominant equilibrium for some \mathbf{p} with $\sum_{i=1}^I p_i \leq 1$, then a^* is robust to incomplete information. This property is a many player, many action generalization of risk dominance. KM proved this result by showing a surprising property of higher order beliefs. Say that an event is \mathbf{p} -believed (for some vector of probabilities \mathbf{p}) if each player i believes it with probability at least p_i ; and the event is common \mathbf{p} -belief if it is \mathbf{p} -believed, it is \mathbf{p} -believed that it is \mathbf{p} -believed, etc... KM show that if vector \mathbf{p} satisfies $\sum_{i=1}^I p_i \leq 1$, and an event has high probability, then with

high probability that event is common \mathbf{p} -belief. A generalization of Lemma 4.2 then proves the result.

Further sufficient conditions for robustness exploit the idea of potential games due to Monderer and Shapley (1996). A function $v : A \rightarrow \mathbb{R}$ is a *potential function* for complete information game g , if

$$v(a_i, a_{-i}) - v(a'_i, a_{-i}) = g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})$$

for all $i = 1, \dots, I$, $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$. This property implies that the game g has identical mixed strategy best response correspondences to the common interest game with common payoff function v . Observe that a^* is thus a Nash equilibrium of g if it is a local maximizer of v , i.e., it is not possible to increase v by changing one player's action. Monderer and Shapley suggested if a game has multiple Nash equilibria, the global maximizer of v (which must of course be a local maximizer and thus a Nash equilibrium) is a natural candidate for selection. If action profile a^* is the strict maximum of a potential function v for complete information game g , we say that a^* is potential maximizer of g . Ui (2000a) shows that a potential maximizing action profile is necessarily robust to incomplete information.¹⁶ Many player, two action, symmetric payoff games are potential games, so this result provides a proof that the strategy profile where all players choose the Laplacian action is robust to incomplete information.¹⁷

The p -dominance sufficient conditions and potential game sufficient conditions for robustness can be unified and generalized. We very briefly sketch the main ideas and refer the reader to Morris (1999) for more details. Action profile a^* is a *characteristic potential maximizer* of the complete information game g if there exists a function $v : 2^{\{1, \dots, I\}} \rightarrow \mathbb{R}$ with $v(\{1, \dots, I\}) > v(S)$ for all $S \neq \{1, \dots, I\}$, and $\mu_i : A_i \rightarrow \mathbb{R}_+$ such that for all i , $a_i \in A_i$ and $a_{-i} \in A_{-i}$,

$$v(\{j : a_j = a_j^*\}) - v(\{j : a_j = a_j^*\} \cup \{i\}) \geq \mu_i(a_i)(g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i}))$$

Here, $v(\cdot)$ is a potential function that depends only on the set of players choosing according to a^* . In this sense, the characteristic potential maximizer condition strengthens the potential maximizer condition. But the earlier equalities are replaced with inequalities and the constants μ_i also add extra degrees of freedom. So the characteristic potential maximizer condition neither implies nor is implied by the potential maximizer condition. Any characteristic potential maximizing action profile is robust to incomplete information. One can use duality arguments

¹⁶Ui uses a slightly weaker version of robustness to incomplete information, where all types in the perturbed game *either* have payoffs given exactly by the complete information game g *or* have a dominant strategy to choose some action.

¹⁷Morris (1997) earlier provided an independent argument showing the robustness of the Laplacian strategy profile.

to show that if a^* is a \mathbf{p} -dominant equilibrium for some \mathbf{p} with $\sum_{i=1}^I p_i \leq 1$, then a^* is a characteristic potential maximizer.¹⁸

Let the actions of each player be ordered, and for any action $a_i \in A_i$, write a_i^- for the action below a_i and a_i^+ for the action above a_i . Action profile a^* is a *local potential maximizer* of the complete information game g if there exists a local potential function $v : A \rightarrow \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i , $\mu_i : A_i \rightarrow \mathbf{R}_+$, such that for all $i = 1, \dots, I$ and $a_{-i} \in A_{-i}$,

$$v(a_i, a_{-i}) - v(a_i^-, a_{-i}) \geq \mu_i(a_i) \begin{bmatrix} g_i(a_i, a_{-i}) \\ -g_i(a_i^-, a_{-i}) \end{bmatrix} \text{ if } a_i > a_i^* \quad (4.2)$$

and $v(a_i, a_{-i}) - v(a_i^+, a_{-i}) \geq \mu_i(a_i) \begin{bmatrix} g_i(a_i, a_{-i}) \\ -g_i(a_i^+, a_{-i}) \end{bmatrix} \text{ if } a_i < a_i^*$

One can show that if a^* is a local potential maximizer, then a^* is both a potential maximizer and a characteristic potential maximizer. Thus it generalizes both conditions. If a^* is a local potential maximizer of g , and g satisfies strategic complementarities and each $g_i(a_i, a_{-i})$ is concave with respect to a_i , then a^* is robust to incomplete information. The following two player, three action, symmetric payoff game satisfies the strategic complementarity and concavity conditions and one can show that $(0, 0)$ is the local potential maximizer and thus robust (the earlier conditions do not help to characterize robustness in this example).

	0	1	2
0	4, 4	0, 0	-6, -3
1	0, 0	1, 1	0, 0
2	-3, -6	0, 0	2, 2

In fact, the local potential maximizer condition can be used to characterize the unique robust equilibrium in generic two player, three action, symmetric payoff games.

4.5. Noise Independent Selection

If an action profile is robust to incomplete information, we know that - roughly speaking - any way that a “small” amount of incomplete information is added cannot prevent that action profile being played in equilibrium. This observation has important implications for global games. Consider a global game where payoffs depend continuously on a random parameter θ (which could be multidimensional)

¹⁸Ui (2000b) extends these ideas with a set based notion of robustness to incomplete information.

and each player observes a noisy signal $x_i = \theta + \sigma \varepsilon_i$. If a^* is a robust equilibrium of the game being played at θ^* , then there will always be an equilibrium of the global game (for small σ) where action profile a^* is almost always played whenever all players observe signals close to θ^* . In other words, there will be no way of adding noise that will prevent action profile a^* being played in the neighbourhood of θ^* in some equilibrium. Thus *if* there is limit uniqueness (say, because there are strategic complementarities and the other assumptions of Frankel, Morris and Pauzner (2000) are satisfied) then a^* must be played in the unique limit for *every* noise distribution. In the language of section 4.1, a^* must be the noise independent selection.

Here is a heuristic argument for the above claim. Fix θ^* and let a^* be a Nash equilibrium of the complete information game at θ^* that is robust to incomplete information. By definition, if a^* is robust to incomplete information in game $u(\cdot, \theta^*)$ if every incomplete information game where payoffs are almost always given by $u(\cdot, \theta^*)$ has an equilibrium where a^* is almost always played. Generically, it will also be true that every incomplete information game where payoffs are almost always *close to* $u(\cdot, \theta^*)$ will have an equilibrium where a^* is almost always played. But now consider an incomplete information game where some types of each player have payoffs close to $u(\cdot, \theta^*)$ (“sane” types), while some types may have very different payoffs (“crazy” types). Suppose that conditional on any player being sane, with probability close to 1 he assigns probability close to 1 to all other players being sane. Now the robustness arguments described above could be adapted to show that this incomplete information game has an equilibrium where, conditional on all players being sane, a^* is almost always played.

Now return to the global game and write $B(\theta^*, \delta)$ for a δ ball around θ^* , i.e., the set of θ within Euclidean distance δ of θ^* . For a generic choice of θ^* , a^* will remain robust to incomplete information close to θ^* , i.e., at all $\theta \in B(\theta^*, \delta)$ for some sufficiently small $\delta > 0$. Now consider a sequence of global games where we let the noise go to zero, i.e., $\sigma \rightarrow 0$. For fixed δ and fixed $q < 1$, we can choose σ sufficiently small such that conditional on a player observing a signal in $B(\theta^*, \delta)$, with probability at least q he will assign probability at least q to all other players observing signals within $B(\theta^*, \delta)$. Labelling the types who observe signals in $B(\theta^*, \delta)$ “sane” and types who observe signals not in $B(\theta^*, \delta)$ “crazy”, the above argument shows that there is an equilibrium where a^* is almost always played in a neighbourhood of θ^* .¹⁹

¹⁹There is a technical problem formalizing this argument. The robustness analysis described in the section 4.4 was carried out in discrete state spaces, where existence of equilibrium in incomplete information games is never a problem. In the uncountable state space setting of global games, it would be necessary to impose extra assumptions to ensure existence.

5. Related Models: Local Heterogeneity and Uniqueness

There are a number of ways that adding local heterogeneity to a population of players can remove multiplicity. In this section, we will attempt to give some intuition for a general logic at work. We start with a familiar example.

There are two players, 1 and 2, and each player i has a payoff parameter x_i . Expected payoffs are given by the following matrix:

	Invest	NotInvest	
Invest	x_1, x_2	$x_1 - 1, 0$	(5.1)
NotInvest	$0, x_2 - 1$	$0, 0$	

If there was common knowledge that $x_1 = x_2 = x \in (0, 1)$, then there would be multiple strict Nash equilibria of the complete information game. Since both pure strategy equilibria are strict, they seem quite stable. It seems surprising that an apparently “small” perturbation could remove either equilibrium.

But now let x be a publicly observed random variable and let $x_1 = x_2 = x$. Let players be restricted to switching strategies, so that player i will invest if his payoff parameter exceeds some cutoff k_i , and not invest otherwise. Thus player i 's strategy is parameterized by a number k_i . Since the game is symmetric, we can write $b^*(k)$ to the optimal cutoff of any player if he expects his opponent to choose cutoff k . Clearly, we have

$$b^*(k) = \begin{cases} 0, & \text{if } k \leq 0 \\ k, & \text{if } 0 \leq k \leq 1 \\ 1, & \text{if } 1 \leq k \end{cases}$$

This function is plotted in figure 5.1 below.

Symmetric equilibria will exist when this best response function crosses the 45° line. So there are a continuum of equilibria: for any $x \in [0, 1]$, there is an equilibrium where each player follows a switching strategy with cutoff x .

If we perturb this best response function, we would expect there to be a finite number of equilibria, i.e., a finite number of points where the function b^* crosses the 45° line. Given the shape of the best response function, it does not seem surprising that there might be natural ways of perturbing the best response function so that there is a unique equilibrium.

The two player example of section 2.1 represented one way of carrying out such a perturbation. There it was assumed that there was a payoff parameter θ and each player i observed a noisy signal $x_i = \theta + \sigma\varepsilon_i$. The payoffs in the above matrix (5.1) then represent the *expected* payoffs of the players, given their signals. Recall that a player observing signal x_i will believe that his opponent's signal x_j is distributed normally with mean x_i and standard deviation $\sqrt{2}\sigma$. If

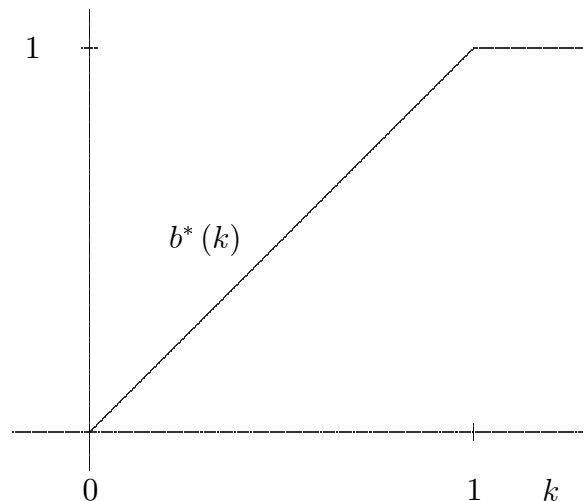


Figure 5.1: Function $b^*(k)$

$\sigma = 0$ in that example, so there is no noise in the signal, we have exactly the scenario described above with best response function b^* . But if $\sigma > 0$, then the best response function rotates clockwise a little bit and crosses the 45° line only at $\frac{1}{2}$ (see figure 2.1) and there is a unique equilibrium.

However, this argument does not really rely on the incomplete information interpretation. The important feature of the argument is the *local heterogeneity* in payoffs: a player with payoff parameter x_i knows that he is interacting with other player(s) who have some perhaps different but nearby payoff parameters; and he knows that those other player(s) in turn know that they are interacting with other player(s) who have some perhaps different but nearby payoff parameters. In the remainder of this section, we will see how a similar logic to the global game argument can arise when players are interacting not with unknown types of an opponent, but with (known) opponents at different locations or at different points in time.²⁰²¹

²⁰This logic also emerges in the the models of Carlsson (1991) and Carlsson and Ganslandt (1998) where players' continuous action choice is subject to a small heterogeneous tremble. The exact connection to global games is not known.

²¹A distinctive feature of these arguments relying on *local* heterogeneity is that a very small amount of heterogeneity is sufficient to imply unique equilibrium in environments where there are multiple strict equilibria without heterogeneity. One can also sometimes obtain uniqueness results assuming global not local heterogeneity, i.e. assuming that each player or type has the same but sufficiently diffuse beliefs about other players or types' payoff parameters. Such global heterogeneity uniqueness arguments rely on the existence of a sufficiently *large* amount

5.1. Local Interaction Games

A continuum of players are evenly distributed on the real line. If a player does not invest, his payoff is 0. If he invests, his payoff is $x + l - 1$, where x is his location and l is a *weighted* average of the proportion of his neighbours investing. In particular, let $f(\cdot)$ be the density of a standard normal distribution with mean 0 and standard deviation $\sqrt{2}\sigma$; a player puts weight $f(z)$ on the actions of players at location $x + z$.

This setup describes a game among a continuum of players. The analysis of this game is identical to the analysis of the continuum player example of section 2.1. In particular, players at locations less than $\frac{1}{2}$ will not invest and players at locations above $\frac{1}{2}$ will invest. This is despite the fact that if players were only interacting with people at the exact same location (i.e., $\sigma = 0$), there would be multiple equilibria at all locations between 0 and 1.

This rather stylized game illustrates the possibility that in local interaction games, play at some locations may be influenced by play at distant locations via the structure of local interaction. A literature on local interaction games has examined this type of effect²². To understand the connection a little better, imagine a local interaction game where payoffs depend in a non-linear way on location. Thus let the payoff to investing be $\psi(x) + l - 1$ (instead of $x + l - 1$). Furthermore, suppose that $\psi(x) < \frac{1}{2}$ for all x and that $\psi(x) < 0$ for some open interval of values of x . For small σ , this game will have a unique equilibrium where no player ever invests. To see why, note that for sufficiently small σ , players inside the open interval where $\psi(x) < 0$ will have a dominant strategy to not invest. But now players close to the edge of that interval will have about $\frac{1}{2}$ their neighbours within that interval, and thus (since $\psi(x) < \frac{1}{2}$ always) will not invest in equilibrium. This argument will iterate to ensure that no investment takes place anywhere.

This argument has very much the flavor of the contagion argument in Ellison (1993). There, a population with constant payoffs interacts with near neighbours on a line. Players choose best responses to some average behavior of their neighbours. But a low rate of mutations ensures small neighbourhoods where each action is played with periodically arise randomly. Once a risk dominant action is played in a small neighbourhood, it will tend to spread to the whole population under the best response dynamics. The initial mutant region where the risk dom-

of heterogeneity. See Baliga and Sjöström (2000) in an incomplete information context (where global heterogeneity corresponds to independent types); Herrendorf, Valentinyi and Waldmann (2000) and Glaeser and Scheinkman (2000) in models of large population interactions; and Frankel (2000) in the context of a dynamic model with payoff shocks.

²²E.g. Blume (1995), Ellison (1993), Young (1998). See Glaeser and Scheinkman (2000) for a recent survey.

inant action is played plays much the same role as the dominant strategy region in the story described above. In this setting with strategic complementarities, best response dynamics mimic iterated deletion of strictly dominated strategies. Morris (1997) describes more formally an exact relationship between a version of Rubinstein’s (1989) e-mail game and a version of Ellison’s contagion effect and describes more generally an exact equivalence between games of incomplete information and local interaction games.²³

The connection between games of incomplete information and local interaction games can be exploited. In evolutionary models, local interaction leads to much faster convergence to stochastically stable states than global interaction, because of the contagious dynamics. But there is a very close connection between which action will spread contagiously in a local interaction game and which action will be played in the limit in a global game. In particular, recall from section 4.1 that some games have a noise independent selection, i.e., an action profile played in the limit of a global game, independent of the noise structure; while in other games, the action played in the limit depends on the noise structure. Translated to a local interaction setting, this result implies that some games will have the same action tend to spread contagiously, independent of the structure of interaction, while in other games fine details of the local interaction structure will determine which action is contagious (see Morris (1999) for details). Thus local interaction may not just speed up convergence to stochastically stable states but may change the stochastically stable states in subtle ways.²⁴

5.2. Dynamic Games

5.2.1. Dynamic Payoff Shocks

A continuum of players each live for an instant of time. If a player does not invest, his payoff is 0. If he invests, his payoff is $x + l - 1$, where x is the date at which he lives and l is a *weighted* average of the proportion of players investing at other points in time. In particular, let $f(\cdot)$ be the density of a standard normal distribution with mean 0 and standard deviation $\sqrt{2}\sigma$; a player puts weight $f(z)$ on the actions of players living at date $x + z$.

This setup describes a game among a continuum of players. The analysis of this game is identical to the analysis of the continuum player example of section 2.1 and thus also the local interaction example of the previous section. In particular,

²³Hofbauer (1998, 1999) introduces an approach to equilibrium selection in a local interaction environment. His “spatially dominant equilibria” seem to coincide with those that are robust to incomplete information.

²⁴Morris (2000) also exploits techniques from the higher order beliefs literature to prove new results about local interaction.

players will not invest before date $\frac{1}{2}$ and will invest after date $\frac{1}{2}$. This is despite the fact that if players were only interacting with people making contemporaneous choices (i.e., $\sigma = 0$), there would be multiple equilibria at all dates between 0 and 1.

This was a very stylized example. But the logic is quite general. In many dynamic strategic environments where choices are made at different points in time, a player's payoff may depend not only on contemporaneous choices but also on choices made by other players at other times. Payoff conditions may be varying through time. Thus players' optimal choices may depend indirectly on environments where payoffs are very different from what they are now. These features may allow us to identify a unique equilibrium. Below, we discuss two approaches that exploit this logic.²⁵

One approach has been developed recently in Burdzy, Frankel and Pauzner (1999), Frankel and Pauzner (1999) and Frankel (2000a).²⁶ A continuum of players are periodically randomly matched in a two player two action game. For simplicity, we can think of them playing the investment game described in matrix (2.1). But assume that the publicly observed common payoff parameter θ evolves through time according to some random process (a random walk in Burdzy, Frankel and Pauzner (1999), a continuous Brownian motion in Frankel and Pauzner (1999)). Furthermore, suppose that each player can only occasionally alter his behavior: revision opportunities arrive according to a Poisson process and arrive slowly relative to changes in the game's payoffs. Under certain conditions on the noise process (roughly equivalent to the sufficiently uniform prior conditions in global games), there is a unique equilibrium where each player invests when θ exceeds $\frac{1}{2}$ and not when θ is less than $\frac{1}{2}$.

This description considerably over-simplifies the analysis. For example, it is natural to assume that players observe the public evolution of θ , so they will be able to infer at any point in time (even if they cannot observe) the proportion of players taking each action. This creates an extra state variable (relative to the global games analysis) and the resulting asymmetry between the past and future complicates the analysis. Nonetheless, the logic is similar to the stylized example described above. In particular, note how the friction in revision opportunities exactly ensures that a player making a choice given some publicly observed θ will take into account the choices that others will make at different times with different

²⁵Morris (1995) describes a third approach. Suppose that players are deciding whether to invest or not invest at different points in time, but they make their decisions in private and their watches are not synchronized. Thus each player will believe that the time on any other player's watch is close to his own, but not identical. Risk dominant play may result even when perfect synchronization would have allowed multiple equilibria.

²⁶See also Frankel and Pauzner (2000) and Levin (2000a) for applications following this approach.

publicly observed θ .²⁷

Levin (2000a) describes another approach that is closer to the stylized example described above. At discrete time t , player t chooses an action. His payoff may depend on the actions of players choosing before him or the player choosing after him, but also depends on a payoff parameter θ . The payoff parameter is publicly observed and evolves according to a random walk. If players act as if they cannot influence or do not care about the action of the decision maker in the next period, then under weak monotonicity conditions (a player's best response is increasing in others' actions and the payoff parameter) and limit dominance conditions (the highest (lowest) action is a dominant strategy for sufficiently high (low) values of θ) there is a unique equilibrium. The no influence assumption makes sense if there are in fact a continuum of players at each date or if actions are observed only with a sufficiently long lag. In Matsui's (1999) currency crisis model, there are overlapping generations of players but there is a natural reason why players do not care about the actions of players preceding them.²⁸

5.2.2. Recurring Incomplete Information

Let θ_t follow a random walk, with $\theta_t = \theta_{t-1} + \eta_t$, where each η_t is independently normally distributed with mean 0 and standard deviation τ . In period t , θ_{t-1} is publicly observed but θ_t is only observed with noise. In particular, each player i observes $x_{it} = \theta_t + \varepsilon_{it}$, where each ε_{it} is independently normally distributed with mean 0 and standard deviation σ . In each period, a continuum of players decide whether to invest with linear payoffs depending of θ_t (the payoff to not investing is 0, the payoff to investing is $\theta_t + l - 1$ where l is the proportion of the population investing).

This dynamic game represents a crude way of embedding the static global games analysis in a dynamic setting. In particular, each period's play of this dynamic game can be analyzed independently and is exactly equivalent to the public signals model of section 3. In particular, θ_{t-1} is the public signal about θ_t while x_{it} is player i 's private signal. A unique equilibrium will exist in this

²⁷Matsui and Matsuyama (1995) earlier analyzed a model with Poisson revision opportunities. However, they assumed that the same game was being played through time (i.e., θ was constant) but examined the stability of different population states. The state where the whole population plays the risk dominant action can be reached in equilibrium from the state where the whole population plays the risk dominated action, but not vica-versa. Hofbauer and Sorger (1999) show that the potential maximizing action of (many action) symmetric potential games tends to be played in the Matsui-Matsuyama environment. Oyama (2000) shows that the $\frac{1}{2}$ -dominant equilibrium is selected in this context. In a private communication, Hofbauer has reported that it also selects the "local potential maximizing action" (see section 4.4) in two player, three action games with strategic complementarities and symmetric payoffs.

²⁸See also Frankel (2000b) on the relationship between some of these models.

dynamic game exactly if $\tilde{\gamma}(\sigma, \tau) \leq 2\pi$ (i.e., σ is small relative to τ). In Morris and Shin (2000), we sketch a continuous time version of this recurring incomplete information model and derive the continuous time sufficient conditions for uniqueness.

In Morris and Shin (1999a), we discuss such a recurring incomplete information model of currency crises. One distinctive implication of that analysis is that by the publicity effect, the previous period's fundamentals may be expected to have a disproportionate influence on current outcomes. Thus for any given actual level of fundamentals, an attack on the exchange rate is more likely when the fundamentals have just risen.

Chamley (1999) considers a richer global game model with recurring incomplete information. A large population of players play a coordination game in each period, but each player has a private cost of taking a risky action that evolves through time. There is correlation in private costs, and dominance regions, so that each period's coordination game has the structure of a global game. But past action conveys information about other players' private costs and thus (because of persistence) their current costs. Chamley identifies sufficient conditions for uniqueness in all periods and discusses a variety of applications.

5.2.3. Herding

In the herding models of Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992), players sequentially make some discrete choice. Players do not care about each other's actions directly, but players have private information, and so each player may partially learn the information of players who choose before him. But if a number of early moving players happen to observe signals favoring one action, late moving players may start ignoring their own private information, leading to inefficient herding because of the negative informational externality.

Herding models share with global game models the feature that outcomes are highly sensitive to fine details of the information structure. However, it is important to note that the mechanisms are quite different. The global games analysis is driven by payoff complementarities and the highly correlated signals generated by the noisy observations technology. However, the sensitivity to the information structure arises in a purely static setting. The herding stories have no payoff complementarities and simple information structures, but rely on sequential choice.

Dasgupta (2000a) analyzes a simple model where it is possible to see both kinds of effects at work. A finite set of players decide sequentially (in an exogenous order) whether to invest or not. Investment conditions are either bad (when each player has a dominant strategy to not invest) or good (in which case it pays to

invest *if all other players invest*). Each player observes a signal from a continuum, with high signals implying a higher probability that investment conditions are good. All equilibria in this model are switching equilibria: each player invests only if all previous players invested *and* his private signal exceeds some cutoff. Such equilibria encompass herding effects: previous players' decisions to invest convey positive information to later players and make it more likely that they will invest. They also encompass higher order belief effects: an increase in a player's signal makes it more likely that he will invest both because he thinks it more likely that investment conditions are good *and* because he thinks it more likely that later players will observe high signals and choose to invest.²⁹

6. Conclusion

Global games rest on the premise that the information received by economic agents are informative, but not so informative so as to achieve common knowledge of the underlying fundamentals. Indeed, as the information concerning the fundamentals become more and more accurate, the actions elicited in equilibrium resemble behavior when the uncertainty concerning the actions of other agents become more and more diffuse. This points to the potential pitfalls if we rely too much on our intuitions that are based on complete information games which allow perfectly coordinated switching of beliefs and actions. Decentralized decision making in market environments cannot be relied on to rule out inefficient outcomes, so that there may be room for policies that mitigate the inefficiencies. The analysis of economic problems using the methods from global games is in its infancy, but the method seems promising.

Global games also present a 'user-friendly' face of games with incomplete information in the tradition of Harsanyi. The potentially daunting task of forming an infinite hierarchy of beliefs over the actions of all players in the game can be given a representation in terms of beliefs (and the behavior that they elicit) which are simple to the point of being naive. Global games go some way to bridging the gap between those who believe that rigorous game theory has a role in economics (as we do) and those who insist on tractable and usable tools for applied economic analysis.

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²⁹For other models combining elements of payoff complementarities and herding, see Chari and Kehoe (2000), Corsetti, Dasgupta, Morris and Shin (2000), Jeitshcko and Taylor (1999) and Marx (2000).

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Appendix A: PROOF OF PROPOSITION 2.2.

We will prove the first half of the result ($s(x) = 0$ for all $x \leq \theta^* - \delta$). The second half ($s(x) = 0$ for all $x \leq \theta^* - \delta$) follows by a symmetric argument. For any given strategy profile $\mathbf{s} = \{s_i\}_{i \in [0,1]}$, we write $\zeta(x)$ for the proportion of players observing signal x who choose action 1; $\zeta(\cdot)$ will always be a continuous function of x .

Write $\pi_\sigma(x, k)$ for the highest possible expected payoff gain to choosing action 1 for a player who has observed a signal x and knows that all other players will choose action 0 if they observe signals less than k .

$$\pi_\sigma(x, k) \equiv \max_{\{\zeta: \zeta(x)=0 \text{ for all } x < k\}} \frac{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) \pi\left(1 - F\left(\frac{k-\theta}{\sigma}\right), \theta\right) d\theta}{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta} \quad (6.1)$$

Lemma 6.1. *There exists $\underline{x} \in \mathbb{R}$ and $\bar{\sigma}_1 \in \mathbb{R}_{++}$ such that $\pi_\sigma(x, k) < 0$ for all $\sigma \leq \bar{\sigma}_1$, $x \leq \underline{x}$ and $k \in \mathbb{R}$.*

PROOF. By property A4*, we can choose $\underline{x} < \underline{\theta}$ and a continuously differentiable function $\bar{\pi} : \mathbb{R} \rightarrow \mathbb{R}$ with $\bar{\pi}'(\theta) = 0$ and $\bar{\pi}(\theta) = -\varepsilon$ for all $\theta \leq \underline{x}$ such that

$$\pi(l, \theta) \leq \bar{\pi}(\theta) \leq -\varepsilon$$

for all $l \in [0, 1]$ and $\theta \in \mathbb{R}$. Now let

$$\begin{aligned} \bar{\pi}_\sigma(x) &\equiv \frac{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) \bar{\pi}(\theta) d\theta}{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta} \\ &= \frac{\int_{z=-\infty}^{\infty} p(x + \sigma z) f(-z) \bar{\pi}(x + \sigma z) dz}{\int_{z=-\infty}^{\infty} p(x + \sigma z) f(-z) dz}, \text{ changing variables to } z = \frac{\theta - x}{\sigma} \end{aligned}$$

Clearly, $\bar{\pi}_\sigma(x)$ is an upper bound on $\pi_\sigma(x, k)$ for all k . Observe that $\bar{\pi}_\sigma(x)$ is continuous in σ ; also $\bar{\pi}_0(x) = \bar{\pi}(x)$ so $\bar{\pi}_0(x) = -\varepsilon$ for all $x \leq \underline{x}$. Also observe

that

$$\begin{aligned}
\left. \frac{d\bar{\pi}_\sigma}{d\sigma}(x) \right|_{\sigma=0} &= \frac{\left[\int_{z=-\infty}^{\infty} p(x+\sigma z) f(-z) dz \right] \left[\int_{z=-\infty}^{\infty} z f(-z) (p'(x+\sigma z) \bar{\pi}(x+\sigma z) + p(x+\sigma z) \bar{\pi}'(x+\sigma z)) dz \right]}{\left[\int_{z=-\infty}^{\infty} p(x+\sigma z) f(-z) dz \right]^2} \\
&\quad - \frac{\left[\int_{z=-\infty}^{\infty} z f(-z) p'(x+\sigma z) dz \right] \left[\int_{z=-\infty}^{\infty} p(x+\sigma z) f(-z) \bar{\pi}(x+\sigma z) dz \right]}{\left[\int_{z=-\infty}^{\infty} p(x+\sigma z) f(-z) dz \right]^2} \Bigg|_{\sigma=0} \\
&= \left[\int_{z=-\infty}^{\infty} z f(-z) dz \right] \frac{\bar{\pi}'(x)}{p(x)}
\end{aligned}$$

Thus by (A6), $\frac{d\bar{\pi}_\sigma}{d\sigma}(x) = 0$ for all $x \leq \underline{x}$. Thus there exists $\bar{\sigma} \in \mathbb{R}_{++}$ such that $\bar{\pi}_\sigma(x) < 0$ for all $\sigma \leq \bar{\sigma}$ and $x \leq \underline{x}$. ■

Lemma 6.2. *There exists $\bar{\sigma}_2 \in \mathbb{R}_{++}$ such that $\pi_\sigma(x, k) < 0$ for all $\sigma \leq \bar{\sigma}_2$, $\underline{x} \leq x < \theta^*$ and $x \leq k \leq \theta^*$.*

$$\begin{aligned}
\pi_\sigma(x, k) &= \frac{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) \pi\left(1 - F\left(\frac{k-\theta}{\sigma}\right), \theta\right) d\theta}{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta} \\
&= \int_{l=0}^1 \psi_\sigma(l; x, k) \pi(l, k - \sigma F^{-1}(l)) dl
\end{aligned}$$

where $\psi_\sigma(l; x, k)$ is the density with c.d.f.

$$\begin{aligned}
\Psi_\sigma(l; x, k) &= \frac{\int_{\theta=-\infty}^{k - \sigma F^{-1}(1-l)} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta}{\int_{\theta=-\infty}^{\infty} p(\theta) f\left(\frac{x-\theta}{\sigma}\right) d\theta} \\
&= \frac{\int_{z=\frac{x-k}{\sigma} + F^{-1}(1-l)}^{\infty} p(x - \sigma z) f(z) dz}{\int_{z=-\infty}^{\infty} p(x - \sigma z) f(z) dz}, \text{ changing variables to } z = \frac{x - \theta}{\sigma}
\end{aligned}$$

Thus as $\sigma \rightarrow 0$, $\Psi_\sigma(l; x, x - \sigma\xi) \rightarrow 1 - F(\xi + F^{-1}(1 - l))$. Thus as $\sigma \rightarrow 0$, $\pi_\sigma(x, x - \sigma\xi) \rightarrow \pi_\sigma^*(x, x - \sigma\xi)$ continuously (where π_σ^* is the variable corresponding to a uniform prior derived in the text). We know that $\pi_\sigma^*(x, x - \sigma\xi) > 0$ for the required values of x and ξ . Since we are interested in values of x in the closed interval $[\bar{x}, \theta^*]$ and since varying ξ generates a compact set of distributions over l , convergence is uniform. ■

Appendix B: THE FINITE PLAYER CASE

As we noted in the linear example of section 2.1, analysis of the continuum and finite players can follow similar methods. Here we briefly note how to extend the uniform prior private values analysis of proposition 2.1 to the finite player case. The extension of the general prior common values analysis of proposition 2.2 is then straightforward.

The setting is as in the section 2.2.1, except that there are now $I \geq 2$ players, and the noise terms in the private signals are identically and independently distributed according to the density $f(\cdot)$. As before, $\pi(l, x)$ is the payoff gain to choosing action 1 rather than action 0, if you have observed signal x and proportion l of your opponents choose action 1. Of course, now (since you have $I - 1$ opponents) l will always be an element of the set $\{0, \frac{1}{I-1}, \frac{2}{I-1}, \dots, 1\}$. Property A3 becomes:

A3(I): **I -Player Single Crossing:** There exists a unique θ_I^* solving $\sum_{k=0}^{I-1} \frac{1}{I} \pi\left(\frac{k}{I-1}, \theta_I^*\right) = 0$.

Observe that as $I \rightarrow \infty$, $\theta_I^* \rightarrow \theta^*$ (i.e., the θ^* of assumption A3). In the special case where $I = 2$, this reduces to $\frac{1}{2}\pi(0, \theta_2^*) + \frac{1}{2}\pi(1, \theta_2^*) = 0$; in other words, θ_2^* is the point where the *risk dominant* action (Harsanyi and Selten (1988)) switches from 0 to 1. Proposition 2.1 remains true as stated for the finite player game, with θ_I^* replacing θ^* (this was essentially shown by Carlsson and van Damme (1993b)). The key step in the proof is showing that in a symmetric strategy profile, each player has uniform beliefs over the proportion of players observing a higher signal. To see why this is true, note that the probability that a player observing signal x assigns to exactly proportion $\frac{n}{I-1}$ of his opponents signal greater than k is

$$\int_{\theta=-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \binom{I-1}{I-1-n} \left[F\left(\frac{k-\theta}{\sigma}\right)\right]^{I-1-n} \left[1 - F\left(\frac{k-\theta}{\sigma}\right)\right]^n d\theta$$

where $F(\cdot)$ is the c.d.f. of $f(\cdot)$. Letting $x = k - \sigma z$ and carrying out the change of variables $\xi = \frac{k-\theta}{\sigma}$, this expression becomes

$$\int_{\xi=-\infty}^{\infty} f(\xi - z) \binom{I-1}{I-1-n} [F(\xi)]^{I-1-n} [1-F(\xi)]^n d\xi$$

This expression is now independent of σ and k , so we may denote this expression by $\psi^I\left(\frac{n}{I-1}; z\right)$. For the same argument to work as in the continuum case, it is enough to show that $\psi^I(\cdot; 0)$ is the uniform distribution. But integration by parts gives:

$$\begin{aligned} \psi^I\left(\frac{n}{I-1}; 0\right) &= \binom{I-1}{I-1-n} \int_{\xi=-\infty}^{\infty} f(\xi) [F(\xi)]^{I-1-n} [1-F(\xi)]^n d\xi \\ &= \binom{I-1}{I-n} \int_{\xi=-\infty}^{\infty} f(\xi) [F(\xi)]^{I-n} [1-F(\xi)]^{n-1} d\xi \\ &= \dots \\ &= \int_{\xi=-\infty}^{\infty} f(\xi) [F(\xi)]^{I-1} d\xi \\ &= \frac{1}{I}. \end{aligned}$$