Inside and Outside Money, Gains to Trade, and IS–LM

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Abstract

We build a one-period general equilibrium model with money. Equilibrium exists, and fiat money has positive value, as long as the ratio of outside money to inside money is less than the gains to trade available at autarky. We show that the nominal effects of government fiscal and monetary policy can be completely described by a diagram identical in form to the IS–LM curves introduced by Hicks to describe Keynes' general theory. IS–LM analysis is thus not incompatible with full market clearing, multiple commodities, and heterogeneous households. We show that as the government deficit approaches a finite threshold, hyperinflation sets in (prices converge to infinity and real trade collapses). If the government surplus is too large, the economy enters a liquidity trap in which nominal GNP sinks and monetary policy is ineffectual.

1 Introduction

Fiat money is a creature of the state, since nobody else can create it. When the state injects it into the private sector in exchange for assets promising the future delivery of money, its arrival foreshadows its departure, and it is called inside money. Money injected into the private sector as a transfer, or in exchange for a commodity (which gives no claim on future repayment), is called outside money.[1]

There is a longstanding puzzle about how to guarantee that outside money has positive value, often called the Hahn paradox. We argue in this paper that if fiat money is the sole medium of exchange, and if the ratio of outside money to inside money is less than the gains to trade available at autarky, then money must have positive value and a full-fledged monetary equilibrium must exist.

The Hahn paradox arises because households do not want to hold money at the end. Equilibrium models usually rely on one of two devices to overcome it. The first device is to assume there is no last period. (See Samuelson [29] or Grandmont–Younes [18] for infinite horizon models, or Grandmont–Younes [17] or Hool [22] for temporary equilibrium models where the last period is really not the last period,

[1] Fiat money as a creature of the state is taken from the title of an article by Lerner [24]. Our definitions of inside and outside money are taken from Gurley–Shaw [19], and stand in contrast to inside money creation as loans between private parties.

since agents have expectations there about the future value of money.) The second device is to *oblige* some agent, either the government or the households themselves, to sell something valuable for money. This device is used for example when the government or some external agent is postulated to sell commodities for money at prearranged prices. Lerner [24], and later Heller [21]and Balasko–Shell [2], assumed that the government is owed in taxes (payable only in money) precisely the sum of the cash balances of all the households. The government is obliged to offer relief from taxes in exchange for money. Finally, Lucas [26], [27], and a long literature following him, assumes that in each period all agents must sell their entire endowment of commodities for money. Magill–Quinzii [28] make a similar assumption, as do Karatzas–Shubik–Sudderth [23].

There are also models of fiat money in equilibrium that have only inside money (see Shubik–Wilson [30], Cass [5], Balasko–Cass [1], Geanakoplos-Mas-Colell [16], and Dubey–Shapley [14]).

None of these papers connects the existence of monetary equilibrium with a *measure* of the gains to trade available at autarky.[2]

Money has value in our model because the assets (bonds), exchanged for inside money when it is injected into the system, sell for endogenous prices (interest rates). In equilibrium they will promise more than they cost (the interest rate will be positive). When their payoffs are discharged, more money leaves the system than entered, and so the outside money is pulled out along with the inside money. As long as the gains to trade are large enough, households will be anxious to get the money, and will voluntarily agree to pay back more than they borrowed.

We give a novel definition of a scalar measure $\gamma(x)$ of the gains to trade available at an arbitrary allocation of goods x, and characterize it in terms of a cycle of trades.

In our model, government deficits increase the stock of outside money, and expansionary monetary policy increases the stock of inside money. Expansionary monetary policy always lowers interest rates and usually increases price levels. As the stock of inside money goes to infinity, price levels will go to infinity. However, if the stock of inside money is extremely low relative to outside money, expansionary open market operations will lower price levels. Thus price level is a U-shaped function of inside money supply. Government deficits raise interest rates and price levels. As the deficit approaches a *finite* threshold, so that the outside/inside money ratio approaches the gains to trade, hyperinflation sets in where prices go to infinity and trade crashes. If the government runs a large budget surplus, it will push the economy into a liquidity trap where the interest rate is zero, and where small changes in government monetary policy have no effect whatsoever.

We first present these results in a stripped-down one-period model in which a central bank injects a fixed stock of inside money into the economy in exchange for bonds promising money at the end of the period. The Treasury branch of the government does nothing else but give a fixed stock of outside money free and clear

[2] Bewley [3], [4] and Levine [25] link the existence of monetary equilibrium in an infinite horizon model to the precautionary demand for money and therefore to the inefficiency of the allocation without money. They do not offer a measure of gains to trade.

to households, who treat it as part of their endowment.

In our second model, we flesh out the Treasury, giving the government five policy instruments (the stock of bank money, the supply of government bonds, lump sum transfers to households, expenditures on inputs for the production of public goods, and ad valorem taxes). The effects of these policy tools on aggregate nominal variables can be completely described by a graphical framework nearly identical in form to the IS–LM diagram used by Hicks in formalizing Keynes' model. This shows that there is nothing incompatible between IS–LM and full market clearing and rational expectations. We also describe the effects of government policy on welfare, consumption, and on price levels, and show that in terms of real variables, the five policy instruments achieve nothing more than is available by using just two of them (open market operations and government spending on commodities by printing money).

Our IS–LM diagram gives a macroeconomic picture of nominal income and the nominal interest rate in a genuine microeconomic model. It is remarkable that an economy with many heterogeneous consumers and commodities can be faithfully summarized by a two-dimensional diagram. This is possible because we use nominal income, not real income, in our IS–LM equations, and because there is only one period in our model. The heterogeneity of consumer tastes is suppressed, because every consumer wants to spend all his money income at the same time.

The subtlety of monetary equilibrium is attenuated in our model by the restriction to one period without uncertainty. Market clearing in the goods markets, namely that aggregate expenditure equals aggregate income (which Hicks called the IS equation), becomes much less interesting when it loses the investment and savings components from which its name derives. Similarly money market clearing (which Hicks called the LM equation) loses much of its complexity because in a one-period model we can retain only the transactions demand for money, necessarily ignoring the precautionary demand and speculative demand. In companion papers we describe monetary equilibrium in time [10], which introduces savings and investment motives and endogenizes the volatility of money, and equilibrium with uncertainty [9], in which the speculative and precautionary demands for money reappear, and in which a liquidity trap can arise without government surplus. We also combine time and uncertainty in an infinite horizon setting in [11].

A crucial ingredient of our model is the Clower [6] cash-in-advance constraint we put on all transactions. In our model of money and time [10], where the trading rounds can come every nanosecond, this is a much less restrictive assumption, and monetary equilibria can still be shown to exist. Obviously many transactions in the real world are carried out by credit cards and by checks, as well as via money. One of the most important virtues of our model is that by making the transactions technology explicit (rather than subsuming it in a reduced-form utility of money), it becomes straightforward to add credit cards to the model, which we do in [8]. There we find that credit cards do not destroy the value of money (indeed equilibrium with a positive value of money is more likely to exist). On the other hand, credit cards do reduce the purchasing power of money, i.e., they lead to inflation.

The idea of introducing fiat money into general equilibrium via a bank came to our attention in the work of Martin Shubik (see [30]). Our contribution is to combine the bank (which we call inside money) with outside money. We first did so in Dubey–Geanakoplos [7]. Theorems 2 and 3 from this paper, connecting the gains to trade with the existence of monetary equilibrium and showing the local uniqueness of monetary equilibrium, already appeared there. The other nine theorems, as well as a simpler existence proof for Theorem 2, appear here for the first time, including the characterization of the measure $\gamma(x)$ of the gains to trade at an arbitrary allocation x.

2 The Model

Consider an economy in which money is the *sole* medium of exchange. Furthermore, suppose that there is just one round of trade between money and commodities. Since the money receipts from commodity sales come after the round is over, let us add the possibility of borrowing money prior to the trading round and repaying it after. Thus the period is divided into three time intervals: borrowing, trading, and repaying.[3]

2.1 The Underlying Economy

We first analyze a pure exchange economy which has only private goods (commodities) $L = \{1, ..., L\}$. (Later we shall add a government sector and public goods.) The agents in the economy are households $H = \{1, ..., H\}$. Each $h \in H$ has an endowment of commodities $e^h \in \mathbb{R}^L_+$ and a utility of consumption $u^h : \mathbb{R}^L_+ \to \mathbb{R}$. We assume: (a) $e^h \neq 0$ for all $h \in H$, i.e., every household has at least some endowment (e.g., its own labor); (b) $\sum_{h \in H} e^h \gg 0$, i.e., every named commodity is present in the aggregate; (c) u^h is continuous, concave, and strictly increasing in each[4] variable, for all $h \in H$. The underlying economy, which constitutes the real sector of our model, is denoted $\mathcal{E} \equiv (u^h, e^h)_{h \in H}$.

2.2 Money and Bank Loans

Money is fiat and gives no direct utility of consumption to the households; they value money only insofar as it enables them to acquire commodities for consumption. Money enters the economy in two ways: as private endowment $m^h \geq 0$ of household $h \in H$ and as a stock $M > 0$ at a (central) bank. Apart from households, the bank is the only other agent in our model, but it has a passive role. It stands ready to lend M to households at an interest rate that is determined endogenously in equilibrium. Both $m \equiv \{m^h\}_{h \in H}$ and M are exogenously fixed as part of the data of the model. The sum $\bar{m} \equiv \sum_{h \in H} m^h$ constitutes the stock of *outside money*, which households own free and clear of debt, at the start of the economy. The bank stock M is *inside money* and is always accompanied by debt when it comes into households' hands. We

[3] In our companion papers we take up multiple trading rounds, uncertainty, and infinite horizon [9], [10], [11].

[4] *Strict* monotonicity is assumed for ease of presentation, and will be weakened (see Section 3.3).

denote the monetary economy by $(\mathcal{E}, m, M) \equiv ((u^h, e^h, m^h)_{h\in H}, M)$; and its private sector by $(\mathcal{E}, m) \equiv (u^h, e^h, m^h)_{h\in H}$.

The period, as was said, is divided into three time intervals. In the first interval, households borrow money from the bank. In effect, households sell IOU notes or bonds to the bank in exchange for cash. In the second interval, they sell commodities for money and simultaneously buy goods with cash. In the third interval, they repay bank loans with money and consume. Default is not permitted.

All commodity markets meet simultaneously in the second interval. Households are required to pay money to purchase commodities at the different markets.[5] It is only in the third interval, after these markets close, that revenue from the sales of commodities comes into households' hands, by which time it is too late to use this revenue for purchases. Those households who find their endowment m^h of money insufficient will need to borrow money from the bank to finance purchases, and will defray the loan out of their sales revenue.[6]

2.3 Macrovariables: Prices and Quantities

Let $p_\ell > 0$ denote the price of commodity $\ell \in L$ in terms of money, and let $r \geq 0$ denote the money rate of interest on the bank loan. Money is borrowed by selling bonds to the bank. Each bond constitutes a promise to pay 1 dollar after commodity trade. Thus the price before commodity trade of a bond is $1/(1+r)$.

The vector $(p, r) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ will be referred to as "market prices." The price of money is $1/p_\ell$ in terms of commodity ℓ, and $(1+r)$ in terms of the bond. The value of money is reflected by these prices. As $p \to \infty$, money loses all value (in terms of commodities). As $r \to -1$, money-now loses all value (in terms of money-later) and as $r \to \infty$ money-later loses all value (in terms of money-now). In this paper our interest is on p, since it is determined by the interaction of the real sector $\mathcal{E}$ and monetary sector (m, M) of the economy, and not so much on r, which is determined entirely by the monetary sector.[7]

We denote money by m (without confusing it with the vector $m \equiv (m^1, ..., m^H)$ of household endowments) and bonds by b. Since money is the sole medium of exchange, the vector q^h of market actions of household h has $2L+1$ components (where $\ell \in L$):

$q^h_{bm} \equiv$ quantity of bonds sold by h to the bank for money
$q^h_{m\ell} \equiv$ money spent by h to purchase ℓ
$q^h_{\ell m} \equiv$ quantity of ℓ sold by h for money

(It is evident, on account of their being just one period, that no household would improve its consumption by depositing money at the bank to earn interest. So, we suppress deposits, i.e., the purchase of bonds q^h_{mb}.)

[5] In another companion paper, "Credit Cards and Money" [8], we allow households to buy on credit, as well as with cash.

[6] The loans are purely short-term, intraperiod transactions loans. This is on account of the fact that there is only one consumption period in the model. Elsewhere we consider long-term, interperiod loans in a multiperiod model. In general, both kinds of loans involve money and carry weight in a modern-day economy.

[7] In the multiperiod setting, which we study in [10], there is a term structure of interest rates determined by the interaction of the real and monetary sectors, and our focus shifts to both p and r.

By real income q we mean the vector of aggregate commodity sales, with components $q_\ell = \sum_{h\in H} q^h_{\ell m}$. By nominal income we mean the value of real income

$$Y = p \cdot q \equiv \sum_{\ell\in L} \sum_{h\in H} p_\ell q^h_{\ell m}.$$

Notice that income corresponds to sales and not to endowments. Since households are not obliged to sell their endowments, real income is genuinely endogenous. Nominal income appears doubly endogenous, since both p and q are endogenous, but often it can be deduced from monetary considerations alone.

Irving Fisher introduced a famous formula for the velocity of money, v, which in our context becomes

$$(M + \bar{m})v = p \cdot q \equiv Y.$$

In a one-period model the velocity of money is not very interesting. If all the money is spent, then $v = 1$ and nominal income is determined. If some of the money is unspent, v may be less than 1 and Y becomes endogenous. This happens in our model in a liquidity trap (Section 11.2).

2.4 The Budget Set of a Household

We consider the case of a perfectly competitive household sector. Each $h \in H$ regards market prices $(p, r) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$ as fixed (uninfluenced by its own actions). The *budget set* $B(p, r, e^h, m^h)$ consists of all market actions and consumptions $(q^h, x^h) \in \mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+$ that satisfy the budget constraints (1), (2), (5), and (3ℓ), (4ℓ), (6ℓ) for all $\ell \in L$. The residual variables $\tilde{x}^h = \tilde{x}^h(q^h, p)$ and $\tilde{m}^h = \tilde{m}^h(q^h, r)$ are determined automatically by q^h, p, r.

$$\tilde{m}^h \equiv \frac{q^h_{bm}}{1+r} \tag{1}$$

$$\sum_{\ell\in L} q^h_{m\ell} \le m^h + \tilde{m}^h \tag{2}$$

$$q^h_{\ell m} \le e^h_\ell \tag{3ℓ}$$

$$\tilde{x}^h_\ell \equiv \frac{q^h_{m\ell}}{p_\ell} \tag{4ℓ}$$

$$q^h_{bm} \le \Delta(2) + \sum_{\ell\in L} p_\ell q^h_{\ell m} \tag{5ℓ}$$

$$x^h_\ell \le (\Delta 3\ell) + \tilde{x}^h_\ell. \tag{6ℓ}$$

Here $\Delta(\alpha)$ is the difference between the right and left sides of inequality (α). The interpretation is clear: (1) says that household h borrows $\tilde{m}^h$ dollars by promising to pay $q^h_{bm} = (1+r)\tilde{m}^h$ dollars after commodity trade, i.e., by selling q^h_{bm} bonds; (2) says that total money spent on purchases cannot exceed the money on hand, i.e., money endowed plus money borrowed; (3ℓ) says that no household can sell more of any commodity than it is endowed with; (4ℓ) says that households purchase commodities

$\tilde{x}^h$ with money at market prices p; (5) says that we are not permitting default, i.e., every household must fully deliver on its bonds; (6ℓ) says that consumption cannot exceed what a household winds up with after trade.

The budget set describes constraints on the flows of money and commodities that a household may send to market. Implicitly, these flows define changes in the household stocks of money and commodities after trade. The budget set ensures that the stocks are always nonnegative.

2.5 Another View of the Budget Set

We denote the set of *budget-feasible consumptions* for household h by

$$B_C(p, r, e^h, m^h) = \{x^h \in \mathbb{R}^L_+ : \exists q^h \in \mathbb{R}^{2L+1}_+ \text{ with } (q^h, x^h) \in B(p, r, e^h, m^h)\}.$$

Note that B_C is homogeneous in p, m^h; i.e., for any $\lambda > 0$,

$$B_C(\lambda p, r, e^h, \lambda m^h) = B_C(p, r, e^h, m^h).^8$$

We can picture the budget-feasible consumptions for a household h, endowed with money and both goods 1 and 2, in the diagram below:

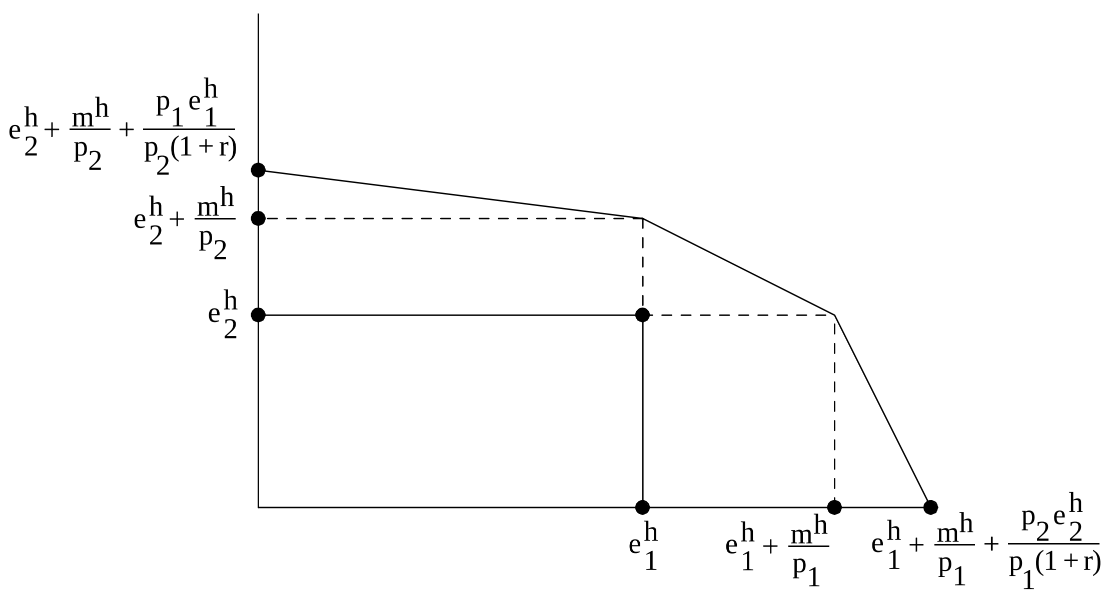

Figure 1: Budget-Feasible Consumptions

For any trade vector $\tau \in \mathbb{R}^L$, and prices $(p, r) \in \mathbb{R}^L_+ \times \mathbb{R}_+$, define the *cost* $C_r(p, \tau)$ of τ by

$$C_r(p, \tau) = p \cdot {}^*\tau - \frac{1}{1+r} p \cdot {}_*\tau$$

[8] Indeed, if $(q^h, x^h) \in B(p, r, e^h, m^h)$, then $(\tilde{q}^h, x^h) \in B(\lambda p, r, e^h, \lambda m^h)$ where $\tilde{q}^h_{\ell m} = q^h_{\ell m}$, $\tilde{q}^h_{bm} = \lambda q^h_{bm}$ and $\tilde{q}^h_{m\ell} = \lambda q^h_{m\ell}$.

where $\cdot$ denotes dot product, and

$$\begin{aligned} {}^*\tau_\ell &= \max\{\tau_\ell, 0\} \\ {}_*\tau_\ell &= -\min\{\tau_\ell, 0\} \end{aligned}$$

give the purchases and sales in $\tau = {}^*\tau - {}_*\tau$.

The cost $C_r(p,\tau)$ discounts the revenue from sales by the interest rate, since money-later (arriving after commodity trade) is worth less to the household than money-now, which can be used for commodity purchases. Using this cost function we can replace the six inequalities describing the budget set with just one, as Lemma 1 shows.

Lemma 1 *Let* $(p,r) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$. *Then for any* $x^h \in \mathbb{R}^L_+$

$$x^h \in B_C(p,r,e^h,m^h) \Leftrightarrow C_r(p,x^h - e^h) \le m^h.$$

Any finite horizon model with outside money, i.e., with positive endowments of fiat money, must somehow dissipate the money through trade. In our model the banking system extracts money every time a household purchases beyond its purely financial wealth m^h. One can imagine other financial institutions which facilitate trade and extract money in other ways. We examine a general model, via an abstract financial cost function $C(p,r,\tau)$, in [13].

2.6 Monetary Equilibrium

A vector of prices and household actions

$$\langle p, r, (q^h, x^h)_{h\in H}\rangle \in \mathbb{R}^L_{++} \times \mathbb{R}_+ \times (\mathbb{R}^{2L+1}_+ \times \mathbb{R}^L_+)^H$$

is a *pre-monetary equilibrium* (preME) of $(\mathcal{E}, m, M)$ if all household actions are in their budget sets, i.e.,

$$(q^h, x^h) \in B(p,r,e^h,m^h) \tag{7}$$

and demand equals supply for the loan market and for all commodity markets, i.e.,

$$\begin{aligned} &\text{(a)} \sum_{h\in H} \tilde{m}^h(q^h, r) = M. \\ &\text{(b)} \sum_{h\in H} \tilde{x}^h_\ell(q^h, p) = \sum_{h\in H} q^h_{\ell m},\ \ell \in L \end{aligned} \tag{8}$$

It is worth noting that in a pre-monetary equilibrium, the total stock of money and commodities held collectively in the hands of the bank and the households is conserved in all three time intervals into which the period is divided. At the start, the bank holds M and households hold $\bar{m}$ of money. Money market clearing (8a) guarantees that the bank stock M flows to households at the end of the first interval. Commodity market clearing (8b) guarantees that the total stock of commodities is conserved and redistributed among the households during the second time interval.

And (8b), multiplied by p_ℓ, shows that the total stock of money is conserved and redistributed among the households during the second time interval. Thus at the end of the first and second intervals, all of $M + \bar{m}$ is with households. The no-default condition (5) implies that the total bonds sold by households do not exceed $M + \bar{m}$. At the end of the third interval in a preME, the bank holds $(1+r)M \leq M + \bar{m}$, and households hold the balance $\bar{m} - rM$.

A preME $\langle p, r, (q^h, x^h)_{h \in H} \rangle$ is a *monetary equilibrium* (ME) iff

$$u^h(x^h) \geq u^h(\underline{x}^h) \text{ for all } (\underline{q}^h, \underline{x}^h) \in B(p, r, e^h, m^h).$$

In any ME, at the end of the third interval, after repaying the bank, no household will be left with unowed cash, otherwise it should have spent more money earlier to purchase commodities, or else curtailed its sale of commodities, improving its utility. Hence at least $M + \bar{m}$ is owed to the bank. But no more could be owed, since default is not permitted. Thus $(1+r)M = M + \bar{m}$ at any ME, i.e., $r = \bar{m}/M$.

This shows that the rate of interest r in (our one-period) monetary equilibrium is determined solely by the stocks of inside and outside money, and is unaffected by the real sector $\mathcal{E}$. In a multiperiod setting [13], there would be a genuine interaction between the real and monetary sectors that determines the interest rates.

In contrast, even with one period, p is determined by a genuine interaction between the real and monetary sectors. Notice that since the components of p at any ME must be finite by definition, money will have positive value at an ME. Thus the existence of an ME is tantamount to a resolution of the Hahn paradox.

3 Gains to Trade

At first glance the cash-in-advance constraint (embodied in (2)) and the presence of the bank seem to provide a way out of the Hahn paradox: the bank, as was said, is an agent that demands money for its own sake, and households will need to hold money at the end in order to repay their loans to the bank. This argument would be fine if we could guarantee that households took out bank loans in the first place. But, unless money already has value to begin with, why should anyone want to take out loans? In a representative agent economy, for instance, nobody would take out loans and money would have no value. Thus the bank, while necessary, does not in and of itself ensure that money will have value. Something more is needed. We show in Theorem 6 that money fails to have value if nothing is added.

One device is to oblige households to put up some positive fraction of their endowment for sale against money (i.e., require in condition (3) of the budget set that $\alpha e^h_\ell \leq q^h_{\ell m} \leq e^h_\ell$ for some $0 < \alpha \leq 1$). Indeed the case when the entire endowment must be put up for sale (i.e., $\alpha = 1$) is considered by Lucas [26], [27], and Magill–Quinzii [28]. Such forced sales, of course, ensure that money will buy something of value in equilibrium (i.e., an ME exists, see Remark 2). But the trouble is that some of these sales *must* be forced. With even the tiniest transactions cost, households would strictly prefer not to sell and buy back the same commodities. For any $\alpha > 0$, if any $e^h \gg 0$, household h would not voluntarily undertake to sell αe^h, for then there would be a commodity ℓ which h would be buying as well as selling. In our model there is no transaction cost; but there is a positive rate of interest at any ME if $\bar{m} > 0$. Households are loath to indulge in wash sales, because they would lose the interest float.

Another device is to introduce a government, ready to defend the sanctity of its fiat money by putting up some exogenous stock of commodities (e.g., gold) for sale against money. By this device we could again get ME without much ado: government sales back the fiat money and guarantee its purchasing power.

We do not have to take recourse to such extraneous and drastic measures as forced sales of commodities, or gold-backed money, in order to guarantee that money has value. What is required is an intrinsic "gains to trade hypothesis."

Fiat money is wanted only for trading commodities. It follows that the value of money should depend on households' motivation to trade commodities with each other. We develop a measure of this motivation called gains to trade and show that, whenever they are strong enough, monetary equilibrium exists. Money is valued and used to move commodities through markets.

Let $\tau^h \in \mathbb{R}^L$ be a trade vector of h (with positive components representing purchases and negative components representing sales). For any scalar $\gamma > -1$, define

$$\tau^h_\ell(\gamma) = \min\{\tau^h_\ell, \tau^h_\ell/(1+\gamma)\}$$

Note $\tau^h_\ell(\gamma) = \tau^h_\ell$ if $\tau^h_\ell < 0$, $\tau^h_\ell(\gamma) = \tau^h_\ell/(1+\gamma)$ if $\tau^h_\ell > 0$. Thus $\tau^h(\gamma)$ entails a diminution of purchases in τ^h by the fraction $\gamma/(1+\gamma)$.

We say that there are *gains to γ-diminished trade* at $x \equiv (x^h)_{h\in H} \in (\mathbb{R}^L_+)^H$ if there exist trades $(\tau^h)_{h\in H}$ such that:

(a) $\sum_{h\in H} \tau^h = 0$

(b) $x^h + \tau^h \in \mathbb{R}^L_+$ for all $h \in H$

(c) $u^h(x^h + \tau^h(\gamma)) > u^h(x^h)$ for all[9] $h \in H$.

In other words, it should be possible — in spite of the "γ-handicap" on trade — for households to Pareto-improve on x. We define $\gamma(x)$ as the supremum of all handicaps that permit Pareto improvement.

Definition The **gains to trade at x** are given by

$$\begin{aligned}\gamma(x) &= \sup\{\gamma : \text{there are gains to } \gamma\text{-diminished trade at } x\} \\ &= \min\{\gamma : \text{there are } \textit{not} \text{ gains to } \gamma\text{-diminished trade at } x\}.\end{aligned}$$

To clarify the definition of gains to trade at e, define the utility functions

$$v^h_\gamma(x) \equiv u^h(e^h + (x - e^h)(\gamma))$$

for $h \in H$. Observe that for fixed γ, every component of $(x - e^h)(\gamma)$ is concave and increasing in x. Since u^h is also concave and increasing, it follows that v^h_γ inherits both these properties. Thus, by standard arguments, the economy $(v^h_\gamma, e^h)_{h\in H}$ has a Walras equilibrium.

[9] Since utilities are strictly monotonic, this is equivalent to requiring that some household is strictly better off and none are worse off.

Lemma 2 *There are no gains to γ-diminished trade at e if and only if the economy $(v_\gamma^h, e^h)_{h\in H}$ has a no-trade Walras equilibrium.*

For smooth economies we can give an explicit formula for $\gamma(x)$. Suppose $x^h \gg 0$ and u^h is continuously differentiable at x^h, for all $h \in H$. With two agents and two goods, $[1+\gamma(x)]^2$ is the ratio of the agents' marginal rates of substitution for the two goods (see Figure 2). It would be incorrect to connect $\gamma(x)$ with the angle between the indifference curves at x in an Edgeworth Box. An important property of $\gamma(x)$ is that rescaling the units of a commodity, say from pounds to ounces, leaves $\gamma(x)$ invariant, though it changes the angle.

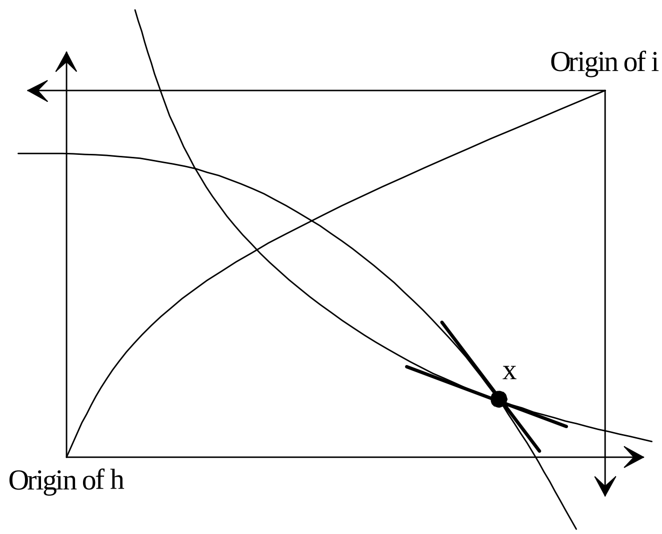

Figure 2: $[1+\gamma(x)]^2$ = ratio of slopes (not the angle between them)

One can check that (if $H = 2$ or $L = 2$),

$$[1+\gamma(x)]^2 = \max_{\substack{h\neq i\\ k\neq \ell}} \left[\left(\frac{\dfrac{\partial u^h(x^h)}{\partial x_k}}{\dfrac{\partial u^h(x^h)}{\partial x_\ell}} \right) \left(\frac{\dfrac{\partial u^i(x^i)}{\partial x_\ell}}{\dfrac{\partial u^i(x^i)}{\partial x_k}} \right) \right].$$

More generally, Pareto improvement may require trade involving more than two households and two commodities. Define a trading cycle c_n as a sequence of distinct commodities $(\ell_1, ..., \ell_n)$ and agents $(h_1, ..., h_n)$ where h_i sells ℓ_i and buys ℓ_{i+1} (where $\ell_{n+1} \equiv \ell_1$). Although each agent h_i trades only a pair of goods, the trading cycle may require many goods and agents because there may be no double coincidence of wants. The next theorem and its corollary show that in calculating the gains to trade available at an allocation x, it suffices to examine trading cycles. Since there are only a finite number of trading cycles, the following formula for the gains to trade can in principle be explicitly calculated.

Theorem 1 *If u^h is continuously differentiable and $x^h \gg 0$ $\forall h \in H$, then*

$$1+\gamma(x) = \max_{2\le n\le L} \max_{c_n \in C_n} \left\{ \bigtimes_{i=1}^{n} \left(\frac{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{\ell_{i+1}}}}{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{\ell_i}}} \right) \right\}^{1/n}$$

where $\ell_{n+1} \equiv \ell_1$ and the second max *is taken over the finite set C_n of all trading cycles $c_n = (\ell_1, ..., \ell_n, h_1, ..., h_n)$ of length n.*[10]

Corollary 1 *Suppose $(u^h)_{h\in H}$ are continuously differentiable and $x^h \gg 0$ $\forall h \in H$. Suppose there are trades $(\tau^h)_{h\in H}$, $\sum_{h\in H} \tau^h = 0$, such that $v_\gamma^h(x^h + \tau^h) > v_\gamma^h(x^h)$ $\forall h \in H$. Then there is a cycle $(\ell_1, ..., \ell_n)$, $(h_1, ..., h_n)$ and trades $\tilde{\tau}^{h_i}$ on the cycle (i.e., $\tilde{\tau}^{h_i}_{\ell_{i+1}} = -\tilde{\tau}^{h_{i+1}}_{\ell_{i+1}} > 0$ for all $i = 1, ..., n$, and $\tilde{\tau}^{h_i}_{\ell} = 0$ if $\ell \neq \{\ell_i, \ell_{i+1}\}$) such that $v_\gamma^{h_i}(x^h + \tilde{\tau}^h) > v_\gamma^{h_i}(x^h)$ $\forall i = 1, ..., n$.*

If utilities are not continuously differentiable, or if some $x_\ell^h = 0$, we can give lower and upper bounds for $\gamma(x)$.

Corollary 2 *Drop the differentiability and interiority assumptions of Theorem 1. Define $\gamma_*(x)$ (or $\gamma^*(x)$) in exactly the same manner as $\gamma(x)$, but with the right-hand derivative in the numerator (or denominator) and the left-hand derivative in the denominator (or numerator). Then $\gamma_*(x) \le \gamma(x) \le \gamma^*(x)$.*

The gains to trade are not given by the distance from x to the Pareto frontier, but as Theorem 1 shows, by the ratio of the slopes of the two indifference curves at x. The number $\gamma(x)$ can be viewed as a local measure of the departure from Pareto optimality. It represents how easy it is to make a *small* Pareto improvement starting from x. If x is Pareto optimal, then $\gamma(x) = 0$; otherwise $\gamma(x) > 0$. Our measure is to be contrasted with a global measure of inefficiency suggested by Debreu, which represents how far from *full* Pareto optimality the allocation is. His coefficient of resource allocation $\delta(x)$ is given by:

$$\begin{aligned} 1+\delta(x) \ = \ \sup \{\lambda : & \text{ there exists a reallocation of } \frac{1}{\lambda}\sum_{h\in H} x^h \\ & \text{which leaves each } h \text{ at utility level at least } u^h(x^h)\}. \end{aligned}$$

Notice that perturbations of utility functions u^h only around x^h will alter our measure $\gamma(x)$, but will have little effect on Debreu's measure $\delta(x)$.

An example makes this local/global distinction transparent. Take $u^h(x_1, x_2) = u^i(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$. For any $x = (x^h, x^i)$, where $x_1^h + 2x_2^h < 2x_1^h + x_2^h$

[10] When the maximum is achieved only on cycles of length at least $n > 2$, there is a failure of double coincidence of wants in the economy. We might even use the minimum length n which achieves the maximum defining $\gamma(x)$ as a measure of this failure. We shall not pursue this point here.

and $x^i_1 + 2x^i_2 > 2x^i_1 + x^i_2$, utilities are differentiable and the local gains to trade are quite steep. By our formula, $\gamma(x) = 1$, since $[1+\gamma(x)]^2 = \frac{2}{1} \cdot \frac{2}{1} = 4$.

If aggregate endowments are equal in both goods, i.e., $x^h_1 + x^i_1 = x^h_2 + x^i_2$, then the Pareto surface consists of the diagonal of the Edgeworth box drawn below in Figure 3. The point x in the picture has $\gamma(x) = 1$, no matter how close it is to the diagonal. In contrast, $\delta(x)$ falls to zero as x approaches the diagonal.

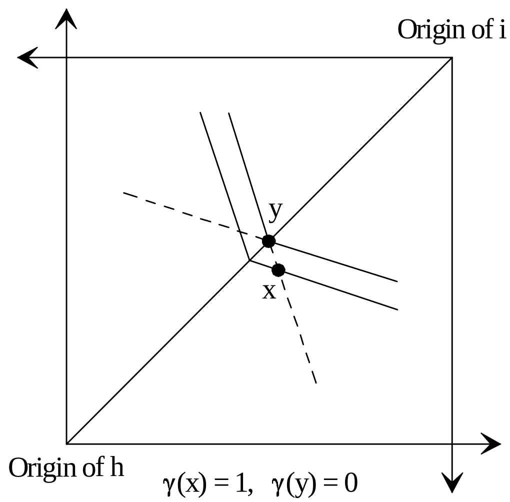

Figure 3

The formula shows that the function $\gamma(x)$ is continuous when the utilities are continuously differentiable and $x \gg 0$. The example shows that, without differentiability, $\gamma(x)$ may not be continuous. But one can check that it is always lower semi-continuous.

We are now ready to state our key condition for the existence of ME.

Gains to Trade Hypothesis $\gamma(e) > \bar{m}/M$; i.e., there are gains to γ-diminished trade at the initial endowment, where $\gamma = \bar{m}/M$.

4 Existence of Monetary Equilibrium

Theorem 2 *Consider a monetary economy $(\mathcal{E}, m, M)$ in which $\bar{m} \equiv \sum_{h \in H} m^h > 0$ and the Gains to Trade Hypothesis holds, i.e., $\gamma(e) > \bar{m}/M$. Then a monetary equilibrium exists and, at any monetary equilibrium, the interest rate $r = \bar{m}/M$.*

Corollary 3 (a) *Suppose $\bar{m} = 0$. Then monetary equilibria exist and coincide (in allocations and price ratios) with the Walras equilibria of the underlying economy.* (b) *Suppose $(p(k), r(k), q(k), x(k))$ is an ME of $(\mathcal{E}, m(k), M(k))$, with $\bar{m}(k) > 0$, and $x(k) \to_{k\to\infty} x$, and $\bar{m}(k)/M(k) \to_{k\to\infty} 0$. Then x is Walrasian for $\mathcal{E}$.*

According to the theorem, increasing the stock of inside money M must eventually guarantee the orderly functioning of markets, if the initial endowment is not Pareto-optimal. Equilibrium exists in $(\mathcal{E}, m, M)$ once M exceeds the finite threshold $\bar{m}/\gamma(e)$. In our model, inside money is indeed "the grease that turns the wheels of commerce."

Part (b) of the corollary assures us that although all trades are conducted via money, and although all prices are quoted in terms of money, as bank money M approaches infinity, the final allocation of goods becomes essentially no different from the Walrasian allocation obtained in an idealized world without any money at all, and in which prices really only have meaning as exchange rates between pairs of commodities.

Levels of bank money beyond $\bar{m}/\gamma(e)$, but short of infinity, give a large domain in which the real sector $\mathcal{E} = (u^h, e^h)_{h \in H}$ and the financial sector (m, M) influence each other, as we show in Section 6.

4.1 Outline of the Proof of Theorem 2

The intuition behind the proof is as follows. Imagine an external agent, say the government, that commits to activating all markets by selling ε units on each side of every market (except for the money side of the bond market, where the bank already commits $M > 0$). In particular, the government sells ε units of each real good for money. Money would then naturally have value because any holder of it could get something real in exchange for it. We quickly show that, with such an external agent, a full blown ε-monetary equilibrium (ε-ME), including commodity prices $p(\varepsilon) < \infty$ and an interest rate $r(\varepsilon)$, necessarily exists. In an ε-ME, as in an ME, the total stock of money and commodities held by households, the bank, and the external agent is conserved in all three time intervals.

We remove the external agent by letting $\varepsilon \to 0$, and examine a limit of ε-ME to see if the households themselves are imputing positive value to money. If $r(\varepsilon)$ and $p(\varepsilon)$ stay bounded as $\varepsilon \to 0$, we can pass to a convergent subsequence which will be a bona fide ME. Clearly $r(\varepsilon) \geq 0$, for otherwise households could arbitrage the bank. Thus money-now must have value in terms of money-later. Note that no $p_\ell(\varepsilon) \to 0$, for otherwise any household h with $m^h > 0$ would be buying more of good ℓ than there is, contradicting the feasibility of ε-ME.

The interest rate $r(\varepsilon)$ must be no bigger than $(\bar{m} + L\varepsilon)/M$, no matter how small $\varepsilon > 0$ is, since default is not allowed. Otherwise, some household would necessarily default, because it would not be able to find the money to pay its debts. Thus $0 \leq \lim r(\varepsilon) \leq \bar{m}/M$.

It remains to show that $p(\varepsilon) \nrightarrow \infty$. Suppose $p(\varepsilon) \to \infty$. Then, since the total money in the system is bounded and since money is the sole medium of exchange, trade in goods $\to 0$ as $p(\varepsilon) \to \infty$. Hence households end up consuming their initial endowment e in the limit. At the same time, notice that with $p(\varepsilon) \to \infty$, the purchasing power of the endowed money m goes to zero and may be ignored. Consider now the limiting price *ratios* (on some subsequence) given by p, where $p_\ell = \lim_{\varepsilon \to 0} p_\ell(\varepsilon) / \sum_{k \in L} p_k(\varepsilon)$. The trading opportunity for any household (at the limit) is effectively to purchase goods solely out of borrowed money and to pay the loan back, at the interest rate $r \equiv \lim r(\varepsilon) \leq \bar{m}/M$ out of his sales revenue (conducting all trade via money, of course, at the prices p). A little reflection reveals that this is tantamount to doing standard Walrasian trades at prices p but consuming only the fraction $1/(1+r)$ of purchases, which in turn may be viewed as consuming the whole Walrasian trade via modified utilities v_r^h. Thus e is a Walras allocation for $(v_r^h)_{h\in H}$ at prices p, and must be Pareto-optimal with respect to $(v_r^h)_{h\in H}$. Since $r \leq \bar{m}/M < \gamma(e)$, e is also Pareto-optimal with respect to $(v_{\gamma(e)}^h)_{h\in H}$. This contradicts the gains-to-trade hypothesis. So $p(\varepsilon) \nrightarrow \infty$, finishing the proof.

In short, once the external agent has given households the confidence that money-now has value (i.e., that $p(\varepsilon) < \infty$), and that money-later has enough value (i.e., that $r(\varepsilon)$ is not too high), they themselves will offer large amounts of goods, and small enough amounts of bonds, for sale against money. Their actions will fulfill their own prophecies, propping up the value of money-now and money-later, if there are gains to $\bar{m}/M$-diminished trade.

Our proof in effect uses the gold-backed device of supporting the value of money, but shows that in the end the gold is not needed, if there are enough potential gains to trade in the underlying real economy. We could have given almost the same proof, using the alternative device of forced commodity sales against money, showing again in the end that such sales are unnecessary.

The proof in the appendix also shows that we can drop the hypothesis of strict monotonicity, which becomes important when we consider a multiperiod model with overlapping generations.

5 Determinacy of Monetary Equilibrium and the Value of Money

If outside money $\bar{m} \equiv \sum_{h\in H} m^h = 0$, the ME are Walrasian, as pointed out in the Corollary to Theorem 1. In this event, it is clear that there is great indeterminacy of the commodity price levels. Households can borrow, hoard, and return money to the bank without spending it and without incurring any interest cost. Hence ME prices can be scaled down arbitrarily (with households hoarding increasing amounts of bank money M) without disturbing the ME.

But the moment $\bar{m} > 0$ we must have the interest rate positive, indeed equal to $\bar{m}/M$. Consequently there is no hoarding at any ME and the above indeterminacy abruptly disappears. In particular, the value of money (given by the price levels) is determinate.

We can state this intuition formally as follows. Fix utilities $(u^h)_{h\in H}$. Let $\mathcal{U}$ be an open set of linear perturbations of the utilities. More precisely for each vector $c \in \mathbb{R}^L$, let $u_c^h(x) = u^h(x) + c \cdot x$ (where $\cdot$ denotes inner product). Take $\mathcal{U}$ to be an open set in $\mathbb{R}^L$, including the origin, and (for household h) identify $c \in \mathcal{U}$ with the utility u_c^h.

Let $\mathbb{R}_{++}^L$ denote the set of possible endowment vectors e^h for any household $h \in H$. Similarly, let $\mathbb{R}_{++}$ denote the set of possible monetary endowments m^h for any household $h \in H$. Finally, also let $\mathbb{R}_{++}$ denote the set of possible levels of bank money. Then we may think of any monetary economy $((c^h, e^h, m^h)_{h\in H}, M)$ as a point in $\Xi \equiv (\mathcal{U} \times \mathbb{R}^L_{++} \times \mathbb{R}_{++})^H \times \mathbb{R}_{++}$.

Theorem 3 *For an open and full measure set Ξ' of vectors $\xi \in \Xi$, the set of* ME for the *economy defined by ξ is finite in number; and this set varies continuously on Ξ'.*

Theorem 3 tells us that as the data of the economy change, equilibrium moves differentiably. We are particularly interested in showing that perturbations of (m, M) cause real changes in $(x^h)_{h\in H}$ and in relative prices p_ℓ/p_k, as well as in nominal price levels and in interest rates. We will also show that changes in the real sector $(u^h, e^h)_{h\in H}$ affect the general price level, as well as price ratios and final consumption.

6 Welfare in a Monetary Equilibrium

As was said, at any monetary equilibrium the interest rate r is the ratio of outside to inside money in the economy, i.e., $r = \bar{m}/M$. (See Theorem 2.) Thus, if $\bar{m} > 0$, households who borrow money lose the interest-float on their marginal purchases, which discourages some trade. The upshot is that ME allocations are not Pareto-optimal, leaving room for further gains to trade.

A simple example will clarify the picture. Suppose that $e^h = e^i = (50, 50)$, $m^h = m^i = 5$, $M = 90$, and $u^h(x_1, x_2) = 10 \log x_1 + 3 \log x_2$ and $u^i(x_1, x_2) = 3 \log x_1 + 10 \log x_2$. In equilibrium (i) household i sells part of its endowment of commodity 1 and buys commodity 2; while household h sells part of its endowment of commodity 2 and buys commodity 1, (ii) both households borrows money from the bank. Since the utilities are differentiable, let

$$\nabla^j_\ell(y) \equiv \text{the partial derivative of } u^j \text{ w.r.t. the variable } x_\ell, \text{ evaluated at } y \in \mathbb{R}^L_+.$$

Then, if x^i and x^h are the households' consumptions at an ME with prices (p, r), we must have

$$\frac{\nabla^i_2(x^i)}{p_2} = (1+r)\frac{\nabla^i_1(x^i)}{p_1}$$

and

$$\frac{\nabla^h_1(x^h)}{p_1} = (1+r)\frac{\nabla^h_2(x^h)}{p_2}$$

Such an equality holds for any "*active*" household j, i.e., any j that borrows money and purchases good ℓ and sells only a *part* of his endowment of k.[11] Therefore gradients of households i and h tilt away from the price ratio p_1/p_2 in opposite

[11] The reason for the equality is clear. If $\nabla^j_\ell(x^j)/p_\ell > (1+r)\nabla^j_k(x^j)/p_k$, then j could improve its utility by borrowing δ more dollars from the bank and spending it to purchase ℓ, while defraying the loan by selling $(1+r)\delta$ dollars' worth more of good k; if the reverse inequality holds, j would benefit by reducing slightly its bank loan and purchase of ℓ, while simultaneously curtailing the concomitant sale of k. There is a "wedge" of size $(1+r)$ between the buying and selling prices of any commodity.

directions. (See Figure 4 below.) The misalignment implies that from the ME it is possible for households i and h to trade further for joint gains. The expansion of trade would no doubt occur in a Walrasian world in which all trade is free. In Walrasian equilibrium, final consumption would be approximately $x^h = (77, 23)$ and $x^i = (23, 77)$. But, in a monetary equilibrium, the cost imposed by the interest-float hampers the possibility.

One can check that in monetary equilibrium $p_1 = p_2 = 2$, $r = 1/9$, $x^h = (75, 25)$, $x^i = (25, 75)$. Agent h spends his \$5 and buys 2.5 units of good 1. He also borrows \$45 from the bank, promising to repay \$50. This is spent to buy 22.5 units more of good 1. To repay the bank he must sell 25 units of good 2, which is purchased by agent i.

The example also reveals that the misalignment is no more than the wedge factor $(1 + r)$, putting a bound on the inefficiency of ME allocations.

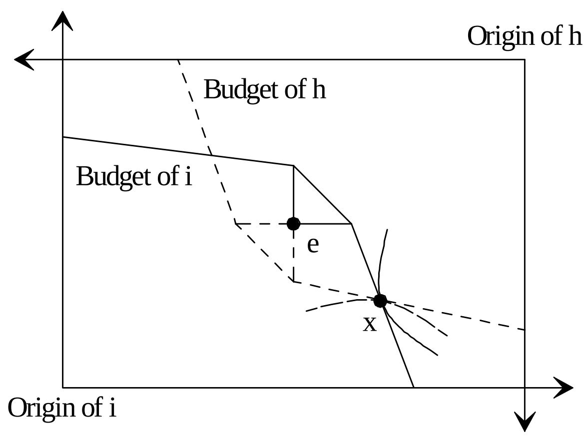

Figure 4: ME Allocation n in the Edgeworth Box

To describe the general situation, we first introduce a condition which will ensure that households never sell all of any commodity they are positively endowed with.

Regularity Condition Assume that each u^h is continuously differentiable, and that

$$\left.\begin{array}{r} u^h(y^h) \geq u^h(e^h) \\ y^h \leq \sum_{i \in H} e^i \\ e^h_\ell > 0 \end{array}\right\} \Rightarrow y^h_\ell > 0$$

The "social welfare" at an ME is summed up in:

Theorem 4 *Let* (p, r, q, x) *be an* ME *of the monetary economy* $(\mathcal{E}, m, M)$ with $\bar{m} > 0$. *Then* $\gamma(x) \leq r$. *If the regularity condition holds,* $\gamma(x) = r = \bar{m}/M$.

Remark 1 Even without regularity, if utilities $(u^h)_{h \in H}$ are differentiable, there are allocations y arbitrarily close to x with $\gamma(y) \geq r$.

7 Non-neutrality of Money

There is no money illusion in our model. If we scale M and the vector $m \equiv (m^h)_{h \in H}$ by the same factor, then clearly the allocation and the interest rate at an ME remain unaffected, and all commodity prices are scaled by the same factor. The scaling is tantamount to a change in units, and it would be surprising indeed if a switch from dollars to cents caused rational agents to behave differently.

But for the above circumstance, money is never neutral. Indeed let $\mathcal{A}(\mathcal{E}, m, M)$ denote the set of all allocations of commodities achieved at monetary equilibria of $(\mathcal{E}, m, M)$. Then we have

Theorem 5 *Consider two monetary economies $(\mathcal{E}, m, M)$ and $(\mathcal{E}, m^*, M^*)$ with the same underlying real sector $\mathcal{E}$, and suppose that the regularity condition holds for $\mathcal{E}$. If $\bar{m}/M \neq \bar{m}^*/M^*$, then the sets $\mathcal{A}(\mathcal{E}, m, M)$ and $\mathcal{A}(\mathcal{E}, m^*, M^*)$ are disjoint.*

Proof Take any $x \in \mathcal{A}(\mathcal{E}, m, M)$ and any $x^* \in \mathcal{A}(\mathcal{E}, m^*, M^*)$. By Theorem 4, the gains to trade at x and at x^* are $\gamma(x) = \bar{m}/M$ and $\gamma(x^*) = \bar{m}^*/M^*$. Since $\bar{m}/M \neq \bar{m}^*/M^*$, x and x^* are distinct. ■

A change in M alone, or in m alone, or in both but in different proportions, will (by Theorem 5) invariably affect real trades. The injection of bank money (with private endowments of money held fixed) corresponds to a form of elementary monetary policy in our model. It is evident that this policy will lower the interest rate (since $r = \bar{m}/M$) and alter commodity allocations, moving them "closer" to Pareto-efficiency since the unexploited gains to trade "left on the table" become smaller. However, as we shall see in Section 9, increases in M will also eventually raise equilibrium price levels p. Households that began with relatively large endowments m^h of money will be hurt, since their cash endowments lose purchasing power. These households could be expected to use their influence on the central bank to resist such expansionary monetary policy. Gifts of fiat money to households constitute fiscal policy. They will cause interest rates to rise, and the ensuing ME allocations are bound to be affected, becoming less (locally) efficient in the process. Of course households that were the primary recipients of the fiscal gifts may be better off than before.

The welfare-reducing impact of fiscal injections is most pronounced in the setting of exchange economies with private goods and complete markets. When there is production and incomplete markets, fiscal injections may be Pareto improving. But we deal with this important issue elsewhere [12]. Fiscal injections can also be Pareto improving when there are public goods. We defer our analysis of monetary and fiscal policy to Section 10.

8 The Necessity of Gains to Trade

We have claimed that the presence of a bank (which "demands" money for its own sake, at the end of commodity trade) is not sufficient for money to have value. We now make this claim precise via Theorems 6 and 7, showing that monetary equilibrium does not exist unless there are sufficient gains to trade.

Let $\tilde{\nabla}_\ell^h(y)$ ($\bar{\nabla}_\ell^h(y)$) denote the left-hand (right-hand) derivative of u^h, with respect to the variable x_ℓ, at the point $y \in \mathbb{R}_+^L$. And let $\mathcal{A}(e)$ be the set of all individually rational and feasible allocations, i.e.,

$$\mathcal{A}(e) = \{y \equiv (y^h)_{h \in H} \in (\mathbb{R}_+^L)^H : \sum_{h \in H} y^h = \sum_{h \in H} e^h, \text{ and } u^h(y^h) \geq u^h(e^h) \text{ for } h \in H\}$$

Put

$$\Gamma(e) = \sup_{y \in \mathcal{A}(e)} \gamma^*(y)$$

i.e., Γ is an upper bound on the gains to trade at any point in $\mathcal{A}$.

Theorem 6

$$(\bar{m}/M) > \Gamma(e) \Rightarrow \textit{no } \text{ME } \textit{exists.}$$

Theorem 6 shows that the value of money depends on household heterogeneity. If all households were identical, $\Gamma(e) = 0$, and monetary equilibrium could not exist. In fact, if e were Pareto efficient, then $\Gamma(e) = 0$ and money would have no value.

Putting together Theorems 2 and 6, we obtain Figure 5.

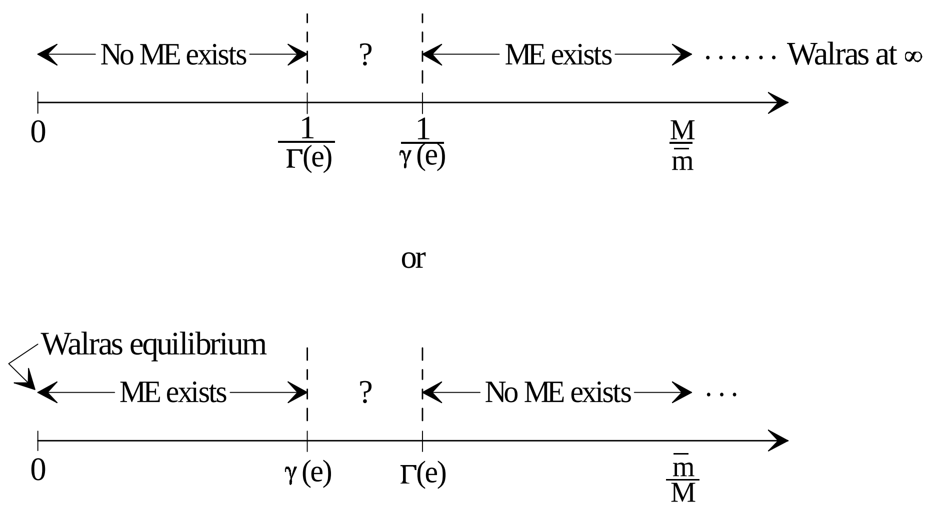

Figure 5

In general, we cannot say whether equilibrium exists in the gap $\gamma(e) \leq \bar{m}/M \leq \Gamma(e)$. But when all utilities u^h are separable, the question mark disappears: the region in question in Figure 5 falls entirely into the domain of nonexistence.

The function $u : \mathbb{R}_+^L \to \mathbb{R}$ is said to be separable if there exist functions $u_\ell : \mathbb{R}_+ \to \mathbb{R}$, for $\ell \in L$, such that $u(x) = \sum_{\ell \in L} u_\ell(x_\ell)$ for all $x \in \mathbb{R}_+^L$. (Cobb–Douglas utilities have this property.)

First we establish the following Theorem, which may be of some interest in its own right. It asserts that for separable utilities, an ME strictly uses up some of the gains to trade available at the initial endowment.

Theorem 7 *Consider a monetary economy* $((u^h, e^h, m^h)_{h \in H}, M)$ *with* $\bar{m} > 0$*, and suppose that* u^h *is strictly concave and separable for all* $h \in H$*. Suppose there exists an* ME $\langle p, r, q, x \rangle$ *with interest rate* r*. Then* $\gamma(e) > r \geq \gamma(x)$*.*

(The second inequality in Theorem 7 was shown in Theorem 4.) From Theorem 7 we immediately get

Theorem 8 *Assume* u^h *is strictly concave and separable for* $h \in H$*. Then an* ME *exists if and only if* $\gamma(e) > \bar{m}/M$*.*

In other words, for strictly concave and separable utilities, the sufficient condition for existence (in Theorem 2) is also necessary (and this incidentally shows that Theorem 2 cannot be strengthened).

9 Hyperinflation

Let us fix $\bar{m}$ and start with M so large that $M/\bar{m}$ is well to the right of $1/\gamma(e)$. By Theorem 2 equilibrium exists, and by Corollary 3 it is nearly Walrasian. Increasing M still further has little real effect; nearly exactly the same real trades are conducted. Since all of $M + \bar{m}$ is spent on practically the same purchases, the price level[12] rises linearly with M, and we have the nominal inflation depicted in Figure 6.

As M falls toward $\bar{m}/\gamma(e)$, what happens to price levels? The decline in money suggests that price levels will continue to fall. But as M falls, $r = \bar{m}/M$ rises, discouraging trade and moving us to less efficient allocations. (By Theorem 4, at any ME allocation x of a regular economy, $\gamma(x) = r$.) Smaller volumes of trade Q make for higher price levels. Which effect dominates?

Suppose the economy has strictly concave and separable utilities. Consider a sequence of ME with bank money $M(n)$ and equilibrium prices $p(n)$. Suppose $M(n)$ converges to $\bar{m}/\gamma(e)$ from above. If $p(n)$ remains bounded, then by passing to a convergent subsequence, we could show the existence of ME at $M = \bar{m}/\gamma(e)$, contradicting Theorem 8. Hence $p(n) \to \infty$ and the price level must look something like the following:

[12] I.e., the price of some fixed good in terms of money. (All price ratios will be bounded, since the interest rate is bounded across all ME.)

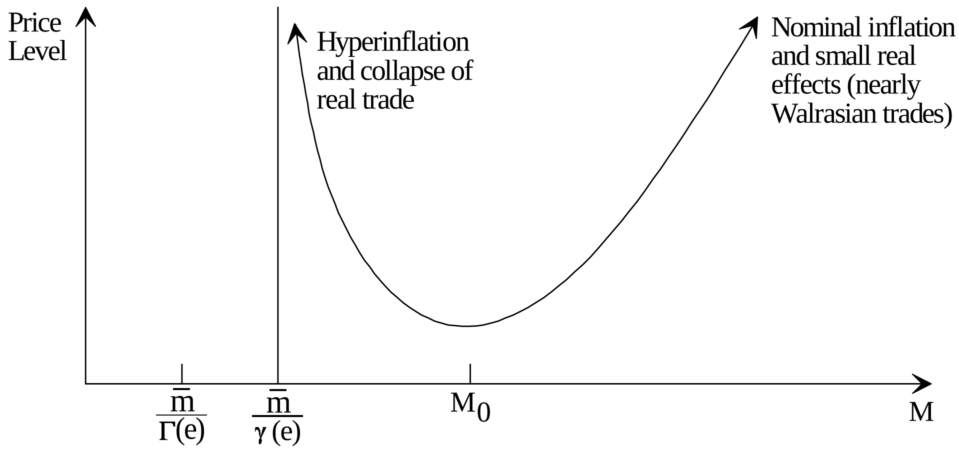

Figure 6: m Fixed

Our analysis has the paradoxical feature that there is some stock M_0 of bank money which minimizes the price level. If the bank eases, and lends more money, inflation will creep in, though the equilibrium allocation will improve somewhat. If the bank tightens its policy, lending less than M_0, inflation will again occur, and eventually price levels will *rise* much more rapidly (i.e., much faster than linearly, since they reach infinity over a finite move $M_0 - \bar{m}/\gamma(e)$). We call this explosion of prices, a hyperinflation.

We can also attribute the hyperflation to changes in the stock of outside money $\bar{m}$, occasioned perhaps by a government eager to distribute money to particular projects.

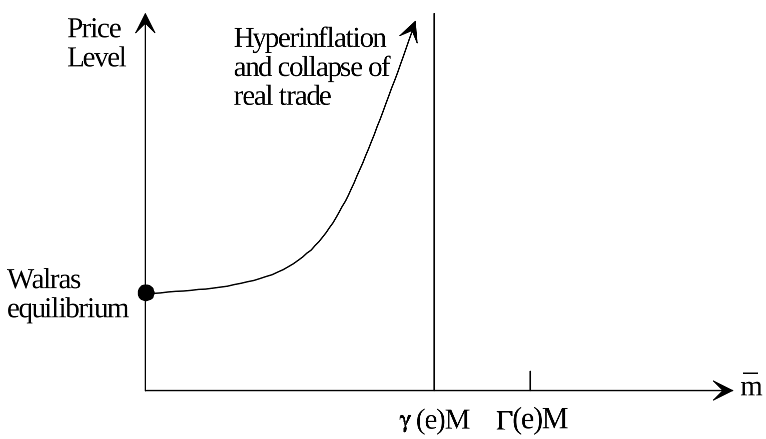

Figure 7: M Fixed

Too big a fiscal injection $\bar{m}$ must result in an explosion of prices, and collapse of trade, as we approach the finite threshold $\gamma(e)M$. Note that we cannot be sure that the price level falls for increases of $\bar{m}$ near 0, since price levels at the Walras equilibrium are now finite.

It is very useful to compare monetary injections ΔM with fiscal injections $\Delta \bar{m}$. As $M \to \infty$ (holding $\bar{m}$ fixed), price levels rise to infinity at the same rate, and we have *nominal* inflation. As $\bar{m}$ increases (holding M fixed), price levels rise at an accelerating pace, reaching infinity when $\bar{m}$ is still finite, and thus giving *hyperinflation*.

We summarize our discussion of hyperinflation in the following theorem.

Theorem 9 **(Hyperinflation)** *Fix the real sector $\mathcal{E} \equiv (u^h, e^h)_{h \in H}$. Assume u^h is strictly concave and separable for $h \in H$. Consider a sequence $\{M(n), (m^h(n))_{h \in H}\}_{n=1}^{\infty}$, indexed by n, such that $M(n)$ is bounded away from 0 and $\sum_{h \in H} m^h(n)/M(n)$ increases monotonically to $\gamma(e)$. Every n defines an economy which, by Theorem 2, has monetary equilibria. Let $p(n)$ be an* ME *commodity price vector of the nth economy. Then $p(n) \to \infty$ as $n \to \infty$.*

We conjecture that hyperinflation occurs, without the separability hypothesis on utilities, at some point m^*/M^* between $\gamma(e)$ ansd $\Gamma(e)$.

10 Government: Treasury and Central Bank

We extend our one-period model by adding a government. Fiat money, after all, is money by government decree, and brought into being by government issue.

The government produces public goods denoted $P = \{1, ..., P\}$. These goods are not marketed, but give utility to households. For simplicity we imagine a production function $F : \mathbb{R}_+^L \to \mathbb{R}_+^P$ mapping inputs of private goods into a unique output of public goods. We suppose F is continuous and $F(0) = 0$.

Public goods have impact on households' utilities. For any vector z of public goods present in the economy, $u_z^h : \mathbb{R}_+^L \to \mathbb{R}$ is the utility of $h \in H$ over his consumption of private goods. We assume that, for every fixed z, $u_z^h(x)$ is concave and strictly monotonic in x; and also that $u_z^h(x)$ is continuous jointly in z, x. (Notice that we do not need to assume that $u_z^h(x)$ is monotonic in z.)

We distinguish the government Treasury department, which chooses $(Q_{bm}, \Delta m, Q_m, \sigma)$, from the government central bank, which sets M. The complete government policy is given by a vector

$$\pi \equiv (M, Q_{bm}, \Delta m, Q_m, \sigma) \in \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}^H \times \mathbb{R}_+^L \times [0, 1).$$

We now describe the four policy instruments of the Treasury.

The Treasury buys all inputs for production, such as labor, from the private sector. Denote these expenditures by $Q_m = (Q_{m\ell})_{\ell \in L}$.

The Treasury also raises taxes and transfers wealth. For simplicity we assume the same ad valorum tax rate on the sale of every good and denote it by $\sigma \in [0, 1)$. An agent who sells a vector q of goods at prices p must pay the Treasury $\sigma p \cdot q$ out of his sales revenue $p \cdot q$. We denote by $\Delta m \equiv (\Delta m^h)_{h \in H}$ the transfer of money (to household h if $\Delta m^h > 0$, and from household h if $\Delta m^h < 0$). We assume these lump sum transfers occur after bank loans are made and before commodity trade. Lump sum transfers Δm^h are rarely observed in reality, especially if $\Delta m^h < 0$. We include them for theoretical reasons.

To finance its expenditures $\bar{Q}_m \equiv \sum_{\ell \in L} Q_{m\ell}$ on inputs for production and on money transfers $\Delta \bar{m} \equiv \sum_{h \in H} \Delta m^h$, the Treasury can borrow money from the bank, in competition with households, by issuing its own bonds Q_{bm}. Let

$$m_\beta \equiv \Delta \bar{m} + \bar{Q}_m - Q_{bm}/(1+r). \tag{9.1}$$

A shortfall $m_\beta > 0$ must be covered by printing money, and a surplus $m_\beta < 0$ must be inventoried.

The Treasury is not allowed to default on its bank loan. Let

$$m_\alpha \equiv Q_{bm} - \text{tax revenue} - \max\{-m_\beta, 0\}. \tag{9.2}$$

If it cannot repay the loan out of its tax revenue and inventory, it must print additional $m_\alpha > 0$; and must dispose of excess money if $m_\alpha < 0$.

The budget deficit of the Treasury is defined by the amount of money it must print to meet its spending

$$\text{deficit} \equiv \max\{m_\beta, 0\} + m_\alpha \equiv -\text{surplus}.$$

If there is no deficit (or surplus), the budget is *balanced*.

We think of the Treasury making a budget plan at the beginning of the period. This plan consists of its issue of bonds Q_{bm}, its transfers $(\Delta m^h)_{h \in H}$ and expenditures $(Q_{m\ell})_{\ell \in L}$, and the tax rate σ. The Treasury can always carry out its plan, no matter how households behave, so long as it is free to print money before trade ($m_\beta > 0$), or hoard money across trading time ($m_\beta < 0$), and print money ($m_\alpha > 0$) or destroy money ($m_\alpha < 0$) at the end. Policies for which the government borrows precisely what it needs to spend, and taxes precisely to cover its debt, i.e., $m_\beta = 0 = m_\alpha$, are called *totally balanced* budget policies. In an uncertain world, however, the Treasury might not be sure of the proceeds from its sale of bonds, or of its tax revenue. There would need to be a mechanism for covering shortfalls and disposing of excess money. Though there is no uncertainty in our model, we wish to investigate whether monetary equilibrium can be maintained even when the Treasury faces imbalances. The residual variables (m_β, m_α), which adjust for imbalances, are determined in our model uniquely from M, $(Q_{bm}, \Delta m, Q_m, \sigma)$, r and tax revenue by equations (9.1) and (9.2). We investigate whether equilibrium exists for arbitrary complete policies $(M, Q_{bm}, \Delta m, Q_m, \sigma)$, even if they entail printing money or destroying money.

In American law, the Treasury cannot literally print money, but must borrow it from the Federal Reserve. But the Federal Reserve can print the money, giving it to the Treasury in exchange for an IOU note. By not redeeming the IOU note, or equivalently by rolling it over in perpetuity, the Treasury prints money by proxy.

10.1 Monetary Equilibrium with Government

We shall denote by $\Pi(m)$ the set of all (complete) policies π that are consistent with the private sector $(\mathcal{E}, m)$. Denote $\bar{m} \equiv \sum_{h \in H} m^h$, $\Delta \bar{m} \equiv \sum_{h \in H} \Delta m^h$, $\bar{Q}_m \equiv$

$\sum_{\ell \in L} Q_{m\ell}$. Then

$$\Pi(m) = \big\{(M, Q_{bm}, \Delta m, Q_m, \sigma) \in \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{+}^{H} \times \mathbb{R}_{+}^{L} \times [0,1) : \\ m + \Delta m \in \mathbb{R}_{+}^{H}, \bar{m} + \Delta \bar{m} + \bar{Q}_m > 0\big\}.$$

Thus the government cannot take more money from any household than it is endowed with ($m + \Delta m \in \mathbb{R}_{+}^{H}$), or wipe out all the outside money in the system ($\bar{m} + \Delta \bar{m} + \bar{Q}_m > 0$). *Throughout, when we consider an economy* $(\mathcal{E}, m, \pi)$, *it will be assumed that* $\pi \in \Pi(m)$.

The budget set $B(p, r, e^h, m^h + \Delta m^h, \sigma)$ of household h is defined exactly as $B(p, r, e^h, m^h)$, but with two amendments. First, replace m^h with $m^h + \Delta m^h$. Second, tax must be deducted from sales revenue (i.e., $\sum_{\ell \in L} p_\ell q_{\ell m}^h$ must be replaced by $\sum_{\ell \in L} (1-\sigma) p_\ell q_{\ell m}^h$).

The vector $\langle p, r, (q^h, x^h)_{h \in H}\rangle$ is a *monetary equilibrium* for the economy $(\mathcal{E}, m, \pi \equiv (M, Q_{bm}, \Delta m, Q_m, \sigma))$ iff

(i) $(q^h, x^h) \in B(p, r, e^h, m^h + \Delta m^h, \sigma)$ and $u_z^h(x^h) \geq u_z^h(\underline{x}^h)$ for all $(\underline{q}^h, \underline{x}^h) \in B(p, r, e^h, m^h + \Delta m^h, \sigma)$ where $z = F(Q_{m1}/p_1, ..., Q_{mL}/p_L)$

(ii) $p_\ell \sum_{h \in H} q_{\ell m}^h = Q_{m\ell} + \sum_{h \in H} q_{m\ell}^h$, $\forall \ell \in L$

(iii) $Q_{bm} + \sum_{h \in H} q_{bm}^h = (1+r)M$

Thus households are optimal in their amended budget sets ((i)) and all markets clear ((ii), (iii)), taking the government's actions into account.

10.2 Gains to Trade and Existence of Monetary Equilibrium

Given private money $(m^h)_{h \in H} \equiv m$ and government policy $\pi \equiv (M, Q_{bm}, \Delta m, Q_m, \sigma)$, consider the equation:

$$g(x) \equiv (1-\sigma)\left[\frac{xM}{Q_{bm} + x} + \sum_{h \in H}(m^h + \Delta m^h) + \sum_{\ell \in L} Q_{m\ell}\right] = x. \qquad (10)$$

Since $\bar{m} + \Delta \bar{m} + \bar{Q}_m > 0$ and $\sigma < 1$, we have $g(0) > 0$. Moreover, since $M > 0$ and the function $\delta(x) \equiv x/(Q_{bm} + x)$ is strictly concave[13] and monotonic and bounded above (by 1), all these properties are inherited by g. It follows that there exists a unique positive scalar which solves the equation. We denote it by $x(m, \pi)$.

[13] Unless $Q_{bm} = 0$, in which case the graph of g is horizontal, still yielding a unique intersection with the 45° line.

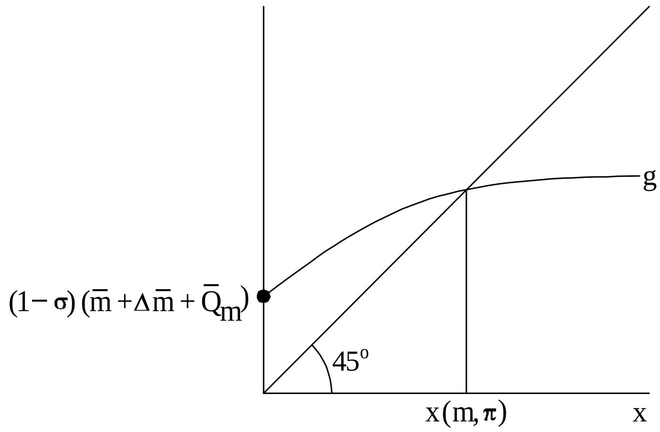

Figure 8

As we shall see in the proof of Theorem 10, the variable x gives the total number of bonds issued by households, and $g(x)$ gives the total money in their hands after commodity trade, assuming $r > 0$.

Define

$$\rho(m, \pi) = \max\left\{0, \frac{Q_{bm} + x(m, \pi)}{M} - 1\right\}. \tag{11}$$

For any z in $\mathbb{R}^P_+$, define the gains to trade $\gamma_z(e)$ in the private sector in the presence of public goods z, exactly as before, using the utilities $(u^h_z)_{h\in H}$. Then $\gamma_0(e)$ represents the gains to trade before any trade or government production occurs.

Theorem 10 *Consider a monetary economy* $(\mathcal{E}, m, \pi)$. *Suppose* $\gamma_0(e) > r^* \equiv r^*(m, \pi) \equiv (\sigma + \rho(m, \pi))/(1 - \sigma)$. *Then an* ME *exists. Moreover if* $\langle p, r, q, x\rangle$ *is an* ME *of* $(\mathcal{E}, m, \pi)$, *then* $r = \rho(m, \pi)$.

The existence theorem for monetary equilibrium with the government differs in two respects from our previous existence Theorem 2. First, the gains to trade $\gamma_0(e)$ must exceed a threshold r^* that is greater than just the interest rate r. Second, the equilibrium interest rate r is no longer a simple ratio of exogenous stocks of outside and inside money.

As in monetary equilibrium without the Treasury, all the money issued by the bank and all the money printed by the Treasury (and not destroyed) must be owed and returned to the bank. Thus the interest rate must be

$$r = \frac{\bar{m} + \max\{m_\beta, 0\} + m_\alpha}{M}.$$

The Treasury can inject money into the system by running a deficit and printing it ($\max\{m_\beta, 0\} + m_\alpha > 0$) or take it out of the system by running a surplus and destroying it ($\max\{m_\beta, 0\} + m_\alpha < 0$). Previously only the bank could inject or withdraw money. The values m_β and m_α are solved endogenously, as part of the monetary equilibrium along with r. Without the Treasury, the equilibrium interest rate r was a simple ratio of the stock of outside money to the stock of inside money. With the Treasury endogenously printing and destroying money (depending on its budget deficit which depends on taxes raised) the interest rate cannot be so easily calculated from the exogenous parameters $(\mathcal{E}, m, \pi)$.

Nevertheless, Theorem 10 shows that the equilibrium interest rate $r = \rho(m, \pi)$ and the impediment to trade $r^* = r^*(m, \pi)$ do not depend on the underlying physical economy $(u^h, e^h)_{h \in H}$ in any way. We emphasize that this is because there is only one time period. With two or more time periods, nominal interest rates would depend on the underlying physical economy [10].

Theorem 10 enables a partition of the policy space $\Pi(m)$ into two regions $\Pi_0(m)$ and $\Pi_+(m)$ such that, if $\pi \in \Pi_0(m)$, the interest rate (at every ME of $(\mathcal{E}, m, \pi)$) is zero; and if $\pi \in \Pi_+(m)$, the interest rate is positive. Define

$$\Pi_0(m) \equiv \{(M, Q_{bm}, \Delta m, Q_m, \sigma) \in \Pi(m) : (\bar{m} + \Delta\bar{m} + \bar{Q}_m) \leq (\sigma/(1-\sigma))(M - Q_{bm})\}$$

and

$$\Pi_+(m) \equiv \Pi(m) \backslash \Pi_0(m).$$

Corollary 4 *Suppose* $\langle p, r, (q^h, x^h)_{h \in H} \rangle$ *is an* ME *of* $(\mathcal{E}, m, \pi)$*. Then*

$$r = 0 \Leftrightarrow \pi \in \Pi_0(m)$$

$$(\text{and so } r > 0 \Leftrightarrow \pi \in \Pi_+(m)).$$

10.3 Welfare and Gains to Trade

Without taxation, we saw that the interest rate created an impediment to trade. Taxation adds another impediment, and exacerbates the interest rate impediment. To see why, consider a household that wishes to sell enough goods to finance purchases worth one dollar. The household must incur a debt of $1 + r$ dollars, and then sell goods on which it will pay the tax σ, bringing its total cost to $(1 + r)/(1 - \sigma)$. This is tantamount to paying an effective interest rate of $r^* = [(1 + r)/(1 - \sigma)] - 1 = (r + \sigma)/(1 - \sigma)$. All our previous theorems on welfare hold (by the same proofs) with this threshold r^* in place of the bank interest rate r. For regular economies (in which utilities are smooth and equilibrium consumption $x_\ell^h > 0$ whenever $e_\ell^h > 0$), r^* measures precisely the unexploited gains to trade at any monetary equilibrium allocation $((x^h)_{h \in H}, z)$.

Government policies which increase r^* move the equilibrium allocation to $((x^h)_{h \in H}, z)$ at which there are more unexploited gains to trade. But welfare might nevertheless be increased if the policy leads to more public goods $z = F(Q_{m1}/p_1, ..., Q_{mL}/p_L)$, and if consumers value these public goods.

11 Monetary and Fiscal Policy

11.1 Nominal Comparative Statics in an "IS–LM" Framework

Macroeconomics seeks to understand aggregate variables such as total output without paying careful attention to the microeconomic details. A complete description of equilibrium in our model would require knowledge of all microeconomic particulars. Changing some household u^h or e^h would typically change equilibrium prices p and consumption $(x^h)_{h\in H}$ for all households. Raising $Q_{m\ell}$ by \$1 and lowering some other $Q_{m\ell'}$ by \$1 would do the same.

We shall show, however, that the equilibrium interest rate r and nominal GNP Y can be calculated without knowledge of microeconomic details. Indeed, write *aggregate* fiscal and monetary policy as the four-dimensional vector

$$\bar{\pi} \equiv (M, Q_{bm}, \mu \equiv \bar{m} + \Delta\bar{m} + \bar{Q}_m, \sigma).$$

We think of the first two coordinates as aggregate monetary policy, and the latter two coordinates as aggregate fiscal policy. We shall show that r and Y can be calculated from $\bar{\pi}$ alone via a graphical analysis identical to the Hicksian IS–LM framework. Each curve corresponds to the locus of points (Y, r) describing goods market clearing, or money market clearing, or bond market clearing.

In this section we assume that government policy is consistent with the existence of monetary equilibrium with a positive interest rate, i.e. (from Theorem 10)

$$\pi \in \Pi_+^*(m) \equiv \left\{\pi \in \Pi_+(m) : \gamma_0(e) > r^*(m,\pi) \equiv \frac{\sigma + \rho(m,\pi)}{1-\sigma}\right\}.$$

Define nominal GNP Y as aggregate spending on commodities: $Y = \sum_{\ell\in L} p_\ell \sum_{h\in H} q^h_{\ell m}$. Agents will always spend all their cash $\bar{m} + \Delta\bar{m}$. Anticipating revenue $(1-\sigma)Y$ from their sale of commodities, and knowing that there is only one consumption period, consumers will borrow and spend an additional $(1-\sigma)Y/(1+r)$. Since government spends $\bar{Q}_m$, market clearing for commodities requires

$$\left[\frac{(1-\sigma)Y}{1+r} + \bar{m} + \Delta\bar{m}\right] + \bar{Q}_m = Y, \text{ or}$$

$$Y = \frac{1+r}{\sigma+r}(\bar{m} + \Delta\bar{m} + \bar{Q}_m) = \frac{1+r}{\sigma+r}\mu. \qquad (10)$$

Note that for $\sigma < 1$, Y declines in r, for any fixed $\mu > 0$. We call (10) the income = spending equation, or IS for short.

Anticipating that they will be spending $Y - \bar{Q}_m$, households will demand precisely this same amount of money if $r > 0$. Since government demand for money is $Q_{bm}/(1+r)$, money market clearing requires that

$$(Y - \bar{Q}_m) + \frac{Q_{bm}}{1+r} = M + \bar{m} + \Delta\bar{m}, \text{ or}$$

$$Y = M + \mu - \frac{Q_{bm}}{1+r} \qquad (11)$$

Anticipating again that their revenue from commodity sales will be $(1-\sigma)Y$, households will offer to sell exactly that many bonds (assuming $r > 0$ so they do not inventory any money). Bond market clearing therefore requires that

$$\begin{aligned} 1+r &= \frac{Q_{bm}+(1-\sigma)Y}{M}, \text{ or} \\ Y &= \frac{1+r}{1-\sigma}M - \frac{Q_{bm}}{1-\sigma}. \end{aligned} \tag{12}$$

These three equations are drawn in the familiar (Y, r) plane below.

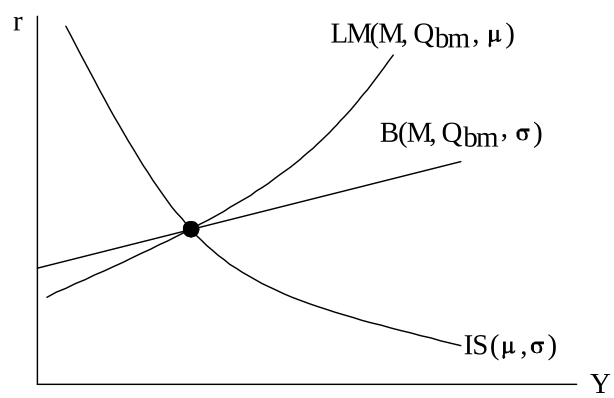

Figure 9

Note that the commodity market clearing equation (10) depends only on fiscal parameters (μ, σ), and is independent of M and Q_{bm}. Similarly bond market clearing does not depend on μ, and money market clearing does not depend on σ. The slopes of the curves are derived from equations (10)–(12). In particular, since $Q_{bm}/(1+r)^2 < M$, the money market clearing curve slopes up faster than the bond market clearing curve near the equilibrium (r, Y).

All the usual Keynesian comparative statics hold. Increasing M by \$1 shifts the LM curve \$1 to the right and the B curve $(1+r)/(1-\sigma) > 1$ dollars to the right, leaving the IS curve unchanged. Thus r declines and Y rises by less than \$1. Increasing government spending by \$1, financed by printing money, moves the C curve $(1+r)/(\sigma+r) > 1$ dollars to the right, and shifts the LM curve exactly \$1 to the right, while leaving the B curve unchanged. Thus r rises and Y rises by more than \$1. Increasing government borrowing by selling $1+r$ more bonds shifts the LM curve \$1 to the left and the B curve $(1+r)/(1-\sigma) > 1$ dollar to the left. Thus r increases and Y decreases by less than \$1. In all three cases r^* moves in the same direction as r, since σ is fixed.

Increasing σ by 1% moves the B curve and the IS curve down by more than 1%, while leaving the LM curve unchanged. Therefore r goes down by more than 1%, and Y goes down. From (10) we have that $Y[(1-\sigma)/(1+r)] = \mu/r^*$. From (12) we know that $Y(1-\sigma)$ drops by a bigger percentage than $1+r$, since Q_{bm} and M are fixed. Hence $Y[(1-\sigma)/(1+r)]$ declines and so r^* increases when σ increases.

At any budget-balanced policy, $r = \bar{m}/M$ and $Y \leq \bar{m}+M$. At any totally budget-balanced policy, $Y = \bar{m} + M$, exactly as was the case without the Treasury. Budget-balanced increases in government expenditures $\bar{Q}_m$ do not raise nominal income Y. In the Keynesian model they do increase Y, because households' marginal propensity to consume is assumed to be less than one (while the government spends all its tax revenue). In our one-period model, households have no reason to save, and their marginal propensity to consume is therefore one. In a totally budget-balanced policy, the government's propensity to spend is also one, while in a balanced-budget policy, the Treasury may borrow and hoard the money, paying the interest by raising taxes, and thus its propensity to spend out of tax revenue may be less than one.

11.2 Liquidity Trap

One new phenomenon introduced by the presence of the government is the possibility that interest rates become zero even when there are initial stocks of outside money $\bar{m} > 0$. If the government surplus, $-\max\{m_\beta, 0\} - m_\alpha = \bar{m}$, then r becomes zero.

At first glance this seems like a knife-edge case, but in fact the region where $r = 0$ is quite robust. Define

$$\Pi_0^*(m) \equiv \left\{ \pi \in \Pi_0(m) : \gamma_0(e) > \frac{\sigma}{1-\sigma} \right\}.$$

Corollary 5 *Fix $(\mathcal{E}, m)$. Then for any government policy $\pi = (M, Q_{bm}, \Delta m, Q_m, \sigma)$ in the nonempty open set $T \equiv \operatorname{int} \Pi_0^*(m)$, monetary equilibrium exists, and at every monetary equilibrium the interest rate is zero.*

For any policy in the liquidity trap, $\pi \in T$, equilibrium leaves $\sigma/(1-\sigma)$ unexploited gains to trade, and leaves $r = 0$. No small policy change that maintains the tax rate σ will be able to budge the interest rate or improve the gains to trade. In the liquidity trap, government monetary policy (M, Q_{bm}) is powerless to improve household trade if the Treasury dares not reduce taxes.

In the liquidity trap households borrow money and hoard some of it. By equation (10), with $r = 0$, total expenditure must be μ/σ. The remaining money must be borrowed and hoarded. As the tax rate σ is increased, households hoard more, $Y =$ GNP declines, and tax revenue $= \sigma Y = \sigma\mu/\sigma = \mu$ stays the same, as does the budget surplus. This explains why the liquidity trap region is robust, and how an open set of government policies can leave the budget surplus unchanged.

If the government insists on further and further increases in the tax rate σ, it will depress nominal GNP. As $\sigma/(1-\sigma)$ approaches $\gamma_0(e)$, real GNP will collapse to zero (if utilities are separable), by Theorem 7.

In the liquidity trap, increasing fiscal expenditures $\Delta\bar{m} + \bar{Q}_m$, without raising taxes, has a somewhat surprising multiplier effect. If the government prints one extra dollar, increasing $\Delta\bar{m} + \bar{Q}_m$ by 1, then aggregate spending Y must increase by $1/\sigma$.

Unfortunately, this remarkable stimulus to GNP springing from government spending is mostly inflation driven. Consider the special case that $1 = \sum_{h\in H} \Delta m^h$ and $\Delta m^h = m^h / \sum_{i\in H} m^i$ for all h, and $\bar{Q}_m = 0 = Q_{bm}$. Then the effect is purely from inflation, with consumption $(x^h)_{h\in H}$ unchanged.

It is worth noting that when the government takes no actions $\Delta\bar{m} = \bar{Q}_m = Q_{bm} = 0$, except to tax $\sigma > 0$, the ME correspond to Walras equilibrium with ad valorem tax σ and lump sum redistributions of the tax revenue $m^h / \sum_{i\in H} m^i$ to each agent $h \in H$. Our existence Theorem 10 thus proves the existence of Walras equilibrium with taxes.

11.3 Hyperinflation

Public goods expenditures $\bar{Q}_m > 0$ and transfers $\Delta\bar{m} > 0$ may be of great value to the economy. But if the Treasury becomes too ambitious by spending or borrowing too much, it will necessarily engender a self-defeating hyperinflation: $p \to \infty$, $x^h \to e^h$, $z \to 0$ as $\bar{Q}_m$ or Q_{bm} approach finite thresholds.

We argue this in the context of separable utilities, as in Section 9. There we showed that if the impediment to trade approached $\gamma(e)$, then hyperinflation necessarily set in. The fact that no money was being hoarded by households was crucial in that argument.

We will suppose $\pi \in \Pi_+(m)$ so that the interest rate is positive and households again do not hoard money. Recall that the impediment to trade is now $r^* \equiv r^*(m,\pi) \equiv [\sigma + \max\{0, (Q_{bm} + x(m,\pi))/M) - 1\}]/(1-\sigma)$. Clearly $r^*(m,\pi)$ approaches $\gamma_0(e)$ at finite thresholds Q^*_{bm} and $\bar{Q}^*_m$. Indeed let $Q^*_{bm} = M(\gamma_0(e)+1)$. Then, for $Q_{bm} \geq Q^*_{bm}$, the impediment $r^*(m,\pi) \geq \gamma_0(e)$ no matter what Δm, Q_m, or σ may be. To compute $\bar{Q}^*_m$, recall $x(m,\pi) = g(x(m,\pi)) \geq (1-\sigma)\bar{Q}_m$; so let $\bar{Q}^*_m = M(\gamma_0(e)+1)/(1-\sigma)$. Then if $\bar{Q}_m \geq \bar{Q}^*_m$, we get $r^*(m,\pi) \geq \gamma_0(e)$ no matter what Δm and Q_{bm} may be (but this second threshold $\bar{Q}^*_m$ does depend on σ).

By its own profligacy, borrowing or spending too much, the Treasury destroys the value of its fiat money, and also its power to produce any public goods or to transfer any real wealth.

11.4 Real Comparative Statics: The Ricardian Equivalence between Policies

In this section we investigate the real consequences of monetary and fiscal policy. This gives a very different picture from the nominal effects described in Section 11.1. For example, we shall show that government expenditures on public goods have the same *real* effect (up to scale) whether they are budget-balanced financed, by debt repaid later out of tax revenue, or deficit financed by printing money. Yet we saw in Section 11.1 that balanced-budget financing does not increase the interest rate r or nominal income Y, whereas printing money does increase both r and Y. This real "Ricardian equivalence" turns out to be delicate to prove. If the Treasury finances its purchases by printing money, it will cause an inflation of commodity prices, which will force it to plan proportionately higher expenditures to maintain the same real purchases. In addition, in order to achieve all the same real effects as the budget-balanced expenditures, the Treasury will be obliged to print still more money to make transfers to compensate the holders of money who will be hurt by the inflation. We describe the precise equivalence between government policies in the Corollary to Theorem 11. The key to the equivalence is the observation that real trade is influenced by r^*, and not by r. Printing and spending money $\Delta\bar{m}$ increases r, and therefore also r^*, while budget-balanced expenditures leave r fixed, but increase σ, and hence also r^*.

By directing government spending toward one public good as opposed to another, or by transferring wealth from one group to another, the government can influence real outcomes without changing the nominal values calculated in the last section. Ricardian equivalence does not apply in these cases. However, we wish to concentrate on the aggregate real effects of aggregate policy. So we restrict attention to aggregate policies for which all Δm^h move in the same proportion, and all Q_m move in the same proportion. Thus we are back to five policy instruments $(M, Q_{bm}, \Delta m, Q_m, \sigma)$.

The Treasury can always create real changes by changing its mix of expenditures between transfers to households, and spending on public goods. If we fix this mix, then Theorem 11 demonstrates that the Treasury, with its three policy tools $(Q_{bm}, (\Delta m, Q_m), \sigma)$, can achieve no more and no less than the same real effects achievable by the central bank with its single policy instrument M. In the nominal IS–LM framework, the Treasury and the central bank had complementary policy tools, one controlling the LM curve, and the other the IS curve (and perhaps the LM curve). In real terms, it turns out that whatever the Treasury can do by printing money (and increasing all expenditures by the same proportion) or by raising commodity taxes (uniformly across goods), the central bank can do by reducing the money stock. Treasury power becomes distinct from the central bank only when it targets a part of the economy, for example by shifting resources from private production to public production.

Let $\langle p, r, q, x\rangle$ be an ME of $(\mathcal{E}, m, \pi)$ where $\pi \equiv (M, Q_{bm}, \Delta m, Q_m, \sigma)$. What are "equivalent" policies $\tilde{\pi} \equiv (\tilde{M}, \tilde{Q}_{bm}, \Delta\tilde{m}, \tilde{Q}_m, \tilde{\sigma})$ for which x remains an ME allocation?

We shall vary $(\Delta m, Q_m)$ linearly with a single parameter $\lambda \in \mathbb{R}_{++}$ as follows:

$$\begin{aligned} \Delta m^h(\lambda) &= \lambda(m^h + \Delta m^h) - m^h, \text{ for } h \in H \\ Q_{m\ell} &= \lambda Q_{m\ell}, \text{ for } \ell \in L. \end{aligned}$$

This enables us to think of a policy as a four-dimensional vector $(\tilde{M}, \tilde{Q}_{bm}, \tilde{\lambda}, \tilde{\sigma})$, where $(\Delta\tilde{m}, \tilde{Q}_m) = (\Delta m(\lambda), Q_m(\lambda))$.

Equivalent policies can be pictured as a smooth surface in four dimensions. In particular this picture reveals that an arbitrary small change in any three policy variables, can be compensated by adjusting the fourth variable, to retain x as an ME allocation.

We describe the surface as a function $\Lambda : D \to \mathbb{R}_+$ where $D \subset \mathbb{R}^3_+$. For any $(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) \in D$, setting $\tilde{\lambda} = \Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma})$ will yield an equivalent policy. Needless to say, the domain D and the function Λ depend upon both $\langle p, r, q, x\rangle$ and $\pi \equiv (M, Q_{bm}, \Delta m, Q_m, \sigma)$.

Denote $r^* \equiv (r+\sigma)/(1-\sigma)$. Then the domain D is given by:

$$D = \left\{ (\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) \in \mathbb{R}_{++} \times \mathbb{R}^2_+ : 0 \le \tilde{\sigma} \le \frac{r^*}{r^*+1}, \tilde{Q}_{bm} < (r^*+1)(1-\tilde{\sigma})\tilde{M} \right\}.$$

We will suppose $B \equiv M - (Q_{bm}/(1+r)) > 0$, i.e., the government is not borrowing *all* the bank money at the ME $\langle p, r, q, x\rangle$ of $(\mathcal{E}, m, M, Q_{bm}, \Delta m, Q_m, \sigma)$. With this proviso, define

$$\Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) = \frac{1}{B}\left(\tilde{M} - \frac{\tilde{Q}_{bm}}{(r^*+1)(1-\tilde{\sigma})}\right).$$

We are ready to state

Theorem 11 *Let $\langle p, r, q, x\rangle$ be an* ME *of $(\mathcal{E}, m, M, Q_{bm}, \Delta m, Q_m, \sigma)$ and assume $M - (Q_{bm}/(1+r)) > 0$. Let $\Lambda : D \to \mathbb{R}_{++}$ be defined as above. Then, for any policy $\tilde{\pi} \in \{(\tilde{M}, \tilde{Q}_{bm}, \Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}), \tilde{\sigma}) : (\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) \in D\}$, there exists an ME $\langle \tilde{p}, \tilde{r}, \tilde{q}, \tilde{x}\rangle$ of $(\mathcal{E}, m, \tilde{\pi})$ such that $\tilde{x} = x$; and $Q_{m\ell}/p_\ell = \tilde{Q}_{m\ell}/\tilde{p}_\ell$ for all $\ell \in L$, where $\tilde{Q}_{m\ell} = \Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma})Q_{m\ell}$ is government's expenditure in $\tilde{\pi}$.*

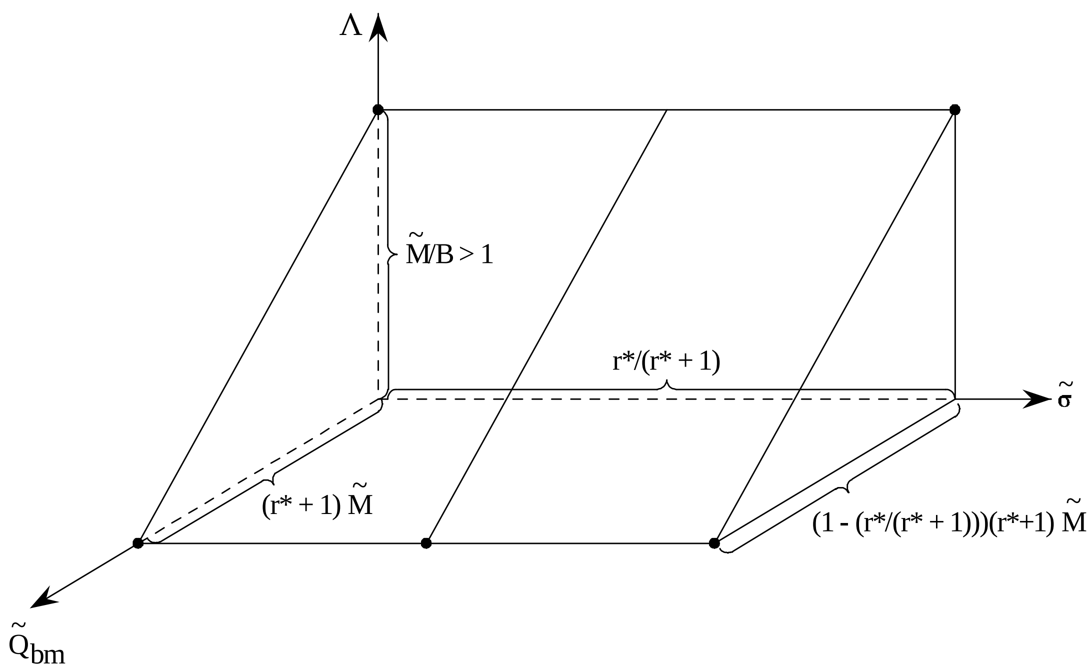

Figure 10: The Graph of Λ for fixed $\tilde{M}$

Corollary Let $\langle p, r, q, x \rangle$ be an ME of $(\mathcal{E}, m, M, Q_{bm}, \Delta m, Q_m, \sigma)$ *and assume* $M - (Q_{bm}/(1+r)) > 0$. *Then there exist policies* π_1, π_2, π_3 *which are equivalent to* $(M, Q_{bm}, \Delta m, Q_m, \sigma)$ *such that the government finances its expenditures solely by*

(1) *printing money* $\tilde{m}_\beta$ *prior to trade in* π_1 *(i.e.,* $\tilde{Q}_{bm}$, $\tilde{\sigma}$ *and* $\tilde{m}_\alpha$ *are zero under* π_1*)*

(2) *borrowing money and repaying it by printing* $\tilde{m}_\alpha$ *after trade in* π_2 *(i.e.,* $\tilde{\sigma}$ *and* $\tilde{m}_\beta$ *are zero under* π_2*)*

(3) *borrowing money and repaying it out of tax revenue in* π_3 *(i.e., via a totally balanced policy where* $\tilde{m}_\beta = \tilde{m}_a = 0$ *under* π_3*)*

(Here $\tilde{m}_\beta$, $\tilde{m}_\alpha$ *are the endogenously determined quantities of money printed in the* ME *achieved under the relevant policy* π_i*.)*

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12 Appendix

Proof of Lemma 1 Let $C_r(p, x^h - e^h) \leq m^h$. We write $\tau \equiv x^h - e^h$. We will construct q^h so that $(q^h, x^h) \in B(p, r, e^h, m^h)$. Let

$$\begin{aligned} q^h_{bm} &= (1+r)\max\{p \cdot {}^*\tau - m^h, 0\} \\ q^h_{\ell m} &= {}_*\tau_\ell \text{ for all } \ell \in L \\ q^h_{m\ell} &= p_\ell\, {}^*\tau_\ell \text{ for all } \ell \in L. \end{aligned}$$

Observe that with $\tilde{m}^h \equiv q^h_{bm}/(1+r)$, $p \cdot {}^*\tau \leq m^h + \tilde{m}^h$, so

$$\sum_{\ell \in L} q^h_{m\ell} = p \cdot {}^*\tau \ \leq m^h + \tilde{m}^h$$

and inequality (2) of the budget set is verified. Since $x^h \in \mathbb{R}^L_+$, ${}_*\tau_\ell \leq e^h_\ell$, verifying inequality ($3\ell$). Observe next that if $q^h_{bm} = 0$, then (5) is automatic. Otherwise,

$$\begin{aligned} q^h_{bm} &= (1+r)[p \cdot {}^*\tau - m^h] \\ &= (1+r)\left[p \cdot {}^*\tau - \frac{1}{1+r} p \cdot {}^*\tau - m^h\right] + p \cdot {}^*\tau \\ &= (1+r)[C_r(p, \tau) - m^h] + p \cdot {}^*\tau \\ &\leq p \cdot {}^*\tau \ = \sum_{\ell \in L} p_\ell q^h_{\ell m} \end{aligned}$$

verifying inequality (5). Finally, letting $\tilde{x}^h_\ell \equiv q^h_{m\ell}/p_\ell = {}^*\tau_\ell$, we get

$$\begin{aligned} x^h_\ell &= (e^h_\ell - \ {}_*\tau_\ell) + {}^*\tau_\ell \\ &= (\Delta 3\ell) + \tilde{x}^h_\ell \end{aligned}$$

verifying inequality (6ℓ). Hence $(q^h, x^h) \in B(p, r, e^h, m^h)$, and so $x^h \in B_C(p, r, e^h, m^h)$.

Conversely, suppose there exists $q^h \in \mathbb{R}^{2L+1}_+$ with $(q^h, x^h) \in B^h(p, r, e^h, m^h)$. Let $\tau \equiv x^h - e^h$, and recall that $\tilde{x}^h_\ell = q^h_{m\ell}/p_\ell$ and (by (3ℓ) and (6ℓ))

$$\tilde{x}^h_\ell - q^h_{\ell m} \geq x^h_\ell - e^h_\ell = {}^*\tau_\ell - {}_*\tau_\ell \,.$$

Hence $\tilde{x}^h_\ell \geq {}^*\tau_\ell$ and $q^h_{\ell m} \geq {}_*\tau_\ell$.

Now,

$$\begin{aligned} C_r(p, x^h - e^h) &= p \cdot {}^*\tau - \frac{1}{1+r} p \cdot {}^*\tau \\ &= \sum_{\ell \in L} p_\ell \tilde{x}^h_\ell - \frac{1}{1+r} \sum_{\ell \in L} p_\ell q^h_{\ell m} - \left[\sum_{\ell \in L} p_\ell(\tilde{x}^h_\ell - {}^*\tau_\ell) - \frac{1}{1+r} \sum_{\ell \in L} p_\ell(q^h_{\ell m} - {}^*\tau_\ell)\right] \\ &\leq \sum_{\ell \in L} p_\ell \tilde{x}^h_\ell - \frac{1}{1+r} \sum_{\ell \in L} p_\ell q^h_{\ell m} \\ &\quad \text{(by the preceding three inequalities)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \in L} q^h_{m\ell} - \frac{1}{1+r} \sum_{\ell \in L} p_\ell q^h_{\ell m} \\
&\leq \sum_{\ell \in L} q^h_{m\ell} - \frac{1}{1+r} \left[(1+r)\tilde{m}^h - (m^h + \tilde{m}^h - \sum_{\ell \in L} q^h_{m\ell} \right] \\
&\quad \text{(by the budget set inequalities (5), (2), and (1))} \\
&= \frac{r}{1+r} \sum_{\ell \in L} q^h_{m\ell} - \frac{r}{1+r}(m^h + \tilde{m}^h) + m^h \\
&\leq \frac{r}{1+r}(m^h + \tilde{m}^h) - \frac{r}{1+r}(m^h + \tilde{m}^h) + m^h \\
&\quad \text{(by the budget set in inequality (2))} \\
&= m^h.
\end{aligned}
$$

■

Proof of Lemma 2 If $\gamma(x) = 0$, the lemma is obvious, since $\gamma(y) \geq 0$ for any y. So suppose $\gamma(x) > \gamma > 0$. Then we can find trades $(\tau^h)_{h \in H}$ such that $\sum_{h \in H} \tau^h = 0$ and for all $h \in H$, $x^h + \tau^h \in \mathbb{R}^L_+$ and $u^h(x^h + \tau^h(\gamma)) > u^h(x^h)$. It follows immediately from the continuity of u^h that for large enough k and $\lambda < 1$ but sufficiently close to 1, $x^h(k) + \lambda\tau^h(\gamma) \geq 0$ and $u^h(x^h(k) + \lambda\tau^h(\gamma)) > u^h(x^h(k))$. ■

Proof of Lemma 3 If no-trade is a Walras equilibrium, then, by the first welfare theorem, e is Pareto optimal with respect to $(v^h_\gamma)_{h \in H}$, and hence there are not gains-to-γ-diminished-trade at e (with respect to $(u^h)_{h \in H}$). Conversely, if there are not gains-to-γ-diminished-trade at e (with respect to $(u^h)_{h \in H}$), then e is Pareto optimal with respect to the utilities $(v^h_\gamma)_{h \in H}$. By the second welfare theorem, no-trade is a Walras equilibrium for $(v^h_\gamma)_{h \in H}$. ■

Proof of Theorem 1 We first prove the theorem assuming that each u^h is strictly concave.

Let $(\gamma_k)_{k=1}^\infty$ be a monotonically increasing sequence $\gamma_k \to_k \gamma(x)$. Let $\langle p(k), (x^h(k))_{h \in H} \rangle$ be a Walras equilibrium of $(v^h_{\gamma_k}, x^h)_{h \in H}$ for each k. Since w.l.o.g. $\sum_{\ell=1}^L p_\ell(k) = 1$ for all k, by passing to convergence subsequences we may suppose that[14] $p(k) \to p \gg 0$ and $x^h(k) \to \tilde{x}^h$ for all $h \in H$. Since the functions $v^h_{\gamma_k}$ converge uniformly to $v^h_{\gamma(x)}$ on compact sets, $\langle p, (\tilde{x}^h)_{h \in H} \rangle$ must be a Walras equilibrium for the economy $(v^h_{\gamma(x)}, x^h)_{h \in H}$. By strict concavity and Lemma 2, $\tilde{x}^h = x^h$ for all $h \in H$.

Furthermore, by Lemma 2 we know that for each k, $\tau^h(k) \equiv \tilde{x}^h(k) - x^h \neq 0$ for at least one $h \in H$ (otherwise there are no more than $\gamma_k < \gamma(x)$ gains-to-trade at x). Hence for each k we can find a trading cycle $(\ell_1(k), \ell_2(k), ..., \ell_{n_k}(k))$ and $(h_1(k), h_2(k), ..., h_{n_k}(k))$ such that each $h_i(k)$ sells $\ell_i(k)$ and buys $\ell_{i+1}(k)$. Since the set of all possible cycles is finite, we may assume the same cycle obtains on our subsequence of k. We denote it $(\ell_1, ..., \ell_n)$, $(h_1, ..., h_n)$.

[14] All price ratios are bounded since utilities are strictly monotonic. Then since $p \neq 0$ by normalization, we have $p \gg 0$.

For each k, and each trader $h \in \{h_1, ..., h_n\}$, defines the net trades $\tilde{\tau}^h(k) \in \mathbb{R}^L$ by

$$\tilde{\tau}^h_\ell(k) = \begin{cases} -\varepsilon^h(k) p_{\ell_{i+1}}(k) & \text{if } \ell = \ell_i \\ \varepsilon^h(k) p_{\ell_i}(k) & \text{if } \ell = \ell_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon^h(k) < \min\{-\tau^{h_i}_{\ell_i}(k)/p_{\ell_i}(k), \tau^{h_i}_{\ell_{i+1}}(k)/p_{\ell_{i+1}}(k)\}$. From utility maximization at Walras equilibrium,

$$v^{h_i}_{\gamma_k}(x^{h_i}(k) - \tilde{\tau}^{h_i}(k)) < v^{h_i}_{\gamma_k}(x^{h_i}(k)) \text{ for all } k.$$

It follows that

$$\varepsilon^h(k) p_{\ell_i}(k) \frac{\partial u^{h_i}(x^{h_i}(k) - \tilde{\tau}^{h_i}(k))}{\partial x_{\ell_{i+1}}} > (1+\gamma_k) \varepsilon^h(k) p_{\ell_{i+1}}(k) \frac{\partial u^h(x^{h_i}(k) - \tilde{\tau}^{h_i}(k))}{\partial x_{i_\ell}}.$$

Rearranging terms,

$$\frac{\dfrac{\partial u^{h_i}(x^{h_i}(k) - \tilde{\tau}^{h_i}(k))}{\partial x_{\ell_{i+1}}}}{\dfrac{\partial u^h(x^{h_i}(k) - \tilde{\tau}^{h_i}(k))}{\partial x_{i_\ell}}} > (1+\gamma_k) \frac{p_{\ell_{i+1}}(k)}{p_{\ell_i}(k)}.$$

Taking products over $i = 1, ..., n$, and passing to the limit as $k \to \infty$,

$$\prod_{i=1}^{n} \frac{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{\ell_{i+1}}}}{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{i_\ell}}} > (1+\gamma(x))^n.$$

Conversely, appealing again to Lemma 2, let $\langle p, (x^h)_{h \in H} \rangle$ be a Walras equilibrium for the economy $(v^h_{\gamma(x)}, x^h)_{h \in H}$. Then for each $h \in H$,

$$\frac{\dfrac{\partial u^h(x^h)}{\partial x_i}}{\dfrac{\partial u^h(x^h)}{\partial x_j}} \leq (1+\gamma(x)) \frac{p_i}{p_j}.$$

Hence on any chain,

$$\prod_{i=1}^{n} \frac{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{\ell_{i+1}}}}{\dfrac{\partial u^{h_i}(x^{h_i})}{\partial x_{i_\ell}}} \leq (1+\gamma(x))^n.$$

This concludes the proof assuming the u^h are strictly concave. But if u^h is not strictly concave, we can replace each u^h with $\tilde{u}^h$ defined by $\tilde{u}^h(x) = u^h(x) + \varepsilon \sum_{\ell \in L} \sqrt{x_\ell}$. Taking limits as $\varepsilon \to 0$ gives our result. ■

Proof of Corollary 1 If $(\tau^h)_{h\in H}$ improves on $(x^h)_{h\in H}$, for the utilities v_γ^h, then any Walras equilibrium $\langle p, (\tilde{x}^h)_{h\in H}\rangle$ for the economy $(v_\gamma^h, x^h)_{h\in H}$ must involve some nonzero trade. From these excess demands at equilibrium we can extract a cycle, as in the proof of Theorem 1. ■

Proof of Corollary 2 Proceed exactly as in the proof of theorem 1, noting that for concave utilities, left-hand derivatives are continuous from the left and exceed right-hand derivatives, which are continuous from the right. ■

Proof of Theorem 2 For any $\varepsilon > 0$, we establish the existence of an ε-monetary equilibrium (ε-ME) whose limit (as $\varepsilon \to 0$) will yield an ME.

An ε-ME may be thought of as the strategic equilibrium of the following generalized game $\mathcal{G}_\varepsilon$. Replace each $h \in H$ by a continuum $(h-1, h]$ of identical households. Each t in the interval $(h-1, h]$ has the characteristics

$$\begin{aligned}(e^t, m^t) &\equiv (e^h, m^h)\\ u^t &\equiv u^h.\end{aligned}$$

The *ambient* strategy-set of each $t \in [h-1, h)$ is $B(\varepsilon) = \{(q^t, x^t) \in \mathbb{R}_+^{2L+1} \times \mathbb{R}_+^L :$ every component is $\leq 1/\varepsilon\}$. Throughout we shall focus on *type-symmetric* strategies. (This permits the use of the notation (q^h, x^h) in three different senses: as the *vector* in $\mathbb{R}_+^{2L+1} \times \mathbb{R}_+^L$, which is the common individual strategy chosen by each household $t \in (h-1, h]$ of type h; as the constant *function* which maps each $t \in (h-1, h]$ to the vector (q^h, x^h) and describes the symmetric strategy-selection of households of type h; and as the *integral* of this constant function on the unit interval $(h-1, h]$, which gives the aggregate strategy by households of type h. The sense in which (q^h, x^h) is used will be indicated, or else will be clear from the context.)

Given a strategy-selection $(q, x) \equiv (q^h, x^h)_{h\in H}$ by all households, market prices $p(\varepsilon, q, x)$, $r(\varepsilon, q, x)$ form according to the rule:

$$\begin{aligned}p_\ell(\varepsilon, q, x) &= \frac{\varepsilon + \sum\limits_{h\in H} q_{m\ell}^h}{\varepsilon + \sum\limits_{h\in H} q_{\ell m}^h}\\ 1 + r(\varepsilon, q, x) &= \frac{\varepsilon + \sum_{h\in H} q_{bm}^h}{M}.\end{aligned}$$

(In the above formulae, read q^h as integral and (q, x) as a function.) In the game $\mathcal{G}_\varepsilon$, we imagine an external agent who puts up ε units of goods, money and bonds as indicated in the formulae. Prices form to clear all markets (taking the external agent into account). In other words, all of M is disbursed to households and the external agent in proportion to their bonds; and at each commodity-money market all the money (or, commodity) received is disbursed to households and the external agent in proportion to the commodity (or, money) sent by them. Of course, given an arbitrary selection (q, x), it may well happen that at the emergent prices households do not balance their budgets. So we are led to consider a generalized game with strategy sets that depend on others' choices. We define the *feasible* strategy-set of each $t \in (h-1, h]$ of type h by:

$$B^h_\varepsilon(q, x) \equiv B(\varepsilon) \cap B(p(\varepsilon, q, x), r(\varepsilon, q, x), e^h, m^h).$$

(Here $(q, x) \equiv$ strategy selection; (e^h, m^h) are t's individual characteristics; $B(p(\varepsilon, q, x), r(\varepsilon, q, x), e^h, m^h)$ is just the budget-set defined earlier.) Given the joint strategies (q, x), each $t \in [h-1, h)$ gets the payoff $u^h(x^h)$.

We define an ε-ME to be a type-symmetric strategic equilibrium of this generalized game $\mathcal{G}_\varepsilon$. We shall shortly prove that ε-ME exists.

Notice that, at any ε-ME, (1) we have a physically closed system, in which all the money or goods sent to market are conserved and redistributed among households and the external agent; (2) all households view $p(\varepsilon, q, x)$, $r(\varepsilon, q, x)$ as fixed, since their individual vector q^h does not affect the integral q^h involved in forming prices; (3) each household chooses optimal strategies (q^h, x^h) in his truncated budget-set (which just consists of those vectors in his standard budget-set that are of size $\leq 1/\varepsilon$ in each component).

Fix $\mu > M + \bar{m}$ and $\eta > \sum_{h \in H} \sum_{\ell \in L} e^h_\ell$, and choose ε small enough to ensure that $M + \bar{m} + L\varepsilon < \mu < 1/\varepsilon$ and $\eta < 1/\varepsilon$. Define

$$u^h_* = u^h(\eta, ..., \eta)$$

and let η^* be chosen to guarantee that

$$u^h(0, ..., 0, \eta^*, 0, ..., 0) > u^h_*$$

for η^* in any component. (W.l.o.g.[15] we may suppose that such a η^* exists.)

Let $B^*(\varepsilon) = ((B(\varepsilon))^H$.

Define the individual "best reply" ("demand") correspondence $\psi^h_\varepsilon : B^*(\varepsilon) \rightrightarrows B(\varepsilon)$ by

$$\psi^h_\varepsilon(q, x) = \arg\max\{u^h(\bar{x}^h) : (\bar{q}^h, \bar{x}^h) \in B^h_\varepsilon(q, x)\}$$

for $h \in H$; and then define ψ_ε from $B^*(\varepsilon)$ to itself by

$$\psi_\varepsilon = \psi^1_\varepsilon \times \cdots \times \psi^H_\varepsilon.$$

Clearly $B^h_\varepsilon(q, x) \equiv B(\varepsilon) \cap B(p(\varepsilon, q, x), r(\varepsilon, q, x), e^h, m^h)$ is non-empty, compact, and convex. On account of the external agent's ε, prices $p(\varepsilon, q, x)$ are always positive, and since $e^h \neq 0$, $B^h_\varepsilon(q, x)$ is a continuous correspondence in (q, x). Hence each ψ^h is non-empty, convex, and upper semi-continuous.

[15] Let $\square$ be the cube in $\mathbb{R}^L_+$ with sides of length η. Recall $u^h : \square \to \mathbb{R}$ is strictly increasing in the variables x_ℓ for $\ell \in L$. Define $\tilde{u}^h : \mathbb{R}^L_+ \to R$ by $\tilde{u}^h(y) = \inf\{L_x(y) : x \in \square,\ L_x$ is an affine function representing a supporting hyperplane to the graph of u^h at the point $(x, u^h(x))\}$. Then it is clear that (a) $\tilde{u}^h$ is concave and strictly monotonic, and coincides with u^h on $\square$, hence ME of our economy are unaltered if we replace u^h by $\tilde{u}^h$; and (b) there exists a η^* such that $\tilde{u}^h(0, ..., 0, \eta^*, 0, ..., 0) > u^h_*$ for η^* in any component.

By Kakutani's Theorem ψ_ε has a fixed point $(q(\varepsilon), x(\varepsilon)) \equiv (q^h(\varepsilon), x^h(\varepsilon))_{h \in H}$, with induced prices $p(\varepsilon)$, $r(\varepsilon)$. The vector $\langle p(\varepsilon), r(\varepsilon), q(\varepsilon), x(\varepsilon)\rangle$ will be called an ε-ME. Select a subsequence of ε-ME as $\varepsilon \to 0$ to ensure that all its components and all ratios of all components converge (possibly to zero or infinity).

Note that $q^h_{bm}(\varepsilon) < \mu < 1/\varepsilon$ (being budget feasible, h must return $q^h_{bm}(\varepsilon)$ to the bank, and there is at most μ units of money in the economy), $q^h_{m\ell}(\varepsilon) \leq$ total expenditure across markets $< \mu$, $q^h_{\ell m}(\varepsilon) \leq e^h_\ell < \eta < 1/\varepsilon$ and $x^h_\ell(\varepsilon) < \eta < 1/\varepsilon$ for $\ell \in L$. Since the limits on their optimal actions are not binding, and their utilities are concave, we obtain

Step 1 For sufficiently small ε, $(q^h(\varepsilon), x^h(\varepsilon))$ is optimal in $B(p(\varepsilon), r(\varepsilon), e^h, m^h)$, not just in $B^h_\varepsilon(q(\varepsilon), x(\varepsilon))$.

Step 2 $r(\varepsilon) \geq 0$, for sufficiently small ε.

Proof Suppose $r(\varepsilon) < 0$. Then let h increase $q^h_{bm}(\varepsilon)$ by a positive δ obtaining $\delta(1 + r(\varepsilon))^{-1} > \delta$ units of bank money. Let him inventory δ to repay this additional loan and spend the surplus to buy $\ell \in L$. This improves his utility, contradicting Step 1.

Step 3 $r(\varepsilon) \leq (\bar{m} + L\varepsilon + \varepsilon)/M < (1 + \bar{m})/M \equiv \tilde{r}$, for small enough ε.

Proof Since all households are budget feasible, all their debts to the bank are honored. So no more than $M + \bar{m} + L\varepsilon$ bonds could have been sold by households. And the external agent sells only ε bonds, so $1 + r \leq (M + \bar{m} + L\varepsilon + \varepsilon)/M$.

Step 4 For small enough ε, $p_k(\varepsilon)/p_\ell(\varepsilon) \leq \eta^*(1 + \tilde{r})/e^*$ for all $k, \ell \in L$, where $e^* \equiv \min_{i \in L} \max_{h \in H} e^h_i$, and $\tilde{r}$ is as in Step 3.

Proof Consider h with $e^h_k = \max\{e^i_k : i \in H\} > 0$. Let $q^h_{bm} = p_k(\varepsilon)e^h_k$, $q^h_{km} = e^h_k$, $q^h_{m\ell} = p_k(\varepsilon)e^h_k/(1 + r(\varepsilon))$ and all other components of $q^h = 0$. From his sale of k, h obtains $p_k(\varepsilon)e^h_k$ units of money, and is able to repay the loan q^h_{bm}. So this action is in his untruncated budget set, and (by Step 1) cannot improve his payoff. But his consumption of ℓ via this action is at least $p_k(\varepsilon)e^h_k/((1 + r(\varepsilon))p_\ell(\varepsilon))$ which must be less than η^* (otherwise h gets more than u^h_* utility, a contradiction). Recalling from Step 3 that $r(\varepsilon) \leq \tilde{r}$, Step 4 follows.

Step 5 Let $m^* = \max\{m^h : h \in H\}$. Then, for small enough ε,

$$\frac{m^*}{\eta^*} \leq p_\ell(\varepsilon).$$

Proof It is clear that u^h_* is an upper bound on the utility of h at an ε-equilibrium (for small enough ε). But if $p_\ell(\varepsilon) < m^*/\eta^*$, then any agent h with $m^h = m^*$ can spend all his private endowment of money to purchase ℓ, consuming at least η^* of ℓ, and thus obtaining more than u^h_* utiles, a contradiction.

Step 6 $p_\ell(\varepsilon) \nrightarrow \infty$ for any $\ell \in L$.

Proof Suppose some $p_\ell(\varepsilon) \to \infty$. By Step 4, $p_k(\varepsilon) \to \infty$ for all $k \in L$. Since $p_k(\varepsilon) < \mu/(\sum_{h\in H} q^h_{km}(\varepsilon))$ we obtain $q^h_{km}(\varepsilon) \to 0$ for all $h \in H$ and $k \in L$, and hence $x^h(\varepsilon) \to e^h$ for all $h \in H$. Let $\hat{p}_\ell(\varepsilon) = p_\ell(\varepsilon)/\sum_{k\in L} p_k(\varepsilon)$ for $\ell \in L$ and $\hat{p} = \lim \hat{p}(\varepsilon)$. By Step 4, $\hat{p} \gg 0$. We also know that (see Step 3) $r \equiv \lim r(\varepsilon) \leq \bar{m}/M$.

Consider the consumption feasible budget sets $B_C(p(\varepsilon), r(\varepsilon), e^h, m^h)$. By homogeneity,

$$B_C(p(\varepsilon), r(\varepsilon), e^h, m^h) = B_C\left(\hat{p}(\varepsilon), r(\varepsilon), e^h, \frac{m^h}{\sum_{\ell\in L} p_k(\varepsilon)}\right).$$

Since $\hat{p}(\varepsilon) \to \hat{p} \gg 0$, $e^h \neq 0$, and $m^h/\sum_{\ell\in L} p_k(\varepsilon) \to 0$, we have (by a standard argument) the set convergence

$$B_C(p(\varepsilon), r(\varepsilon), e^h, m^h) \to B_C(\hat{p}, r, e^h, 0).$$

Hence, since $x^h(\varepsilon)$ is u^h-optimal in $B_C(p(\varepsilon), r(\varepsilon), e^h, m^h)$, $\lim x^h(\varepsilon) = e^h$ must be u^h-optimal in $B_C(\hat{p}, r, e^h, 0)$. Therefore $(\hat{p}, e)$ is Walrasian for the economy $(v^h_r, e^h)_{h\in H}$, where $v^h_r(x) \equiv u^h(e^h + (x - e^h)(r))$. By Lemma 3, there are not gains-to-r-diminished trade at e, with respect to the utilities $(u^h)_{h\in H}$. Hence $\gamma(e) \leq r$, but by Step 3, $r \leq \bar{m}/M$, and so $\gamma(e) \leq \bar{m}/M$, contradicting the gains-to-trade hypothesis.

Step 7 All markets clear at $\langle p, r, q, x\rangle \equiv \lim_{\varepsilon\to 0}\langle p(\varepsilon), r(\varepsilon), q(\varepsilon), x(\varepsilon)\rangle$.

Proof By Steps 2 and 3, $0 \leq r \leq \lim(\bar{m} + L\varepsilon/M) = \bar{m}/M < \infty$, and by Steps 5 and 6, $0 \ll p \ll \infty$. By definition, all markets clear at an ε-ME, once we include the actions of the external agent. But since his actions go to zero, and all prices are finite and positive, it follows that markets clear without the external agent at the limit $\langle p, r, q, x\rangle$.

Step 8 An ME exists.

Proof Utility maximization at $\langle p, r, q, x\rangle$ follows from the continuity of u^h that it is the upper semi-continuity and lower semi-continuity of B_C. ■

Step 9 At any ME $\langle p, r, q, x\rangle$ of $(\mathcal{E}, m, M)$, $r = \bar{m}/M$.

Proof This was proved just after the definition of ME in Section 1.6. ■

Proof of the Corollary First consider case (b). By Theorem 1, $r(k) = \bar{m}(k)/M(k)$, hence $r(k) \to 0$. Define $\hat{p}_\ell(k) \equiv p_\ell(k)/\sum_{j \in L} p_j(k)$ for $\ell \in L$. It is easily verified that x is Walrasian with prices $\hat{p} \equiv \lim \hat{p}(k)$.

Next consider case (a). If $\langle p, r, q, x \rangle$ is an ME, then r must be zero (by an argument analogous to Step 2), hence $\langle p, x \rangle$ is Walrasian. On the other hand if $\langle p, x \rangle$ is Walrasian, choose $\lambda > 0$ so that $\lambda p \cdot \sum_{h \in H}$. Then $(\lambda p, x)$ is achieved at an ME by letting each h borrow and spend $\lambda p \cdot e^h$ to buy the goods x^h. These actions clearly constitute an ME. ■

Remarks

Dropping Strict Monotonicity: the Having–Wanting Chain It is important to note that strict monotonicity of u^h in every commodity can be dropped from the hypotheses of our model. What is needed is a version of resource relatedness. We formulate this in terms of a "having–wanting" chain. If $e^h_\ell > 0$, say that "agent h has ℓ"; and if $u^h(x)$ is everywhere strictly increasing in the variable x_ℓ, say that "agent h wants ℓ." Consider a directed graph on node-set L with arc (ℓ, k) if there exists an agent h who has ℓ and wants k. Our existence theorem holds if we assume for every (ℓ, k) in $L \times L$, with $\ell \neq k$, that there is a directed path (chain) from ℓ to k. The only change required is in the proof of Step 4, which we now indicate.

Exactly as in the proof of Step 4, if arc (ℓ, k) exists then the upper bound of Step 4 is valid. Since any two goods ℓ and k are connected by a chain of length at most $L - 1$, $(\eta^*(1 + \tilde{r})/e^*)^{L-1}$ will be an upper bound for $p_\ell(\varepsilon)/p_k(\varepsilon)$ (for every (ℓ, k) in $L \times L$).

Forced Sales of Commodities For $0 < \alpha \leq 1$, define an α-ME exactly like an ME, but with condition (3) of the budget set amended to read

$$\alpha e^h_\ell \leq q^h_{\ell m} \leq e^h_\ell.$$

Then an α-ME exists *without* the gains-to-trade hypothesis. To see this, repeat the proof of Theorem 1 up to Step 5 (but with households being forced to obey the above inequality) and then notice that $p_\ell(\varepsilon) \leq \mu/\sum_{h \in H} \alpha e^h_\ell$, so $p(\varepsilon) \nrightarrow \infty$, hence the limit is an α-ME. The case $\alpha = 1$ corresponds to the hypothesis made in Lucas [26] and Magill–Quinzii [28]. Theorem 1 shows that money has value even without such α-forced sales.

Proof of Theorem 3 See Dubey–Geanakoplos [7]. ■

Proof of Theorem 4 The proof that $\gamma(x) \leq r$ relies on properties of the cost function C_r which are of interest on their own. Recall that for any trade vector $\tau \in \mathbb{R}^L$, ${}^*\tau_j = \max\{0, \tau_j\}$ denotes purchases, and ${}_*\tau_j = -\min\{0, \tau_j\}$ denotes sales; and that we defined the present-value cost function $C_r : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$C_r(p, \tau) \equiv p \cdot {}^*\tau - \frac{1}{1+r} p \cdot {}^*\tau = \frac{1}{1+r}[p \cdot \tau + rp \cdot {}^*\tau].$$

Recall also that we defined $\tau(\gamma) \equiv \frac{1}{1+\gamma}\,{}^*\tau - {}_*\tau \leq \tau$, for any $\gamma \geq 0$.

Lemma 4 *The cost function $C_r(p,\tau)$ is continuous, convex in τ, homogenous of degree one in p and τ separately, and satisfies*

$$\begin{aligned} C_r(p,\tau+\tilde{\tau}) &\leq C_r(p,\tau)+C_r(p,\tilde{\tau}) \\ C_r(p,\tau(r)) &= \frac{p\cdot\tau}{1+r} \leq C_r(p,\tau) \end{aligned}$$

for any trade vectors $\tau, \tilde{\tau} \in \mathbb{R}^L$, and any $p \geq 0$, $r \geq 0$.

Proof of Lemma 4 Continuity and homogeneity are evident, as is the second displayed inequality. Convexity in τ and homogeneity in τ guarantee the first inequality. To verify convexity in τ, write

$$C_r(p,\tau) = \frac{1}{1+r}\sum_{\ell\in L} p_\ell[\tau_\ell + r\max\{\tau_\ell,0\}].$$

The ℓth term is convex in τ_ℓ, for each $\ell \in L$, hence C_r is convex in τ. Finally, the equality is straightforward from the definitions. ■

We are now ready to prove Theorem 4.

If $\bar{m} = 0$, the theorem follows from Case (a) of the Corollary. So assume $\bar{m} > 0$. Note that this implies, by Theorem 2, that $r > 0$.

We show first that $\gamma(x) \leq r$. If $\gamma(x) > r$, then there exist gains-to-r-diminished-trade at x, i.e., there exist $(\tau^h)_{h\in H}$ such that $\sum_{h\in H}\tau^h = 0$, $x^h + \tau^h \in \mathbb{R}^L_+$ and $u^h(x^h+\tau^h(r)) > u^h(x^h)$ for $h \in H$. It follows that for some h, $p\cdot\tau^h \leq 0$, where p is the ME price vector. From Lemma 4 we must then have $C_r(p,\tau^h(r)) \leq 0$. Let $\bar{\tau}^h = x^h - e^h$ be the ME trade of household h. From Lemma 1, $C_r(p,\bar{\tau}) \leq m^h$. By Lemma 4,

$$C_r(p,\bar{\tau}^h+\tau^h(r)) \leq C_r(p,\bar{\tau}^h) + C_r(p,\tau^h(r)) \leq m^h,$$

hence by Lemma 1 again, $x^h + \tau^h(r) = e^h + \bar{\tau}^h + \tau^h(r) \in B_C(p,r,e^h,m^h)$. But this contradicts the optimization behavior of h in the ME.

To show that $\gamma(x) = r$ under the regularity condition, it is useful to concentrate on "active" households h, namely households that borrow money, sell goods, and buy other goods. By the regularity condition, no active household consumes zero units of any commodity he is positively endowed with.

Since all of the bank money $M > 0$ is borrowed at the ME (see condition (8)), there exists at least one household h_1 with borrowed money $\tilde{m}^{h_1} > 0$. Since the interest rate $r = \bar{m}/M > 0$, every household h spends all the money on hand (i.e., $\tilde{m}^h + m^h$) on purchases at the ME. Hence h_1 could not be hoarding money, and so he must be selling goods to repay the bank, i.e., h_1 is an active seller. Suppose h_1 buys ℓ_2 and sells ℓ_1. Since $r > 0$, h_1 is not indulging in "wash sales," i.e., buying and selling the same commodity. Then some other household h_2 must be selling ℓ_2, and in this case he too must be borrowing money from the bank (otherwise, why sell?) and spending it on purchases of some commodity other than ℓ_2.

Consider a directed graph with a node for each commodity; and arc $(\ell, k) \in L \times L$ if there exists an active household who buys k and sells ℓ.

We have shown that there exists at least one arc; and that if there is an incoming arc at any node, there must also be an outgoing arc at that node. Since L is finite, there is a cycle $(\ell_1, \ell_2), ..., (\ell_i, \ell_{i+1}), ..., (\ell_n, \ell_1)$ and active households $h_1, ..., h_n$ such that h_i buys ℓ_{i+1} and sells ℓ_i (with $\ell_{n+1} \equiv \ell_1$).

The regularity condition implies (as discussed in Section 5)

$$\frac{\nabla^{h_i}_{\ell_{i+1}}(x^{h_i})}{p_{\ell_{i+1}}} = \frac{(1+r)\nabla^{h_i}_{\ell_i}(x^{h_i})}{p_{\ell_i}}$$

for $i = 1, ..., n$. Consider any scalar $\tilde{\gamma} < r$. Then for small enough ε, household h_i would benefit by selling $\varepsilon p_{\ell_{i+1}}$ more units of commodity ℓ_i, and buying εp_{ℓ_i} more units of commodity ℓ_{i+1} but consuming only $\varepsilon p_{\ell_i}/(1+\tilde{\gamma})$ more units of ℓ_{i+1} (since $\tilde{\gamma} < r$, and at r he is indifferent).

So define trades accordingly on the cycle, i.e., let

$$(\tilde{t}^{h_i})_\ell = \begin{cases} -\varepsilon p_{\ell_{i+1}} & \text{if } \ell = \ell_i \\ \varepsilon p_{\ell_i} & \text{if } \ell = \ell_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, ..., n$. For small enough ε, the $(\tilde{t}^{h_i})_{i=1}^n$constitute a feasible trade at $x = (x^h)_{h \in H}$ and we have

$$u^{h_i}(x^{h_i} + \tilde{t}^{h_i}(\tilde{\gamma})) > u^{h_i}(x^{h_i})$$

for $i = 1, ..., n$. This shows that there are gains-to-$\tilde{\gamma}$-diminished trades at x, i.e., $\gamma(x) > \tilde{\gamma}$ for all $\tilde{\gamma} < r$, proving Theorem 3. ■

Proof of Theorem 6 Suppose $\langle (p, r), (q^h, x^h)_{h \in H} \rangle$ is an ME. Then $r = \bar{m}/M > 0$. Consider a cycle $(\ell_1, \ell_2), ..., (\ell_n, \ell_1)$ in which household h_i buys commodity ℓ_{i+1} and sells commodity ℓ_i (with $\ell_{n+1} \equiv 1$). As shown in the proof of Theorem 4, such a cycle always exists. Then, letting $\tilde{\nabla}$ (or, $\bar{\nabla}$) denote left (or, right) hand derivative, we must have

$$\frac{\tilde{\nabla}^{h_i}_{\ell_{i+1}}(x^{h_i})}{p_{\ell_{i+1}}} \geq (1+r)\frac{\bar{\nabla}^{h_i}_{\ell_i}(x^{h_i})}{p_{\ell_i}}$$

for $i = 1, ..., n$; otherwise h_i would do better to reduce both his purchase of ℓ_{i+1} and the concomitant sale of ℓ_i (by a little). Hence

$$\frac{\tilde{\nabla}^{h_i}_{\ell_{i+1}}(x^{h_i})}{\tilde{\nabla}^{h_i}_{\ell_i}(x^{h_i})} \geq (1+r)\frac{p_{\ell_{i+1}}}{p_{\ell_i}}$$

for $i = 1, ..., n$. Taking products of the left and right sides, we obtain

$$1 + \Gamma^*(e) \geq \left\{ (1+r)^n \prod_{i=1}^{n} \frac{p_{\ell_i}}{p_{\ell_{i+1}}} \right\}^{1/n} \equiv (1+r)1$$

which implies

$$\Gamma^*(e) \geq r$$

a contradiction, since $r = \bar{m}/M$ by Theorem 2. This proves Theorem 6. ■

Proof of Theorem 7 Let $x \equiv (x^h)_{h \in H}$ denote the ME allocation. W.l.o.g., rescaling units of commodities if necessary, suppose $p_\ell = 1$ for all $\ell \in L$. (Recall that $\gamma(e)$ remains unaltered by rescaling.) Also, relabelling households and commodities if necessary, there exists (as shown in the proof of Theorem 3) a "cycle" of households such that household k sells commodity k and buys commodity $k+1$ for $k = 1, ..., n$ (where $n + 1 \equiv 1$). We can *reduce* trade on this cycle by a small $\delta > 0$ without affecting other households' trades. Define trade vectors T^k for $k = 1, ..., n$ by

$$T_\ell^k = \begin{cases} \delta & \text{if } \ell = k \\ -\delta & \text{if } \ell = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Also define $\tilde{T}^k \equiv -T^k$. (See Figure 11.) Notice that the bundle $\tilde{x}^k$ given by

$$\tilde{x}_\ell^k = \begin{cases} x_\ell^k + \delta & \text{if } \ell = k \\ x_\ell^k - \frac{\delta}{1+r} & \text{if } \ell = k+1 \\ x_\ell^k & \text{otherwise} \end{cases}$$

is feasible in the budget set of $k = 1, ..., n$; and also (for small enough δ)

$$\tilde{x}_k^k < e_k^k,\ \tilde{x}_{k+1}^k > e_{k+1}^k \tag{13}$$

for $k = 1, ..., n$, since no agent conducts wash sales at any ME.

By the strict concavity of u^k, the convexity of budget sets and the fact that x^k maximizes u^k on k's budget set, we get

$$u^k(\tilde{x}^k) < u^k(x^k) \equiv u^k(\tilde{x}^k + \tilde{T}^k(r))$$

for $k = 1, ..., n$; hence (since u^k is continuous), for some $\tilde{\gamma} > r$,

$$u^k(\tilde{x}^k) < u^k(\tilde{x}^k + \tilde{T}^k(\tilde{\gamma})) \tag{14}$$

for $k = 1, ..., n$.

Now we invoke separability of the u^h. Denote $u^k(x) \equiv \sum_{\ell=1}^{L} u_\ell^k(x_\ell)$. Then

$$u_{k+1}^k\left(e_{k+1}^k + \frac{\delta}{1+\tilde{\gamma}}\right) - u_{k+1}^k(e_{k+1}^k) \equiv \Delta_{k+1}$$
$$u_k^k(e_k^k) - u_k^k(e_k^k - \delta) \equiv \Delta_k$$
$$u_{k+1}^k\left(\tilde{x}_{k+1}^k + \frac{\delta}{1+\tilde{\gamma}}\right) - u_{k+1}^k(\tilde{x}_{k+1}^k) = \tilde{\Delta}_{k+1}$$
$$u_k^k(\tilde{x}_k^k) - u_k^k(\tilde{x}^k - \delta) = \tilde{\Delta}_k$$

By (13), and the concavity of the u_ℓ^h,

$$\begin{aligned} \Delta_{k+1} &\geq \tilde{\Delta}_{k+1} \\ \Delta_k &\leq \tilde{\Delta}_k \end{aligned}$$

Now since u^k is separable,

$$u^k(\tilde{x}^k + \tilde{T}^k(\tilde{\gamma})) - u^k(\tilde{x}^k) = \tilde{\Delta}_{k+1} - \tilde{\Delta}_k$$

and

$$u^k(e^k + \tilde{T}^k(\tilde{\gamma})) - u^k(e^k) = \Delta_{k+1} - \Delta_k$$

So

$$\begin{aligned} u^k(e^k + \tilde{T}^k(\tilde{\gamma})) - u^k(e^k) &\geq u^k(\tilde{x}^k + \tilde{T}^k(\tilde{\gamma})) - u^k(\tilde{x}^k) \\ &> 0 \end{aligned}$$

for $k = 1, ..., n$ (the strict inequality coming from (14)).

But $\sum_{k=1}^n \tilde{T}^k = 0$ by construction, so we have shown that there are gains-to-$\tilde{\gamma}$-diminished trades at e. Since $\tilde{\gamma} > r$, this proves the lemma. ■

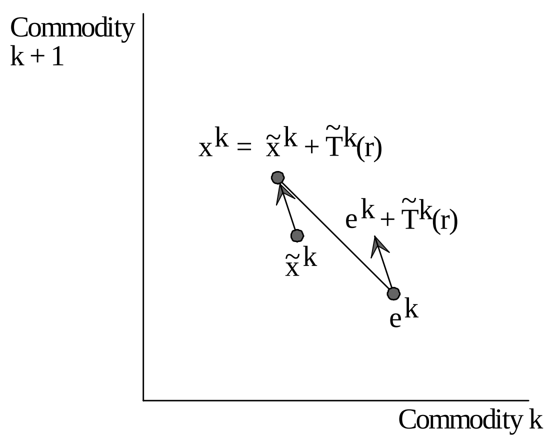

Figure 11: By separability and concavity, the increase in utility from e^k to $e^k + \tilde{T}^k(r)$ is at least as high as from $\tilde{x}^k$ to $\tilde{x}^k + \tilde{T}^k(r)$

Proof of Remark 1 First assume that utilities are strictly concave. Let $\langle p, r, q, x \rangle$ be an ME of $(\mathcal{E}, m, M)$. Define $\tilde{x}^h$ and $\tilde{\tau}^h$ as in the proof of Lemma 5. Then, as shown in that proof, there exist gains-to-$\tilde{\gamma}$-diminished-trade at $\tilde{x} \equiv (\tilde{x}^h)_{h \in H}$ for some $\tilde{\gamma} > r$. Moreover, $\tilde{x}$ can be made arbitrarily close to x by choosing δ small. This proves that there exist y arbitrarily close to x with $\gamma(y) > r$, provided the u^h are strictly concave.

If u^h is not strictly concave, consider $\tilde{u}^h$ defined by $\tilde{u}^h(z) = u^h(z) + \varepsilon \sum_{\ell \in L} (z_\ell)^{1/2}$ and take limits.

Proof of Theorem 8 Suppose there exists a monetary equilibrium with interest rate r. Then, by Theorem 7, $\gamma(e) > r$; and by Theorem 1, $r = \bar{m}/M$. Thus $\gamma(e) > \bar{m}/M$. (This proves "only if"; "if" follows from Theorem 1.) ■

Proof of Theorem 9 Since $r(n) \equiv \sum_{h\in H} m^h(n)/M(n)$ is bounded above as $n \to \infty$, all price ratios $p_\ell(n)/p_k(n)$ (for $\ell, k \in L$) are bounded as $n \to \infty$ (by the argument in Step 4 of the proof of Theorem 1). Suppose $p(n) \nrightarrow \infty$. Then take a subsequence along which the price vectors converge to some limit p^*. A standard argument shows that p^* is an ME price of the limit economy. But the interest rate in the limit economy is $r = \lim_{n\to\infty} \sum_{h\in H} m^h(n)/M(n) = \gamma(e)$. However, by the lemma, $r < \gamma(e)$, a contradiction. ■

Proof of Theorem 10 Define an ε-ME $\langle p(\varepsilon), r(\varepsilon), q(\varepsilon), x(\varepsilon)\rangle$ as before, with the external agent putting up ε at each market and with the government's market actions fixed according to Q. It exists as before. The interest rate $r(\varepsilon) \geq 0$, otherwise households would arbitrage as in Step 2 (of the proof of Theorem 2). Moreover the government prints at most $\sum_{\ell\in L} Q_{m\ell} + \sum_{h\in H} \Delta m^h$ before commodity trade at any ε-ME; and at most Q_{bm} after commodity trade. Thus the total money in the system, as also the interest rate $r(\varepsilon)$, is bounded above independent of ε. This implies that price-ratios $p_\ell(\varepsilon)/p_k(\varepsilon)$ are bounded as in Step 4. But, since the government spends at least $\bar{Q}_m/L > 0$ on some commodity, the price of some $\ell \in L$ is bounded from below by $\bar{Q}_m/(L(\sum_{h\in H} e^h_\ell + \varepsilon)) > \bar{Q}_m/(L(\sum_{h\in H} e^h_\ell + 1)) > 0$. Hence Step 5 also holds, bounding prices away from 0.

Let $\langle p, r, q, x\rangle \equiv \lim_{\varepsilon\to 0}\langle p(\varepsilon), r(\varepsilon), q(\varepsilon), x(\varepsilon)\rangle$ with possibly $p = \infty$. First we claim that $r \leq \max\{0, ([Q_{bm} + x(m,\pi)]/M) - 1\} = \rho(m,\pi)$. If $r = 0$, there is nothing to show. So suppose $r > 0$. Denote $\sum_{h\in H} q^h_{bm} \equiv \bar{q}_{bm}$ and $\sum_{h\in H} q^h_{bm}(\varepsilon) \equiv \bar{q}_{bm}(\varepsilon)$. The external agent borrows $\varepsilon/(1 + r(\varepsilon)) \leq \varepsilon$ units of bank money, since he sells only ε units of the bond and since $r(\varepsilon) \geq 0$. Hence households borrow $(\bar{q}_{bm}(\varepsilon)/(Q_{bm} + \bar{q}_{bm}(\varepsilon)))(M - \varepsilon/(1 + r(\varepsilon))) \to \bar{q}_{bm}M/(Q_{bm} + \bar{q}_{bm})$ units of bank money. Since $r(\varepsilon) \to r > 0$, households spend all the money at hand on commodity trade, i.e., they collectively spend $\delta(\bar{q}_{bm})M + m^h + \Delta m^h$ in the limit (where $\delta(x) \equiv x/(Q_{bm} + x)$). The external agent spends $L\varepsilon \to 0$, which we ignore. The government spends $\bar{Q}_m \equiv \sum_{\ell\in L} Q_{m\ell} > 0$. Thus (letting $\bar{m} \equiv \sum_{h\in H} m^h$ and $\Delta\bar{m} \equiv \sum_{h\in H} \Delta m^h$) total expenditure on commodity trade converges to $\delta(\bar{q}_{bm})M + \bar{m} + \Delta\bar{m} + \bar{Q}_m$. Of this, a fraction σ is taxed away by the government. Hence only the amount $(1 - \sigma)[\delta(\bar{q}_{bm})M + \bar{m} + \Delta\bar{m} + \bar{Q}_m] = g(\bar{q}_{bm})$ accrues to households *and* the external agent. Since households have spent all their cash, this is their only source of funds to repay the bank. From Figure 11, we know $\bar{q}_{bm} \leq g(\bar{q}_{bm})$ implies that $\bar{q}_{bm} \leq x(m,\pi)$. Hence

$$1 + r = \frac{Q_{bm} + \bar{q}_{bm}}{M} \leq \frac{Q_{bm} + x(m,\pi)}{M},$$

establishing the claim on r.

Now suppose $p(\varepsilon) \to \infty$. Then all commodity sales $\to 0$ since the money in the system is bounded above. Hence $x^h(\varepsilon) \to e^h$ for all $h \in H$, and public goods produced $F((Q_{m\ell}/p_\ell(\varepsilon))_{\ell\in L}) \to F(0) = 0$.

Let $\tilde{p} = \lim(p(\varepsilon)/\sum_{\ell\in L} p_\ell(\varepsilon))$. Since all price ratios stay bounded in $p(\varepsilon)$, $\tilde{p} \gg 0$.

Let $\hat{r} \equiv (\sigma+r)/(1-\sigma)$. Notice that, in the limit, all trades τ such that $e^h+\tau \in \mathbb{R}^L_+$, and

$$\tilde{p} \cdot {}^*\tau = \frac{1}{1+\hat{r}}\hat{p} \cdot {}_*\tau$$

are feasible for household h. As before, this implies that no trade is a Walras equilibrium for utilities $(v^h_{\hat{r}})_{h\in H}$ (where $v^h_{\hat{r}}$ is defined using u^h_0), and hence there are *no* gains-to-$\hat{r}$-diminished-trade at e for the utilities $(u^h_0)_{h\in H}$, contradicting the gains-to-trade hypothesis that $\gamma_0(e) > [\sigma + \rho(m,\pi)]/(1-\sigma)$, since $r \leq \rho(m,\pi)$ as we saw.

We conclude that $p(\varepsilon) \nrightarrow \infty$. It is now straightforward to verify that $\langle p, r, q, x\rangle$ is a bona-fide ME.

We still must show that $r = \rho(m,\pi)$ at any ME $\langle p, r, q, x\rangle$ of $(\mathcal{E}, m, \pi)$. Suppose there is no hoarding at the ME by households and that they spend all their borrowed and privately held money $\delta(\bar{q}_{bm})M + \bar{m} + \Delta\bar{m}$ on purchases of commodities, where $\bar{q}_{bm} \equiv \sum_{h\in H} q^h_{bm}$. Government spends $\bar{Q}_m$. Thus exactly $g(\bar{q}_{bm}) \equiv (1-\sigma)(\delta(\bar{q}_{bm})M + \bar{m} + \Delta\bar{m} + \bar{Q}_m)$ is in the hands of households after trade. Since they will not default, or be left holding worthless cash in equilibrium, $g(\bar{q}_{bm}) = \bar{q}_{bm}$. This shows $r = \rho(m,\pi)$.

If there is hoarding by households at the ME, then $r = 0$ and spending is strictly less than $\bar{q}_{bm} = \delta(\bar{q}_{bm})M$. Households will clearly spend at least as much as their privately held money. (If they spend less, they could do so exclusively out of their private money, thus avoiding all debt, and freeing the rest of their private money for more purchases.) So let $\alpha\delta(\bar{q}_{bm})M$ be the fraction of borrowed money that households collectively spend where $0 \leq \alpha < 1$. We must have

$$(1-\sigma)(\alpha\delta(\bar{q}_{bm})M + \bar{m} + \Delta\bar{m} + \bar{Q}_m) = \alpha\bar{q}_{bm},$$

since the hoarded money $(1-\alpha)\delta(\bar{q}_{bm})M$ pays off an equal amount of bonds $(1-\alpha)\bar{q}_{bm}$. This implies

$$(1-\sigma)(\bar{m} + \Delta\bar{m} + \bar{Q}_m) = \sigma\alpha\bar{q}_{bm} < \sigma\bar{q}_{bm} = \sigma(M - Q_{bm}).$$

(The first equality holds since $\bar{q}_{bm} = \delta(\bar{q}_{bm})M$; the inequality holds since $\alpha < 1$; and the last equality holds since $\bar{q}_{bm} + Q_{bm} = M$ on account of $r = 0$.) We conclude that if

$$\bar{m} + \Delta\bar{m} + \bar{Q}_m \geq (\sigma/(1-\sigma))(M - Q_{bm})$$

then there is no hoarding of money by households, and hence $r = \rho(m,\pi)$.

Suppose households hoard money. Then $r = 0$ and $q_{bm} + Q_{bm} = M$ and

$$\bar{m} + \Delta\bar{m} + \bar{Q}_m < (\sigma/(1-\sigma))(M - Q_{bm}).$$

Put $z \equiv M - Q_{bm}$. Rewriting the above inequality,

$$\begin{aligned} & (1-\sigma)[\bar{m} + \Delta\bar{m} + \bar{Q}_m] < \sigma z \\ \Leftrightarrow\ & (1-\sigma)[z + \bar{m} + \Delta\bar{m} + \bar{Q}_m] < z \\ \Leftrightarrow\ & g(z) < z \\ \Leftrightarrow\ & x(m,\pi) < z = M - Q_{bm}. \end{aligned}$$

(The second "⇔" follows since $zM/(Q_{bm}+z) = z$ for $z \equiv M - \bar{Q}_{bm}$; the third "⇔" follows from Figure 11 for g.) But then $\rho(m,\pi) = 0$ by definition (see (11) of Section 10.2) and again $r = \rho(m, M, \pi)$. ■

Proof of Corollary to Theorem 10 (This follows the last three implications in the Proof of Theorem 10.) Indeed

$$\begin{aligned} r = 0 &\Leftrightarrow x(m,\pi) \leq M - Q_{bm} \\ &\Leftrightarrow g(M - Q_{bm}) \leq M - Q_{bm} \\ &\Leftrightarrow (1-\sigma)(M - Q_{bm} + \bar{m} + \Delta\bar{m} + \bar{Q}_m) \leq M - Q_{bm} \\ &\Leftrightarrow \bar{m} + \Delta\bar{m} + \bar{Q}_m \leq (\sigma/(1-\sigma))(M - Q_{bm}). \end{aligned}$$

(The first "⇔" comes from Theorem 10; the second "⇔" from Figure 10 for g; the third "⇔" from substituting $M - Q_{bm}$ for x in $g(x)$.) ■

Proof of Theorem 11 We shall construct $\langle \tilde{p}, \tilde{r}, \tilde{q}, \tilde{x} \rangle$. First define $\tilde{r} \equiv (1-\tilde{\sigma})r^* - \tilde{\sigma}$, where $r^* \equiv (r+\sigma)/(1-\sigma)$. Then $\tilde{r} \geq 0$ since $\tilde{\sigma} \leq r^*/(r^*+1)$. By the definition of $\tilde{r}$

$$\frac{\tilde{r}+\tilde{\sigma}}{1-\tilde{\sigma}} = \frac{r+\sigma}{1-\sigma} \tag{i}$$

Next denote $M_1 \equiv Q_{bm}/(1+r)$, $M_2 \equiv \tilde{Q}_{bm}/(1+\tilde{r})$ and observe

$$\Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) = (\tilde{M} - M_2)/(M - M_1) > 0.$$

(We omit the straightforward algebra that checks this.) For brevity, denote $\Lambda(\tilde{M}, \tilde{Q}_{bm}, \tilde{\sigma}) \equiv \tilde{\lambda}$. Now define, for $h \in H$ and $\ell \in L$

$$\begin{aligned} \Delta\tilde{m}^h &= \tilde{\lambda}(m^h + \Delta m^h) - m^h && \text{(ii)} \\ \tilde{q}^h_{m\ell} &= \tilde{\lambda} q^h_{m\ell} && \text{(iii)} \\ \tilde{q}^h_{\ell m} &= q^h_{\ell m} && \text{(iv)} \\ \tilde{q}^h_{bm} &= \tilde{\lambda}\frac{(1+\tilde{r})}{(1+r)} q^h_{bm} && \text{(v)} \\ \tilde{x}^h_{\ell m} &= x^h_{\ell m} && \text{(vi)} \\ \tilde{Q}_{m\ell} &= \tilde{\lambda} Q_{m\ell} && \text{(vii)} \\ \tilde{p}_\ell &= \tilde{\lambda} p_\ell && \text{(viii)} \end{aligned}$$

Notice that $(\Delta\tilde{m}^h, \tilde{Q}_{m\ell}) = (\Delta m^h(\tilde{\lambda}), Q_{m\ell}(\tilde{\lambda}))$ as required.

We submit that $\langle \tilde{p}, \tilde{r}, \tilde{q}, \tilde{x} \rangle$ is an ME of $(\mathcal{E}, m, \tilde{M}, \tilde{Q}_{bm}, \Delta\tilde{m}, \tilde{Q}_{m\ell}, \tilde{\sigma})$. To verify this, notice (by (ii) and (viii)) that $m^h + \Delta\tilde{m}^h$ has the same purchasing power at $\tilde{p}$, as $m^h + \Delta m^h$ has at p. Moreover, by (i), the impediment to trade is the same for each household at the first scenario $\langle p, r, q, x \rangle$ and at the second scenario $\langle \tilde{p}, \tilde{r}, \tilde{q}, \tilde{x} \rangle$. Hence households face the same budget set in the two scenarios and their optimal consumptions are invariant.

All sales of commodities remain the same (by (iv)), and expenditures are scaled by $\tilde{\lambda}$ in the second scenario (by (iii) and (vii)), hence the definition of prices $\tilde{p}$ in (viii) shows that commodity markets clear in the second scenario (since they already cleared in the first scenario by assumption). Finally $\sum_{h\in H} \tilde{q}^h_{bm} = \tilde{\lambda}(1+\tilde{r}) \sum_{h\in H} q^h_{bm}/(1+r) = ((\tilde{M} - M_2)/(M - M_1))(1 + \tilde{r})(\sum_{h\in H} q^h_{bm}/(1 + r)) = ((\tilde{M} - M_2)/(M - M_1))(1 + \tilde{r})(M - M_1) = (\tilde{M} - M_2)(1 + \tilde{r})$. (The third equality holds because the bond market clears in the first scenario by assumption; i.e., households borrow $M - M_1$ with their bonds $\sum_{h\in H} q^h_{bm}$ at interest rate r, given that the government is borrowing $M_1 = Q_{bm}/(1+r)$.) Hence $\tilde{Q}_{bm} + \sum_{h\in H} \tilde{q}^h_{bm} = (1+\tilde{r})M_2 + (1+\tilde{r})(\tilde{M} - M_2) = (1+\tilde{r})\tilde{M}$ showing that the bond market clears in the second scenario as well.

This proves that $\langle \tilde{p}, \tilde{r}, \tilde{q}, \tilde{x} \rangle$ is an ME of $(\mathcal{E}, m, \tilde{M}, \tilde{Q}_{bm}, \Delta\tilde{m}, \tilde{Q}_{m\ell}, \tilde{\sigma})$. But by (vii) and (viii), it is evident that $\tilde{Q}_{m\ell}/\tilde{p}_\ell = Q_{m\ell}/p_\ell$ for all $\ell \in L$. ■

Proof of Corollary to Theorem 11 Fix $\tilde{M} = M$, and consider Figure 10. Let $D(M) = \{(\tilde{Q}_{bm}, \tilde{\sigma}) : (M, \tilde{Q}_{bm}, \tilde{\sigma}) \in D\}$, and consider the restriction of Λ to $D(M)$. The policy π_1 is clearly given by the point $(0, 0)$ in the restricted domain $D(M)$ of Λ. (If $\bar{Q}_m = 0$ then π_1 suffices for π_2 and π_3, so we assume $\bar{Q}_m > 0$.) Now fix $\tilde{\sigma} \in [0, r^*/(r^* + 1)]$, and go up the "$\tilde{\sigma}$-vertical line" in $D(M)$ consisting of points $(\tilde{Q}_{bm}, \tilde{\sigma})$, where $\tilde{Q}_{bm}$ increases in $[0, (r^*+1)(1-\tilde{\sigma})M)$. It is evident from our formulae that (on the $\tilde{\sigma}$-vertical line):

(i) the interest rate $\tilde{r} = r^*(1 - \tilde{\sigma}) - \tilde{\sigma}$ is constant (It falls linearly from r^* to 0 as we shift the vertical line, by raising $\tilde{\sigma}$ from 0 to $r^*/(r^* + 1)$.)

(ii) government expenditure falls linearly from $(M/B)\bar{Q}_m$ to 0

(iii) total expenditure falls linearly to 0

(iv) money borrowed by the government rises linearly from 0 to M (not hitting M since the points $((r^*+1)(1-\tilde{\sigma})M, \tilde{\sigma})$, on the sloping boundary of the trapezium, are excluded from $D(M)$).

The two straight lines, given by (ii) and (iv) for fixed $\tilde{\sigma}$, intersect when $\tilde{Q}_{bm} = Q^*_{bm}(\tilde{\sigma}) \equiv M\bar{Q}_m(r^*+1)(1-\tilde{\sigma})/(B+\bar{Q}_m)$; and therefore the policy $(Q^*_{bm}(\tilde{\sigma}), \tilde{\sigma}, \Lambda(Q^*_{bm}(\tilde{\sigma}), \tilde{\sigma}))$ has $\tilde{m}_\beta = 0$ for all $\tilde{\sigma} \in [0, r^*/(r^* + 1)]$. Let π_2 correspond to $\tilde{\sigma} = 0$. It is easy to check that it satisfies the requirements of part (b) of the Corollary.

Again consider the points $(\tilde{Q}_{bm}, \tilde{\sigma})$ on the $\tilde{\sigma}$-vertical line for fixed $\tilde{\sigma} \in (0, r^*/(r^* + 1))$. As $\tilde{Q}_{bm} \downarrow 0$, the total expenditure and tax revenue converge to positive numbers by (ii), (iii); but the amount borrowed from, and owed to, the bank converges to 0. Hence $\tilde{m}_\alpha < 0$ for small $\tilde{Q}_{bm}$. On the other hand, as $\tilde{Q}_{bm} \uparrow (r^* + 1)(1 - \tilde{\sigma})M$ the amount borrowed by the government converges to M, while expenditures and tax revenue converge to 0 (by (ii), (iii)). Since the interest rate is positive (by (i)), the only way for the government to meet its interest payment is to print money at the end. Thus $\tilde{m}_\alpha > 0$ as $\tilde{Q}_{bm} \uparrow (r^* + 1)(1 - \tilde{\sigma})M$. Since clearly $\tilde{m}_\alpha$ varies linearly on the $\tilde{\sigma}$-vertical line, there is a unique $\tilde{Q}_{bm} = Q^{**}_{bm}(\tilde{\sigma})$ at which $\tilde{m}_\alpha = 0$. Any of the policies $(Q^{**}_{bm}(\tilde{\sigma}), \tilde{\sigma}, \Lambda(Q^{**}_{bm}(\tilde{\sigma}), \tilde{\sigma}))$ for $\tilde{\sigma} \in (0, r^*/(r^* + 1))$ suffices for π_3.

The locus of points $(Q^*_{bm}(\tilde{\sigma}), \tilde{\sigma})$ and $(Q^{**}_{bm}(\tilde{\sigma}), \tilde{\sigma})$, for which $\tilde{m}_\beta = 0$ and $\tilde{m}_\alpha = 0$ respectively, form lines in $D(M)$. We shall show that $Q^{**}_{bm}(\tilde{\sigma}) < Q^*_{bm}(\tilde{\sigma})$ for $\tilde{\sigma}$ close to 0, and $Q^{**}_{bm}(\tilde{\sigma}) > Q^*_{bm}(\tilde{\sigma})$ for $\tilde{\sigma}$ close to $r^*/(r^*+1)$, so that the two lines intersect yielding a totally balanced policy which is equivalent to the original policy.

To this end, recall that $\tilde{m}_\alpha(\tilde{Q}_{bm}, \tilde{\sigma})$ varies linearly in $\tilde{Q}_{bm}$ for fixed $\tilde{\sigma} \in (0, r^*/(r^*+1))$. Denote $\tilde{m}^0_\alpha(\tilde{\sigma}) \equiv \tilde{m}_\alpha(0, \tilde{\sigma})$, $\tilde{m}^+_\alpha(\tilde{\sigma}) \equiv \tilde{m}_\alpha((r^*+1)(1-\tilde{\sigma})M, \tilde{\sigma})$. For $\tilde{Q}_{bm} = 0$, the government finances its expenditure by printing $\tilde{m}_\beta > 0$ since it borrows nothing from the bank, hence $\tilde{m}^0_\alpha(\tilde{\sigma}) = -$tax revenue $\downarrow 0$ as $\tilde{\sigma} \downarrow 0$. But interest rate $\tilde{r} \uparrow r^*$ as $\tilde{\sigma} \downarrow 0$ by (i); and by (iii) and (iv) expenditures/tax revenue $\downarrow 0$, while money borrowed by the government $\uparrow M$, as $\tilde{Q}_{bm} \uparrow (r^*+1)(1-\tilde{\sigma})M$, so $\tilde{m}^+_\alpha(\tilde{\sigma})$ is approximately r^*M for small $\tilde{\sigma}$. Since $Q^{**}_{bm}(\tilde{\sigma})$ is defined as the zero point of the line joining $\tilde{m}^0_\alpha(\tilde{\sigma})$ and $\tilde{m}^+_\alpha(\tilde{\sigma})$, we see that $Q^{**}_{bm}(\tilde{\sigma}) \downarrow 0$ as $\tilde{\sigma} \downarrow 0$. But $Q^*_{bm}(\tilde{\sigma}) \to M\bar{Q}_m(r^*+1)/(B+\bar{Q}_m)$ as $\tilde{\sigma} \downarrow 0$. This proves $Q^{**}_{bm}(\tilde{\sigma}) < Q^*_{bm}(\tilde{\sigma})$ for $\tilde{\sigma}$ close to 0.

Finally as $\tilde{\sigma} \uparrow r^*/(r^*+1)$ and $\tilde{Q}_{bm} = 0$, by (i) tax revenue converges to $(r^*/(r^*+1))\times$ expenditures) $\geq (r^*/(r^*+1))(M/B)\bar{Q}_m \equiv K$ (say) for positive K, i.e., $\tilde{m}^0_\alpha(\tilde{\sigma}) \leq -K$ as $\tilde{\sigma} \uparrow r^*/(r^*+1)$. However $\tilde{r} \downarrow 0$ as $\tilde{\sigma} \uparrow r^*/(r^*+1)$ by (i). So, for $\tilde{Q}_{bm}$ close to its upper bound $(r^*+1)(1-\tilde{\sigma})M$ (where expenditures/tax revenue are nearly zero by (iii), and the government has borrowed almost all of M by (iv)), $\tilde{m}_\alpha$ is approximately $\tilde{r}M \to 0$ as $\tilde{\sigma} \uparrow r^*/(r^*+1)$. Thus $\tilde{m}^+_\alpha(\tilde{\sigma})$ is close to zero as $\tilde{\sigma} \uparrow r^*/(r^*+1)$. We conclude that $Q^{**}_{bm}(\tilde{\sigma}) \uparrow (r^*+1)(1-\tilde{\sigma})M$ as $\tilde{\sigma} \uparrow r^*/(r^*+1)$ and becomes bigger than $Q^*_{bm}(\tilde{\sigma}) = M\bar{Q}_m(r^*+1)(1-\tilde{\sigma})/(B+\bar{Q}_m)$. ■