The Equivalence of the Dekel-Fudenberg Iterative Procedure and Weakly Perfect Rationalizability

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Abstract

Two approaches have been proposed in the literature to refine the rationalizability solution concept: either assuming that players make small errors when playing their strategies, or assuming that their is a small amount of payoff uncertainty. We show that both approaches lead to the same refinement if errors are made according to the concept of weakly perfect rationalizability, and there is payoff uncertainty as in Dekel and Fudenberg [*Journal of Economic Theory* 52 (1990), 243–267]. For both cases, the strategies that survive are obtained by starting with one round of elimination of weakly dominated strategies followed by many rounds of elimination of strictly dominated strategies.

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1 Introduction

The solution concept of rationalizability has been introduced independently by Bernheim [2] and Pearce [10]. In some games it fails to eliminate all intuitively unreasonable outcomes, for instance in games with weakly dominated strategies (see e.g., Pearce [10], Herings and Vannetelbosch [9]). Therefore, one has looked for refinements that strengthen the rationalizability concept but still do not assume common expectations of the behavior of the players. Two different approaches have mainly been investigated in the literature. Both approaches propose to select outcomes which are robust to the introduction of small perturbations. One approach consists of assuming that players make small mistakes when determining their strategic choices, while the other one consists of assuming a small amount of payoff uncertainty.

The approach of strategy perturbations, i.e. the assumption that players make small mistakes, has produced many refinements like perfect rationalizability (see Bernheim [2]), cautious rationalizability (see Pearce [10]), proper rationalizability (see Schuhmacher [11]), weakly perfect rationalizability (see Herings and Vannetelbosch [9]), and trembling-hand rationalizability (see Herings and Vannetelbosch [9]). An analysis as well as a detailed description of these refinements can be found in Herings and Vannetelbosch [9]. Recently, a general framework for studying refinements of rationalizability has been introduced in Gul [7], who develops a solution concept called τ-theory. In this theory it is modelled in a coherent way that players may behave irrationally with a small probability, which is related to the assumption that players make mistakes with a small probability.

The approach of payoff perturbations, i.e., the assumption of a small amount of payoff uncertainty, has been studied by Dekel and Fudenberg [6], who obtained the following substantial result. Under the assumption that there is a little bit of uncertainty about the payoffs, rationalizability is equivalent to one round of deletion of weakly dominated strategies, followed by iterated deletion of strategies that are strictly dominated. In what follows, this rule for deleting dominated strategies will be referred to as the Dekel–Fudenberg iterative procedure.

There are also other approaches that lead to the Dekel–Fudenberg iterative procedure. Börgers [3] has shown that if it is approximate common knowledge that players maximize expected utility using full support conjectures, then the players choose strategies that correspond to the Dekel–Fudenberg iterative procedure. Brandenburger [4] has obtained a similar result to Börgers [3]. But, instead of approximate common knowledge, Brandenburger used a lexicographic analogue, called common first-order knowledge. Gul [7] shows that the Dekel–Fudenberg iterative procedure is the weakest perfect τ-theory. For the class of generic extensive-form games with perfect information, Ben–Porath [1] shows that the set of outcomes that are consistent with common certainty of rationality at the beginning of the game coincides with the set of outcomes that survive the Dekel-Fudenberg iterative procedure.

These results suggest that the Dekel–Fudenberg iterative procedure is a well motivated strengthening of rationalizability. To give further underpinning of this claim, we show that the Dekel–Fudenberg iterative procedure receives also support from the most common approach to refine rationalizability, namely by assuming that players make errors with a small probability. In this note we show that the concept of weakly perfect rationalizability coincides with the Dekel–Fudenberg iterative procedure. For such an equivalence result to hold, however, it is necessary that players believe that their opponents might make correlated mistakes. We provide a counterexample to equivalence if players make uncorrelated errors instead.

2 Definitions and Notations

We consider a normal-form game $\Gamma(I, S, U)$. The set I is a finite set of players. Each player i has a finite pure-strategy set S_i and a payoff function $U_i : S \to \mathbb{R}$, where $S = \prod_{i \in I} S_i$ and $U = (U_i)_{i \in I}$.

As general notation, we denote by $\Delta(X)$ the set of all Borel probability measures on X. For finite X, we denote by $\Delta^0(X)$ the set of all Borel probability measures giving positive probability to each member of X.

Given $c_i \in \Delta(S_i)$, we denote by $c_i(s_i)$ the probability that c_i assigns to pure strategy s_i. Player i's opponents in the game $\Gamma(I, S, U)$ are denoted by $-i$.

Given a product set T, which is the Cartesian product of individual strategy sets, T_i denotes the strategy set of player i. The Cartesian product $\prod_{j \neq i} T_j$ is denoted by T_{-i}. For $c_{-i} \in \Delta(S_{-i})$, $c_{-i}(s_{-i})$ denotes the probability that c_{-i} assigns to the pure strategy profile s_{-i}.

2.1 The Dekel–Fudenberg Iterative Procedure

To define the Dekel–Fudenberg iterative procedure we need to define the notions of strict and weak dominance first.

Definition 1 (strict dominance) Let a product set $T \subseteq S$ of pure strategy profiles in the game $\Gamma(I, S, U)$ be given. A pure strategy $s_i \in T_i$ of player i is strictly dominated in T if there exists $c_i \in \Delta(T_i)$ such that $U_i(c_i, s_{-i}) > U_i(s_i, s_{-i})$ for all $s_{-i} \in T_{-i}$.

Given a product set T of pure strategy profiles, the pure strategies of player i that are not strictly dominated in T are denoted by $\mathrm{B}_i(T)$. The pure strategy profiles that are not strictly dominated are denoted by $\mathrm{B}(T) = \prod_{i \in I} \mathrm{B}_i(T)$.

Definition 2 (weak dominance) Let a product set $T \subseteq S$ of pure strategy profiles in the game $\Gamma(I, S, U)$ be given. A pure strategy $s_i \in T_i$ of player i is weakly dominated in T if there exists $c_i \in \Delta(T_i)$ such that $U_i(c_i, s_{-i}) \geq U_i(s_i, s_{-i})$ for all $s_{-i} \in T_{-i}$, and $U_i(c_i, s_{-i}) > U_i(s_i, s_{-i})$ for some $s_{-i} \in T_{-i}$.

Given a product set T of pure strategy profiles, the pure strategies of player i that are not weakly dominated in T are denoted by $\mathrm{W}_i(T)$. The pure strategy profiles that are not weakly dominated are denoted $\mathrm{W}(T) = \prod_{i \in I} \mathrm{W}_i(T)$.

The Dekel–Fudenberg iterative procedure for removing dominated strategies consists of one round of deletion of weakly dominated strategies, followed by an arbitrarily large number of rounds of deletion of strictly dominated strategies. This procedure can be motivated by assuming small payoff uncertainty, see Dekel and Fudenberg [6], who give the following intuition for this result: "Each player knows his/her own payoffs, and so by our rationality postulate will not choose a weakly dominated strategy. In order to do a second round of deletion players must know that all the others will not choose certain strategies. A small amount of payoff uncertainty cannot alter strong dominance relationships, but can break weak ones, so that after the first round we can only proceed with the iterated deletion of strongly dominated strategies" (Dekel and Fudenberg [6, p.245]).

Definition 3 (Dekel-Fudenberg iterative procedure) Let $P^1 = \mathrm{W}(S)$. For $k \geq 2$, $P^k = \mathrm{B}\left(P^{k-1}\right)$. The set $P^\infty = \lim_{k\to\infty} P^k$ is the set of pure strategy profiles generated by the Dekel-Fudenberg iterative procedure.

Evidently, $\emptyset \neq P^k \subseteq P^{k-1} \subseteq ... \subseteq P^1$. Since the set S_i is finite for each player i, there exists some integer n such that $P^k = P^n$ for all $k \geq n$. Therefore, the limit set P^∞ is well-defined and non-empty.

2.2 Weakly Perfect Rationalizability

Weakly perfect rationalizability has been introduced by Herings and Vannetelbosch [9]. Here, we adapt our original definition such that it allows the players to hold correlated conjectures. Correlated weakly perfect rationalizability weakens weakly perfect rationalizability because allowing correlated conjectures about the strategies of the opponents makes more strategies rationalizable. Correlated strategies or conjectures appear to make more sense in the context of the non-equilibrium approach than in the equilibrium approach (see e.g., Brandenburger and Dekel [5] or Hammond [8]). The motivation for weakly perfect rationalizability is that each player makes mistakes but subject to an explicit constraint: the mistake technology puts a positive weight less than ε on strategies that are not best responses.[1]

For our results it is crucial whether a player may believe that her opponents make correlated mistakes or not. It is not unreasonable for a player i to conjecture that her opponents make correlated mistakes. Suppose, for instance, that her opponents implement a correlated strategy $c_{-i} \in \Delta(S_{-i})$ by means of a mediator. The mediator randomly selects a pure strategy profile $s_{-i} \in S_{-i}$ with probability $c_{-i}(s_{-i})$. Then the mediator recommends a player j, $j \neq i$, confidentially to use strategy s_j if s_{-i} is the pure strategy profile selected. If the mediator makes errors and chooses with positive probability not exceeding ε any pure strategy profile $\overline{s}_{-i} \in S_{-i}$ by mistake,

[1]This mistake technology is different from the one used in the perfect rationalizability concept due to Bernheim [2], where each player has to choose each of her pure strategies with a certain strictly positive minimum probability. It is shown in Herings and Vannetelbosch [9] that for the case of uncorrelated beliefs perfect rationalizability is a refinement of weakly perfect rationalizability.

then this leads the opponents of player i to make correlated mistakes, even if the mediator makes no errors when making recommendations and the players make no errors in playing the recommended strategy. On the other hand, if the mediator makes no errors in randomly selecting a pure strategy profile $s_{-i} \in S_{-i}$ with probability $c_{-i}(s_{-i})$, but makes uncorrelated mistakes when doing his recommendations, or players make uncorrelated mistakes when carrying out the recommendation, then this leads to uncorrelated mistakes of the players. In this section we allow for the possibility of correlated mistakes.

Weakly perfect rationalizability will be defined as an iterative procedure. At any stage k, player i has a set $D_i^k(\varepsilon) \subset \Delta(S_i)$ of mixed strategies that are still rational for her to play at stage k. Here $\varepsilon \geq 0$ is related to the mistake technology. She conjectures her opponents to play a correlated mixed strategy profile in $\Delta(\prod_{j \neq i} D_j^k(\varepsilon))$, which is subject to correlated mistakes. A mixed strategy is rational for player i at stage $k+1$, and belongs to $D_i^{k+1}(\varepsilon)$, if it is a best response against some Borel probability measure over such correlated mixed strategy profiles subject to mistakes. The set $D_i^k(\varepsilon)$ will be shown to satisfy the pure strategy property.

Definition 4 A subset D_i of $\Delta(S_i)$ satisfies the pure strategy property if $c_i(s_i) > 0$ for some $c_i \in D_i$ implies that s_i belongs to D_i. The product set D satisfies the pure strategy property if D_i satisfies the pure strategy property for all i.

If the opponents of a player i would not make any mistakes, then a correlated conjecture of player i on the play of her opponents is a Borel probability measure $\gamma_{-i} \in \Delta(\prod_{j \neq i} D_j^k(0))$. Analogously to Pearce [10], for the purposes of expected utility maximization, we can replace the collection of Borel probability measures $\Delta(\prod_{j \neq i} D_j^k(0))$ by the set $\Delta(\prod_{j \neq i} S_j^k(0))$, where $S_j^k(0)$ is the set of pure strategies in $D_j^k(0)$. Player i is rational if she maximizes her utility against an element of $\Delta(\prod_{j \neq i} S_j^k(0))$.

Suppose player i conjectures that her opponents make correlated mistakes with positive probability not exceeding ε.[2] Let e_{-i} be a measure on S_{-i} describing the mistake technology. For $s_{-i} \in S_{-i}$, $e_{-i}(s_{-i})$ is the probability by which the pure strategy profile s_{-i} is played by mistake. It holds that $0 < e_{-i}(s_{-i}) \leq \varepsilon$, $s_{-i} \in S_{-i}$. Now, if player i conjectures that her opponents try to coordinate on the correlated mixed strategy profile $c_{-i} \in \Delta(S_{-i})$ and expects them to make mistakes according to e_{-i}, then player i should optimize against the probability measure $\bar{c}_{-i} \in \Delta^0(S_{-i})$ satisfying

$$\bar{c}_{-i}(\bar{s}_{-i}) = (1 - \sum_{s_{-i} \in S_{-i}} e_{-i}(s_{-i}))c_{-i}(\bar{s}_{-i}) + e_{-i}(\bar{s}_{-i}), \quad \bar{s}_{-i} \in S_{-i}.$$

Suppose we are at stage k, and each player i has a set $D_i^k(\varepsilon) \subset \Delta(S_i)$ of mixed strategies that correspond to rational play, where each set $D_i^k(\varepsilon)$ satisfies the pure

[2] It is always assumed that ε is smaller than one over the total number of pure strategy profiles in S.

strategy property. We denote the pure strategies in $D_i^k(\varepsilon)$ by $S_i^k(\varepsilon)$. If the mistake technology is as described before, then player i should maximize utility against an element of $\Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon))$, where

$$\Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon)) = \{c_{-i} \in \Delta^0(S_{-i}) \mid c_{-i}(s_{-i}) \leq \varepsilon \text{ if } s_j \notin S_j^k(\varepsilon) \text{ for some } j \neq i\}.$$

The set $\Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon))$ contains all correlated, completely mixed strategy profiles that put weight less than or equal to ε on any pure strategy profile containing a pure strategy not in $S_j^k(\varepsilon)$ for some player j. Any probability measure of beliefs of player i on correlated strategies profiles subject to mistakes, will never assign weight exceeding ε to a pure strategy profile $s_{-i} \in S_{-i}$ such that $s_j \notin S_j^k(\varepsilon)$ for some j. To prove that the strategy profiles in $\Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon))$ are the ones to consider, it remains to show that any strategy profile in $\Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon))$ can indeed be conjectured.

Consider any $c_{-i} \in \Delta^\varepsilon(\prod_{j\neq i} S_j^k(\varepsilon))$. We define $\overline{S}_{-i} = \{s_{-i} \in S_{-i} \mid c_{-i}(s_{-i}) > \varepsilon\}$ and denote the cardinality of $\overline{S}_{-i}$ by $\#\overline{S}_{-i}$. For each $\overline{s}_{-i} \in \overline{S}_{-i}$, let $\widetilde{c}_{-i}^{\overline{s}_{-i}} \in \Delta^0(S_{-i})$ be defined by

$$\begin{aligned}
\widetilde{c}_{-i}^{\overline{s}_{-i}}(\overline{s}_{-i}) &= (1 - \frac{\varepsilon^2}{c_{-i}(\overline{s}_{-i})}) \sum_{s_{-i}\in\overline{S}_{-i}} c_{-i}(s_{-i}), \\
\widetilde{c}_{-i}^{\overline{s}_{-i}}(\widehat{s}_{-i}) &= \frac{1}{\#\overline{S}_{-i} - 1} \frac{\varepsilon^2}{c_{-i}(\overline{s}_{-i})} \sum_{s_{-i}\in\overline{S}_{-i}} c_{-i}(s_{-i}), \quad \widehat{s}_{-i} \in \overline{S}_{-i} \setminus \{\overline{s}_{-i}\}, \\
\widetilde{c}_{-i}^{\overline{s}_{-i}}(s_{-i}) &= c_{-i}(s_{-i}), \quad s_{-i} \in S_{-i} \setminus \overline{S}_{-i}.
\end{aligned}$$

The mixed correlated conjecture $\widetilde{c}_{-i}^{\overline{s}_{-i}}$ corresponds to the conjectured play of the pure strategy profile $\overline{s}_{-i}$ subject to error. Notice that $\widetilde{c}_{-i}^{\overline{s}_{-i}}(\widehat{s}_{-i}) \leq \varepsilon$ for all $\widehat{s}_{-i} \in S_{-i} \setminus \{\overline{s}_{-i}\}$. Because of the pure strategy property of each $D_j^k(\varepsilon)$, player i may indeed conjecture her opponents to play $\widetilde{c}_{-i}^{\overline{s}_{-i}}$. If player i conjectures $\widetilde{c}_{-i}^{\overline{s}_{-i}}$ with probability $c_{-i}(\overline{s}_{-i})/(\sum_{s_{-i}\in\overline{S}_{-i}} c_{-i}(s_{-i}))$, then she should indeed optimize against c_{-i}.

Now we can define weakly perfect rationalizability by the following iterative procedure.

Definition 5 (weakly perfect rationalizability) Let $\varepsilon > 0$ be given. Let $D^0(\varepsilon) = \prod_{i\in I} \Delta(S_i)$. For $k \geq 1$, $D^k(\varepsilon) = \prod_{i\in I} D_i^k(\varepsilon)$ is inductively defined as follows: c_i belongs to $D_i^k(\varepsilon)$ if $c_i \in \Delta(S_i)$ and there is $c_{-i} \in \Delta^\varepsilon(\prod_{j\neq i} S_j^{k-1}(\varepsilon))$ such that c_i is a best response against c_{-i} within $\Delta(S_i)$. The set $D^\infty(\varepsilon) = \lim_{k\to\infty} D^k(\varepsilon)$ is the set of ε-weakly perfectly rationalizable strategy profiles and $D^\infty = \lim_{\varepsilon\to 0^+} D^\infty(\varepsilon)$ the set of weakly perfectly rationalizable strategy profiles.

In Definition 5 the limit set D^∞ is given by

$$\lim_{\varepsilon\to 0^+} D^\infty(\varepsilon) = \left\{ c \in \prod_{i\in I} \Delta(S_i) \,|\, \exists \{\varepsilon^t\}_{t=0}^\infty \to 0^+, \exists \{c^t\}_{t=0}^\infty \to c, c^t \in D^\infty(\varepsilon^t) \right\}.$$

It is trivial to verify that indeed each set $D_i^k(\varepsilon)$ satisfies the pure strategy property.

Instead of the algorithmic definition given here, it is possible to define ε-weakly perfect rationalizability in an axiomatic way following the seminal contribution of Pearce [10]. The following three axioms characterize ε-weakly perfect rationalizability.

A1. Each player i forms a subjective prior over her opponents choice of strategy, i.e., a prior over mixed strategy profiles $c_{-i} \in \Delta(S_{-i})$ played subject to an error e_{-i} satisfying $0 < e_{-i}(s_{-i}) \leq \varepsilon$, $s_{-i} \in S_{-i}$.

A2. Each player maximizes her utility relative to her prior.

A3. A1 and A2 are common knowledge.

Similarly to Herings and Vannetelbosch [9] it is possible to obtain the following result.

Theorem 1 *For every normal-form game* $\Gamma(I, S, U)$, D^∞ *is non-empty, closed, satisfies the pure strategy property, and contains all perfect Nash equilibria.*

Although Theorem 1 shows that weakly perfect rationalizability is a well-defined solution concept, it does not give us an easy characterization of the strategy profiles that survive.

3 The Equivalence Theorem

Theorem 2 gives an easy characterization of the set of weakly perfect rationalizable strategies. It states that weakly perfect rationalizability coincides with the Dekel-Fudenberg iterative procedure. In the proof of Theorem 2 we will frequently use the following lemma from Pearce [10].

Lemma 1 *A strategy* s_i *is strictly dominated in* T *if and only if it is not a best response against a correlated conjecture on* T_{-i}*. A strategy* s_i *is weakly dominated in* T *if and only if it is not a best response against a completely mixed correlated conjecture on* T_{-i}*.*

Theorem 2 *For every normal-form game* $\Gamma(I, S, U)$, $P^\infty = S^\infty$.

Proof. Let $\overline{\varepsilon} = (\prod_{i \in I} \# S_i)^{-1}$. For every product set T of pure strategy profiles, if a pure strategy $s_i \in T_i$ is strictly dominated in T, then by Lemma 1 it is not a best response against any correlated conjecture on T_{-i}. By continuity of the payoff function, there is $\varepsilon(i,T)$, $0 < \varepsilon(i,T) \leq \overline{\varepsilon}$, such that a pure strategy $s_i \in T_i$ that is strictly dominated in T is not a best response against any conjecture in

$$\{c_{-i} \in \Delta^0(S_{-i}) \mid c_{-i}(s_{-i}) \leq \varepsilon(i,T) \text{ if } s_j \notin T_j \text{ for some } j \neq i\}.$$

We denote the minimum over all players i and all product sets T of $\varepsilon(i,T)$ by $\widehat{\varepsilon}$. We show by induction on k that $P_i^k = S_i^k(\varepsilon)$, for $\varepsilon \in (0, \widehat{\varepsilon}]$.

Clearly, $P_i^0 = S_i^0(\varepsilon) = S_i$.

Strategy s_i belongs to P_i^1 if and only if it is not weakly dominated in S. By Lemma 1 s_i is not weakly dominated in S if and only if it is a best response against a completely mixed correlated conjecture on S_{-i}. Strategy s_i is a best response against a completely mixed correlated conjecture on S_{-i} if and only if it is a best response against a conjecture in $\Delta^\varepsilon(\prod_{j \neq i} S_j^0(\varepsilon))$, which is the case if and only if s_i belongs to $S_i^1(\varepsilon)$. So, $P_i^1 = S_i^1(\varepsilon)$.

Now, let $k \geq 2$ and let $P^{k-1} = S^{k-1}(\varepsilon)$.

Consider any $s_i \in P_i^k$. By Lemma 1, s_i is a best response against some correlated conjecture $\widehat{c}_{-i}$ on $\prod_{j \neq i} P_i^{k-1} = \prod_{j \neq i} S_j^{k-1}(\varepsilon)$. Clearly, $s_i \in P_i^1$, since $s_i \in P_i^k \subseteq P_i^1$. So by Lemma 1 s_i is a best response against a completely mixed correlated conjecture $\overline{c}_{-i} \in \Delta^0(S_{-i})$. There is a convex combination of $\widehat{c}_{-i}$ and $\overline{c}_{-i}$ belonging to $\Delta^\varepsilon(\prod_{j \neq i} S_j^{k-1}(\varepsilon))$. It is sufficient to put a weight low enough on $\overline{c}_{-i}$. It follows that s_i is a best response against this convex combination, so $s_i \in S_i^k(\varepsilon)$.

Consider any $s_i \in S_i^k(\varepsilon)$. Then s_i is a best response against some $c_{-i} \in \Delta^\varepsilon(\prod_{j \neq i} S_j^{k-1}(\varepsilon))$. By the construction of $\widehat{\varepsilon}$, s_i is not strictly dominated in $\prod_j S_j^{k-1}(\varepsilon) = P^{k-1}$. So $s_i \in P_i^k$. ■

For every normal-form game $\Gamma(I, S, U)$, a pure strategy survives one round of deletion of weakly dominated strategies followed by iterated deletion of strategies that are strictly dominated if and only if it is weakly perfectly rationalizable.

Theorem 2 allows us to advocate Dekel–Fudenberg iterative procedure for deleting strategies since it is obtained both under the assumption that there is some small uncertainty about the payoffs (see Dekel and Fudenberg [6]) and under the assumption that there is some small uncertainty about the strategies (Theorem 2).

In Herings and Vannetelbosch [9] it is shown that, for the case of uncorrelated conjectures, the concepts of perfect rationalizability [2], cautious rationalizability [10], proper rationalizability [11], weakly perfect rationalizability [9], and trembling-hand rationalizability [9] are different in two-person games. For those games there is no distinction between correlated and uncorrelated conjectures. Weakly perfect rationalizability is the only existing refinement of rationalizability based on strategy perturbations that coincides with the Dekel-Fudenberg iterative procedure.

4 Uncorrelated Mistakes

We will show by means of an example that if players make uncorrelated mistakes, then weakly perfect rationalizability does not coincide with the Dekel–Fudenberg iterative procedure. It is obvious that we need at least three players for such an example, since the issue does not arise in games with less than three players. Consider the following three-player game in Figure 1.

	Y_1	Y_2
X_1	$1, 1, 1$	$0, 0, 0$
X_2	$0, 1, 1$	$0, 0, 0$

Z_1

	Y_1	Y_2
X_1	$1, 1, 1$	$1, 0, 0$
X_2	$0, 1, 0$	$0, 0, 1$

Z_2

Figure 1: Correlation of mistakes matter

It is rather obvious that only player 1's pure strategy X_1, player 2's pure strategy Y_1, and player 3's pure strategies Z_1 and Z_2 survive the Dekel–Fudenberg iterative procedure. Indeed, for $k \geq 1$, $P_1^k = \{X_1\}$, $P_2^k = \{Y_1\}$, and $P_3^k = \{Z_1, Z_2\}$. By Theorem 2 this coincides with the strategies selected by weakly perfect rationalizability.

Now consider a version of weakly perfect rationalizability, where players are sure that their opponents make uncorrelated mistakes. Although this goes counter the intuition underlying the conjectured use of correlated strategy profiles, it is a possibility we want to scrutinize. Since the possible conjectures of players are now more restricted, it is obvious that $\widetilde{S}_1^k(\varepsilon) = \{X_1\}$ and $\widetilde{S}_2^k(\varepsilon) = \{Y_1\}$, $k \geq 1$, where a tilde is used to indicate that we are considering the case with uncorrelated mistakes. If player 3 conjectures that players 1 and 2 are going to play the strategy profile $\{X_2, Y_1\}$, then, if the mistake probability ε is sufficiently small, player 3 chooses Z_1. Similarly, if player 3 conjectures that players 1 and 2 are going to coordinate on the strategy profile $\{X_2, Y_2\}$, then player 3 chooses Z_2. Consequently, $\widetilde{S}_3^1(\varepsilon) = \{Z_1, Z_2\}$.

At stage 2, player 3 knows that players 1 and 2 will coordinate on the strategy profile $\{X_1, Y_1\}$. But if players 1 and 2 make uncorrelated mistakes, then player 3 will optimize against a conjecture $c_{-3} \in \Delta^0(S_{-3})$ for which $c_{-3}(X_2, Y_2) < c_{-3}(X_2, Y_1)$ and $c_{-3}(X_2, Y_2) < c_{-3}(X_1, Y_2)$. Against such a conjecture it is always optimal for player 3 to use strategy Z_1. It follows that $\widetilde{S}_3^k(\varepsilon) = \{Z_1\}$, $k \geq 2$.

Weakly perfect rationalizability with uncorrelated mistakes does not coincide with the Dekel–Fudenberg iterative procedure.

5 Two Examples

We analyze two examples to conclude. The first example in Figure 2 is due to Börgers [3]. Börgers' example is a counterexample to Dekel and Fudenberg's [6, Footnote 4] assertion that in two-player normal-form games perfect rationalizability coincides with the Dekel–Fudenberg iterative procedure. Indeed, it can be shown that only player 1's pure strategies X_1, X_2 and player 2's pure strategy Y_2 are perfectly rationalizable (see Börgers [3, pp. 274-275]). Meanwhile, player 1's pure strategies X_1, X_2, X_3 and player 2's pure strategies Y_1, Y_2, Y_3 survive the Dekel–Fudenberg iterative procedure and are weakly perfectly rationalizable. Indeed, $P_1^k = \{X_1, X_2, X_3\}$ and $P_2^k = \{Y_1, Y_2, Y_3\}$, and, it holds that $S_1^k(\varepsilon) = \{X_1, X_2, X_3\}$ and $S_2^k(\varepsilon) = \{Y_1, Y_2, Y_3\}$, $k \geq 1$.

	Y_1	Y_2	Y_3
X_1	3, 0	1, 0	0, 0
X_2	0, 0	1, 0	3, 0
X_3	2, 0	0, 0	2, 0
X_4	0, 3	0, 2	0, 0
X_5	0, 0	0, 2	0, 3

Figure 2: Börgers' example

The second example shows the importance of allowing the players to have correlated conjectures in order to derive our equivalence result. The example is a three-player game (see Figure 3) and is taken from Herings and Vannetelbosch [9], where we have shown that weakly perfect rationalizability without allowing correlated conjectures supports the following pure strategies: $\{X_1, X_2\}$ for player 1, $\{Y_1, Y_2\}$ for player 2, and $\{Z_1, Z_2\}$ for player 3. Next, consider the Dekel–Fudenberg iterative procedure. It is easily seen that $P_1^1 = \{X_1, X_2, X_3\}$, $P_2^1 = \{Y_1, Y_2\}$, and $P_3^1 = \{Z_1, Z_2\}$. It is not possible in the first iteration to eliminate any pure strategy of player 1, since all strategies of player 1 are equally good against $(c_2, c_3) = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$. In the second iteration of Dekel–Fudenberg procedure it is impossible to eliminate any other pure strategy of player 2 or 3. Consequently, for $k \geq 1$, $P_1^k = \{X_1, X_2, X_3\}$, $P_2^k = \{Y_1, Y_2\}$, and $P_3^k = \{Z_1, Z_2\}$. Given Theorem 2, we have $S_1^\infty = \{X_1, X_2, X_3\}$, $S_2^\infty = \{Y_1, Y_2\}$, and $S_3^\infty = \{Z_1, Z_2\}$. That is, X_3 is correlated weakly perfectly rationalizable but not uncorrelated weakly perfectly rationalizable. Intuitively, compared to strategies X_1 and X_2, strategy X_3 is good against the conjectures (Y_1, Z_1), (Y_2, Z_2), and (Y_3, Z_3), but bad against all other pure strategy combinations. Correlation allows any combination of the first three conjectures to arise with very high probability, which is not possible when conjectures are uncorrelated.

Z_1:

	Y_1	Y_2	Y_3
X_1	2,1,1	1,1,1	0,0,1
X_2	0,1,1	1,1,1	0,0,1
X_3	2,1,1	0,1,1	0,0,1

Z_2:

	Y_1	Y_2	Y_3
X_1	1,1,1	0,1,1	0,0,1
X_2	1,1,1	2,1,1	0,0,1
X_3	0,1,1	2,1,1	0,0,1

Z_3:

	Y_1	Y_2	Y_3
X_1	1,1,0	1,1,0	0,0,0
X_2	1,1,0	1,1,0	0,0,0
X_3	0,1,0	0,1,0	2,0,0

Figure 3: A three-player game

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