Uniqueness, Stability, and Comparative Statics in Rationalizable Walrasian Markets*

Donald J. Brown
Department of Economics, Yale University
Chris Shannon
Department of Economics, University of California, Berkeley

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Abstract

This paper studies the extent to which qualitative features of Walrasian equilibria are refutable given a finite data set. In particular, we consider the hypothesis that the observed data are Walrasian equilibria in which each price vector is locally stable under tâtonnement. Our main result shows that a finite set of observations of prices, individual incomes and aggregate consumption vectors is rationalizable in an economy with smooth characteristics if and only if it is rationalizable in an economy in which each observed price vector is locally unique and stable under tâtonnement. Moreover, the equilibrium correspondence is locally monotone in a neighborhood of each observed equilibrium in these economies. Thus the hypotheses that equilibria are locally stable under tâtonnement, equilibrium prices are locally unique and equilibrium comparative statics are locally monotone are not refutable with a finite data set.

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1 Introduction

The major theoretical questions concerning competitive equilibria in the classical Arrow–Debreu model — existence, uniqueness, comparative statics, and stability of price adjustment processes — have been largely resolved over the last forty years. With the exception of existence, however, this resolution has been fundamentally negative. The conditions under which equilibria can be shown to be unique, comparative statics globally determinate or tâtonnement price adjustment globally stable are quite restrictive. Moreover, the Sonnenschein–Debreu–Mantel theorem shows in striking fashion that no behavior implied by individual utility maximization beyond homogeneity and Walras' Law is necessarily preserved by aggregation in market excess demand. This arbitrariness of excess demand implies that monotone equilibrium comparative statics and global stability of equilibria under tâtonnement will only result from the imposition of a limited set of conditions on preferences and endowments. Based on these results, many economists conclude that the general equilibrium model has no refutable implications or empirical content.

Of course no statement concerning refutable implications is meaningful without first specifying what information is observable and what is unobservable. If only market prices are observable, and all other information about the economy such as individual incomes, individual demands, individual endowments, individual preferences and aggregate endowment or aggregate consumption is unobservable, then indeed the general equilibrium model has no testable restrictions. This is essentially the content of Mas-Colell's version of the Sonnenschein–Debreu–Mantel theorem. Mas-Colell [6] shows that given an arbitrary nonempty compact subset C of strictly positive prices in the simplex, there exists an economy $\mathcal{E}$ composed of consumers with continuous, monotone, strictly convex preferences such that the equilibrium price vectors for the economy $\mathcal{E}$ are given exactly by the set C.

In many instances, however, it is unreasonable to think that only market prices are observable; other information such as individual incomes and aggregate consumption may be observable in addition to market prices. Brown and Matzkin [2] show that if such additional information is available, then the Walrasian model does have refutable implications. They demonstrate by example that with a finite number of observations — in fact two — on market prices, individual incomes and aggregate consumptions, the hypothesis that these data correspond to competitive equilibrium observations can be rejected. They also give conditions under which this hypothesis is accepted and there exists an economy rationalizing the observed data.

This paper considers the extent to which qualitative features of Walrasian equilibria are refutable given a finite data set. In particular, we consider the hypothesis that the observed data are Walrasian equilibria in which each price vector is locally stable under tâtonnement. Based on the Sonnenschein–Debreu–Mantel results and the well-known examples of global instability of tâtonnement such as Scarf's [8], it may seem at first glance that this hypothesis will be easily refuted in a Walrasian setting. Surprisingly, however, we show that it is not. Our main result shows that a finite set of observations of prices, individual incomes and aggregate consumption vectors is rationalizable in an economy with smooth characteristics if and only if it is rationalizable in a distribution economy in which each observed price is locally stable under tâtonnement. Moreover, the equilibrium correspondence is locally monotone in a neighborhood of each observed equilibrium in these economies, and the equilibrium price vector is locally unique.

The conclusion that if the data is rationalizable then it is rationalizable in a distribution economy, i.e., an economy in which individual endowments are collinear, is not subtle. If we do not observe the individual endowments and only observe prices and income levels, then one set of individual endowments consistent with this data is collinear, with shares given by the observed income distribution. Since distribution economies by definition have a price-independent income distribution, this observation may suggest that our results about stability and comparative statics derive simply from this fact. Kirman and Koch [4] show that this intuition is false, however. They show that the additional assumption of a fixed income distribution places no restrictions on excess demand: given any compact set $K \subset \mathbb{R}^{\ell}_{++}$ and any smooth function $g : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$ which is homogeneous of degree 0 and satisfies Walras' Law, and given any fixed income distribution $\alpha \in \mathbb{R}^{n}_{++}$, $\sum_{i=1}^{n} \alpha_i = 1$, there exists an economy $\mathcal{E}$ with smooth, monotone, strictly convex preferences and initial endowments $\omega_t = \alpha_t \omega$ such that excess demand for $\mathcal{E}$ coincides with g on K. Hence any dynamic on the price simplex can be replicated by some distribution economy.

This paper shows that rationalizable data is always rationalizable in an economy in which market excess demand has a very particular structure. Using recent results of Quah [7], we show that if the data is rationalizable then it is rationalizable in an economy in which each individual demand function is locally monotone at each observation. The strong properties of local monotonicity, in particular the fact that local monotonicity of individual demand is preserved by aggregation in market excess demand and the fact that local monotonicity implies local stability in distribution economies, allow us to conclude that if the data is rationalizable in a Walrasian setting, then it is rationalizable in an economy in which each observation is locally stable under tâtonnement. Thus global instability, while clearly a theoretical possibility in Walrasian markets, cannot be detected in a finite data set consisting of observations on prices, incomes, and aggregate consumption.

The paper proceeds as follows. In Section 2 we discuss conditions for rationalizing individual demand in economies with smooth characteristics. By developing a set of "dual" Afriat inequalities, we show that if the observed data can be rationalized by an individual consumer with smooth characteristics then it can be rationalized by a smooth utility function which generates a locally monotone demand function. In Section 3 we discuss the implications of locally monotone demand and use these results together with the results from Section 2 to show that local uniqueness, local stability, and local monotone comparative statics are not refutable in Walrasian markets.

2 Rationalizing Individual Demand

Given a finite number of observations $(p^r, x^r), r = 1, \ldots, N$, on prices and quantities, when is this data consistent with utility maximization by some consumer with a non-satiated utility function? We say that a utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ **rationalizes** the data $(p^r, x^r), r = 1, \ldots, N$, if $\forall r$

$$p^r \cdot x^r \geq p^r \cdot x \Rightarrow U(x^r) \geq U(x),\ \forall x \in \mathbb{R}^\ell_+.$$

Using this terminology, we can restate the question above: given a finite data set, when does there exist a non-satiated utility function which rationalizes these observations? The classic answer to this question was given by Afriat [1].

Theorem (Afriat). *The following are equivalent:*

(a) *there exists a non-satiated utility function which rationalizes the data*

(b) *the data satisfies Cyclical Consistency*

(c) *there exist numbers U^i, $\lambda^i > 0$, $i = 1, ..., N$ which satisfy the "Afriat inequalities":*

$$U^i - U^j \leq \lambda^j p^j \cdot (x^i - x^j),\ i, j = 1, ..., N$$

(d) *there exists a concave, monotone, continuous, non-satiated utility function which rationalizes the data.*

In particular, the equivalence of (a) and (d) shows that the hypothesis that preferences are represented by a concave utility function can never be refuted based on a finite data set, since if the data is rationalizable by any non-satiated utility function then it is rationalizable by a concave, monotone, and continuous one. Moreover, Afriat showed explicitly how to construct such a utility function which rationalizes a given data set. For each $x \in \mathbb{R}^\ell_+$, define

$$U(x) = \min_r \{U^r + \lambda^r p^r \cdot (x - x^r)\}.$$

This utility function is indeed continuous, monotone and concave, and rationalizes the data by construction.

As Chiappori and Rochet [3] note, however, this utility function is piecewise linear and thus neither differentiable nor strictly concave. Such a utility function does not generate a smooth demand function, and for a number of prices does not even generate single-valued demand. This utility function is thus incompatible with many standard demand-based approaches to the question of rationalizability or estimation, as well as with our questions about comparative statics and asymptotic stability.

Whether or not a given set of observations can be rationalized by a smooth utility function which generates a smooth demand function will obviously depend on the nature of the observed data. Two situations in which such a rationalization is impossible are depicted in Figures 1 and 2. The observations in Figure 1 cannot be rationalized by any smooth utility function, and those in Figure 2 cannot be rationalized by any demand function. If the data satisfies SARP, then situations like that in Figure 2 are eliminated; Chiappori and Rochet [3] show that if in addition situations like that in Figure 1 are ruled out then the data can be rationalized by a smooth, strongly concave utility function.

Insert Figures 1 and 2 here

More formally, the data satisfies the **strong strong axiom of revealed preference (SSARP)** if it satisfies SARP and if for all $i, j = 1, \ldots, N$,

$$p^i \neq p^j \Rightarrow x^i \neq x^j.$$

Chiappori and Rochet [3] show that given a finite set of data satisfying SSARP, there exists a strictly increasing, C^∞, strongly concave utility function defined on a compact subset of $\mathbb{R}^\ell_+$ which rationalizes this data. Although SSARP is a condition on the observable data alone, it can be equivalently characterized by the "strict Afriat inequalities". That is, the data satisfy SSARP if and only if there exist numbers $U^i, \lambda^i > 0$, $i = 1, \ldots, N$ such that

$$U^i - U^j < \lambda^j p^j \cdot (x^i - x^j),\ i, j = 1, \ldots, N,\ \ i \neq j.$$

Our work makes use of a modification of the results of Chiappori and Rochet [3]. Since our data consists of prices and income levels, we find it more natural to first recast the question of rationalizability in terms of indirect utility, and develop a set of dual Afriat inequalities characterizing data which can be rationalized by a consumer with smooth characteristics. An important benefit of this dual characterization is that it allows us to conclude not only that the data can be rationalized by a smooth demand function but by a demand function which is locally monotone in a neighborhood of each observation (p^r, I^r).

Definition. Let $\omega \in \mathbb{R}^\ell_{++}$ be given. An individual demand function $f(p, I)$ is **locally monotone** at $(\bar{p}, \bar{I})$ if there exists a neighborhood $\mathcal{W}$ of $(\bar{p}, \bar{I})$ such that

$$(p - q) \cdot (f(p, I) - f(q, I)) < 0$$

for all $(p, I), (q, I) \in \mathcal{W}$ such that $p \neq q$.

Our first result can then be stated as follows.

Theorem 1. *Let $(p^r, x^r), r = 1, \ldots, N$ be given. There exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data such that the implied demand function is locally monotone at (p^r, I^r) for each $r = 1, \ldots, N$ where $I^r = p^r \cdot x^r$ if and only if there exist numbers V^i, λ^i, and vectors $q^i \in \mathbb{R}^\ell$, $i = 1, \ldots, N$ such that:*

(a) *for $i \neq j$,*

$$V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j),\ i, j = 1, \ldots, N$$

(b) $\lambda^j > 0$, $q^j \ll 0$, $j = 1, \ldots, N$

(c) $q^j = -\lambda^j x^j$, $j = 1, \ldots, N$.

Conditions (a) and (b) constitute our "dual strict Afriat inequalities". Condition (c) here is just an expression of Roy's identity in this context. To see this, note that the vector (q^j, λ^j) corresponds to the gradient of the rationalizing indirect utility function V evaluated at (p^j, I^j); this is essentially the content of (a). If $q^j = -\lambda^j x^j$, then x^j is indeed demand at the price-income pair (p^j, I^j) by Roy's identity.

The proof of Theorem 1 relies on two intermediate results. The first is a version of Lemma 2 in Chiappori and Rochet [3] modified to apply to our dual Afriat inequalities.

Lemma 1. *If there exist numbers V^i, λ^i and vectors q^i, $i = 1, \ldots, N$ satisfying the dual strict Afriat inequalities, then there exists a convex C^∞ function $W : \mathbb{R}^{\ell+1}_+ \to \mathbb{R}$ which is strictly increasing in I and strictly decreasing in p such that $W(p^i, I^i) = V^i$, $DW(p^i, I^i) = (q^i, \lambda^i)$, and $\frac{\partial^2 W}{\partial^2 I}(p^i, I^i) = 0$ for every $i = 1, \ldots, N$.*

Proof: For each $(p, I) \in \mathbb{R}^{\ell+1}_{++}$ define

$$Y(p, I) = \max_t \{V^t + (q^t, \lambda^t) \cdot [(p, I) - (p^t, I^t)]\}.$$

Then Y is convex, continuous, strictly increasing in I and strictly decreasing in p. Moreover, $Y(p^r, I^r) = V^r$ for each r. To see this, note that by definition $\forall r\ \exists\ s$ such that

$$Y(p^r, I^r) = V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

By the dual strict Afriat inequalities, if $r \neq s$

$$V^r - V^s > q^s \cdot (p^r - p^s) + \lambda^s (I^r - I^s) = (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

So $\forall s \neq r$

$$V^r > V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

Thus $Y(p^r, I^r) = V^r$ for each $r = 1, \ldots, N$.

This argument shows that for every $s \neq r$,

$$Y(p^r, I^r) > V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

Since Y is continuous, $\forall r$ there exists $\eta_r > 0$ such that

$$Y(p, I) = V^r + (q^r, \lambda^r) \cdot [(p, I) - (p^r, I^r)]$$

for all $(p, I) \in B((p^r, I^r), \eta_r)$, i.e., Y is piecewise linear.

Following Chiappori and Rochet [3], we will smooth Y by convolution. Let $\eta = \min_r \eta_r$, and $y = (p, I)$. Define

$$\rho(y) = \begin{cases} \exp\left[-\frac{1}{1-||y||^2}\right]\left[\int \exp\left[-\frac{1}{1-||z||^2}\right] dz\right]^{-1}, & \text{if } ||y|| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\rho_\eta(y) = \frac{1}{\eta}\rho\left(\frac{y}{\eta}\right).$$

Then ρ_η is nonnegative, symmetric, and C^∞, and $\rho_\eta(y) = 0$ if $y \notin B(0, \eta)$.

Now define

$$W(y) \equiv Y * \rho_\eta(y) = \int Y(y - \xi)\rho_\eta(\xi)d\xi.$$

Then W is C^∞ and convex, as

$$\begin{aligned} W(\gamma y + (1-\gamma)y') &= \int Y(\gamma y + (1-\gamma)y' - \xi)\rho_\eta(\xi)d\xi \\ &\leq \int [\gamma Y(y - \xi) + (1-\gamma)Y(y' - \xi)]\rho_\eta(\xi)d\xi \\ &= \gamma \int Y(y - \xi)\rho_\eta(\xi)d\xi + (1-\gamma)\int Y(y' - \xi)\rho_\eta(\xi)d\xi \\ &= \gamma W(y) + (1-\gamma)W(y') \end{aligned}$$

for $\gamma \in [0, 1]$. Moreover, W is strictly decreasing in p and strictly increasing in I. To see this, suppose $p < p'$. Then

$$W(p, I) - W(p', I) = \int [Y((p, I) - \xi) - Y((p', I) - \xi)]\rho_\eta(\xi)d\xi > 0.$$

Similarly, $W(p, I') - W(p, I) > 0$ for $I' > I$.

Furthermore, for each $r = 1, \ldots, N$,

$$\begin{aligned} W(p^r, I^r) &= \int Y((p^r, I^r) - \xi)\rho_\eta(\xi)d\xi \\ &= \int_{B(0,\eta)} (V^r - (q^r, \lambda^r) \cdot \xi)\rho_\eta(\xi)d\xi \\ &= V^r - (q^r, \lambda^r) \cdot \int_{B(0,\eta)} \xi\rho_\eta(\xi)d\xi \\ &= V^r \end{aligned}$$

since $\rho_\eta(\xi) = \rho_\eta(-\xi)$ for all ξ.

Similarly,

$$\begin{aligned} DW(p^r, I^r) &= \int_{B(0,\eta)} DY((p^r, I^r) - \xi)\rho_\eta(\xi)d\xi \\ &= \int_{B(0,\eta)} (q^r, \lambda^r)\rho_\eta(\xi)d\xi \\ &= (q^r, \lambda^r) \end{aligned}$$

and

$$\begin{aligned} D^2W(p^r, I^r) &= \int_{B(0,\eta)} D^2Y((p^r, I^r) - \xi)\rho_\eta(\xi)d\xi \\ &= \int_{B(0,\eta)} 0 \ \rho_\eta(\xi)d\xi = 0. \end{aligned}$$

In particular, $\frac{\partial^2 W}{\partial I^2}(p^r, I^r) = 0$ for each r. $\square$

The intuition behind this result is straightforward. The dual Afriat function $Y(p, I)$ gives a convex indirect utility function rationalizing the data which has kinks whenever two or more of the dual Afriat equations are equal. Since the data satisfy the strict dual Afriat inequalities, none of these kinks occurs at an observation, so smoothing the function Y in a sufficiently small neighborhood of each kink gives a smooth function which is equal to the original dual Afriat function in a neighborhood of each observation, and thus in particular is locally linear in each such neighborhood.

The second important result we use, due to Quah [7], gives conditions on indirect utility analogous to the Mijutschin-Polterovich conditions on direct utility under which individual demand is locally monotone.

Theorem (Quah). *Let $h : \mathbb{R}_+^{\ell+1} \to \mathbb{R}$ be a smooth, convex, strictly quasiconvex indirect utility function satisfying the property*[1]

$$\varepsilon(\bar{p}, \bar{I}) \equiv \frac{\bar{I} h_{II}(\bar{p}, \bar{I})}{h_I(\bar{p}, \bar{I})} < 2.$$

Then h generates a demand function which is locally monotone in a neighborhood of $(\bar{p}, \bar{I})$.

In particular, if the indirect utility function h is linear in income in a neighborhood of $(\bar{p}, \bar{I})$, then $\varepsilon(\bar{p}, \bar{I}) = 0$ and Quah's theorem shows that h generates a demand function which is locally monotone in a neighborhood of $(\bar{p}, \bar{I})$.

[1] Here subscripts denote partial derivatives.

The proof of Theorem 1 then combines these two ideas and exploits the duality between direct and indirect utility.

Proof of Theorem 1: Let $M = \max_{s,t} \|p^s - p^t\|^2$. By assumption there exist two distinct observations, so $M > 0$. Since there exists a solution to the strict dual Afriat inequalities, there exists $\varepsilon > 0$ sufficiently small so that $\forall\, i \neq j$,

$$V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j) + \varepsilon M. \qquad (*)$$

Define

$$\tilde{q}^j = q^j - \varepsilon p^j,\ j = 1, \ldots, N$$

and

$$\tilde{V}^j = V^j - \tfrac{1}{2}\varepsilon \|p^j\|^2,\ j = 1, \ldots, N.$$

Then $\forall\, i \neq j$, we claim that

$$\tilde{V}^i - \tilde{V}^j > \tilde{q}^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j). \qquad (**)$$

To verify $(**)$, let $i \neq j$. Then

$$\begin{aligned}
&\tilde{V}^i - \tilde{V}^j - \tilde{q}^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) \\
&= \left(V^i - \tfrac{1}{2}\varepsilon \|p^i\|^2\right) - \left(V^j - \tfrac{1}{2}\varepsilon \|p^j\|^2\right) - (q^j - \varepsilon p^j) \cdot (p^i - p^j) - \lambda^j (I^i - I^j) \\
&= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) - \tfrac{1}{2}\varepsilon \|p^i\|^2 + \tfrac{1}{2}\varepsilon \|p^j\|^2 + \varepsilon p^j \cdot (p^i - p^j) \\
&= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) + \tfrac{1}{2}\varepsilon (\|p^j\|^2 - \|p^i\|^2) - \tfrac{1}{2}\varepsilon 2 p^j \cdot (p^j - p^i) \\
&= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) + \tfrac{1}{2}\varepsilon (p^j - p^i) \cdot (p^j + p^i - 2p^j) \\
&= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) - \tfrac{1}{2}\varepsilon (p^j - p^i) \cdot (p^j - p^i) \\
&= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j (I^i - I^j) - \tfrac{1}{2}\varepsilon \|p^j - p^i\|^2 \\
&> 0
\end{aligned}$$

by $(*)$.

Now by Lemma 1, there exists a convex, C^∞ function $W : \mathbb{R}_+^{\ell+1} \to \mathbb{R}$ which is strictly increasing in I, strictly decreasing in p and satisfies:

$$\begin{aligned}
&W(p^i, I^i) = \tilde{V}^i,\ i = 1, \ldots, N \\
&DW(p^i, I^i) = (\tilde{q}^i, \lambda^i),\ i = 1, \ldots, N \\
&\frac{\partial^2 W}{\partial I^2}(p^i, I^i) = 0,\ i = 1, \ldots, N.
\end{aligned}$$

Define $V(p, I) = W(p, I) + \frac{1}{2}\varepsilon \|p\|^2$. Then V is C^∞, convex in (p, I), strictly convex in p, strictly increasing in I and strictly decreasing in p for ε sufficiently small. Moreover,

$$\begin{aligned}
&V(p^i, I^i) = V^i,\ i = 1, \ldots, N \\
&DV(p^i, I^i) = (q^i, \lambda^i),\ i = 1, \ldots, N \\
&\frac{\partial^2 V}{\partial I^2}(p^i, I^i) = 0,\ i = 1, \ldots, N.
\end{aligned}$$

By Quah's theorem, the demand function

$$x(p,I) \equiv -\frac{1}{D_I V(p,I)} D_p V(p,I)$$

generated by this indirect utility function is locally monotone at (p^i, I^i) for each $i = 1, \ldots, N$.

To convert this indirect utility function into a direct utility function, for each $x \in \mathbb{R}^\ell_+$ define

$$\begin{array}{rll} U(x) \equiv & \min\limits_{(p,I)\in\Delta\times\mathbb{R}_+} & V(p,I) \\ & \text{s.t.} & p \cdot x \le I. \end{array}$$

Then U is smooth, strictly quasiconcave, monotone, and rationalizes the data. Moreover, $x(p,I)$ is the demand function generated by U. To establish this claim, we must show that for each $(p,I) \in \Delta \times \mathbb{R}_+$,

$$\begin{array}{rll} V(p,I) = & \max\limits_{x\ge 0} & U(x) \\ & \text{s.t.} & p \cdot x \le I. \end{array} \qquad (\dagger)$$

Let $\bar{x}$ solve $(\dagger)$. Then $\bar{x}$ solves the first order conditions

$$\begin{aligned} DU(\bar{x}) &= \lambda p \\ p \cdot \bar{x} &= I, \end{aligned}$$

where $\lambda > 0$. By definition,

$$\begin{array}{rll} U(\bar{x}) \equiv & \min\limits_{(p',I')\in\Delta\times\mathbb{R}_+} & V(p',I') \\ & \text{s.t.} & p' \cdot \bar{x} \le I'. \end{array} \qquad (\ddagger)$$

so by the envelope theorem, $DU(\bar{x}) = \gamma p(\bar{x})$, where $\gamma > 0$ is the Lagrange multiplier for $(\ddagger)$ and $(p(\bar{x}), I(\bar{x}))$ is the solution to $(\ddagger)$. Thus $\gamma p(\bar{x}) = \lambda p$, which implies that $p(\bar{x}) = p$ since $p, p(\bar{x}) \in \Delta$. Then $I(\bar{x}) = p(\bar{x}) \cdot \bar{x} = p \cdot \bar{x} = I$, so $U(\bar{x}) \equiv V(p(\bar{x}), I(\bar{x})) = V(p,I)$. Finally, by Roy's identity $x(p,I)$ is the demand function generated by U, which also implies that U rationalizes the data. □

The conditions which are necessary and sufficient for the existence of a smooth, monotone utility function rationalizing a given finite data set – the dual strict Afriat inequalities – are exactly the same conditions which are necessary and sufficient for the existence of smooth preferences rationalizing the data for which demand is locally monotone at each observation. Thus it follows immediately from Theorem 1 that any finite data set that can be rationalized by smooth preferences can be rationalized by smooth preferences giving rise to locally monotone demand.

Corollary. *Let* $(p^r, x^r), r = 1, \ldots, N$ *be given. There exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data if and only if there exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data such that the implied demand function is locally monotone at* (p^r, I^r) *for each* $r = 1, \ldots, N$, *where* $I^r = p^r \cdot x^r$.

This result will provide the foundation for the study of rationalizing equilibria which is contained in the next section.

3 Rationalizing Walrasian Equilibria

In this section, we turn to the question of rationalizing observations as equilibria. Here we consider a finite number of observations $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$ of prices, income levels for each of T consumers, and aggregate consumption. Our main result shows that such a finite data set can never be used to refute the hypotheses that equilibria are locally unique or locally stable under tâtonnement, or that equilibrium comparative statics are locally monotone.

Theorem 2. *Let* $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$, *be given. This data can be rationalized by an economy in which each consumer has a smooth, strictly quasiconcave, monotone utility function if and only if it can be rationalized by an economy in which each consumer has a strictly quasiconcave, monotone utility function and in which each observed equilibrium* p^r *is locally unique and locally stable under tâtonnement and in which the equilibrium correspondence is locally monotone at* (p^r, ω^r) *for each* r.

To establish this result, consider a given finite data set $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$. When can these observations be rationalized as Walrasian equilibria? Since we do not observe individual consumption bundles or utilities, these observations are **rationalizable as Walrasian equilibria** if for each observation $r = 1, \ldots, N$ there exist consumption bundles x_t^r for each consumer $t = 1, \ldots, T$ such that the individual observations (p^r, x_t^r), $r = 1, \ldots, N$, are rationalizable for each consumer, $p^r \cdot x_t^r = I_t^r$ for each r and t, and such that markets clear in each observation, that is, $\sum_{t=1}^{T} x_t^r = \omega^r$ for each r. Putting this definition together with the dual strict Afriat inequalities characterizing individual rationalizability yields the following result.

Lemma 2. *Let* $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$ *be a finite set of observations. There exist smooth, strictly quasiconcave, monotone utility functions rationalizing this data and initial endowments* $\{\omega_t^r\}_{t=1}^T$ *such that* p^r *is an equilibrium price vector for the economy* $\mathcal{E}^r$ *if and only if there exist numbers* V_t^r, λ_t^r *and vectors* q_t^r *for* $t = 1, \ldots, T$

and $r = 1, \ldots, N$ *such that:*

(a) *the dual strict Afriat inequalities hold for each consumer* $t = 1, \ldots, T$

(b) $p^r \cdot q_t^r = -\lambda_t^r I_t^r$ *for* $t = 1, \ldots, T$ *and* $r = 1, \ldots, N$

(c) *"markets clear":*

$$\sum_{t=1}^{T} x_t^r = \omega^r \quad \forall r$$

where $x_t^r = -\frac{1}{\lambda_t^r} q_t^r$ *for each* r *and* t*.*

Proof: Follows immediately from Theorem 1 and the definition of rationalizability. □

Moreover, note that given a finite set of observations $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$, without observations of initial endowments we can without loss of generality assume that each observation of the income distribution $\{I_t^r\}_{t=1}^T$ is derived from collinear individual endowments. More precisely, for each r and t define

$$\alpha_t^r = \frac{I_t^r}{p^r \cdot \omega^r}$$

and $\alpha^r = (\alpha_1^r, \ldots, \alpha_T^r)$. Given utility functions $\{U_t\}_{t=1}^T$, the **distribution economy** $\mathcal{E}_{\alpha^r}$ is the economy in which consumer t has preferences represented by the utility function U_t and endowment $\alpha_t^r \omega^r$. Using this observation we can now restate Lemma 2 as follows.

Lemma 2′. *Let* $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$, *be a finite set of observations. There exist smooth, strictly quasiconcave, monotone utility functions rationalizing this data such that* p^r *is an equilibrium price vector for the distribution economy* $\mathcal{E}_{\alpha^r}$ *if and only if there exist numbers* V_t^r, λ_t^r *and vectors* q_t^r *for* $t = 1, \ldots, T$ *and* $r = 1, \ldots, N$ *such that:*

(a) *the dual strict Afriat inequalities hold for each consumer* $t = 1, \ldots, T$

(b) $p^r \cdot q_t^r = -\lambda_t^r I_t^r$ *for* $t = 1, \ldots, T$ *and* $r = 1, \ldots, N$

(c) *"markets clear":*

$$\sum_{t=1}^{T} x_t^r = \omega^r \quad \forall r$$

where $x_t^r = -\frac{1}{\lambda_t^r} q_t^r$ *for each* r *and* t*.*

The dual strict Afriat inequalities in (a) are exactly the conditions characterizing observations which can be rationalized by consumers with smooth characteristics, and we showed in Theorem 1 that such observations can always be rationalized by utility functions generating locally monotone demand. The results we derive regarding the refutability of local stability and local comparative statics then follow from the striking properties of locally monotone individual demand functions.

First, unlike almost every other property of individual demand such as the weak axiom or the Slutsky equation, local monotonicity aggregates. If f_t is an individual demand function which is locally monotone at $(\bar{p}, \bar{I}_t)$ for each $t = 1, \ldots, T$, then market excess demand

$$F(p) \equiv \sum_{t=1}^{T} f_t(p, \bar{I}_t) - \omega$$

is locally monotone at $\bar{p}$.

Furthermore, local monotonicity at equilibrium implies local stability of tâtonnement, at least in distribution economies, as the following result shows.

Theorem (Malinvaud). *Let $\bar{p}$ be an equilibrium price vector for a distribution economy $\mathcal{E}_\alpha$ with income distribution $\{\alpha_t\}_{t=1}^{T}$. If each consumer's demand function is locally monotone at $(\bar{p}, \bar{I}_t)$, where $\bar{I}_t = \alpha_t(\bar{p} \cdot \omega)$, then the tâtonnement price adjustment process is locally stable at $\bar{p}$.*

In addition, local monotonicity at equilibrium implies both local uniqueness[2] and monotone local comparative statics in distribution economies.

Theorem (Malinvaud). *Let $\bar{p}$ be an equilibrium price vector for a distribution economy $\mathcal{E}_\alpha$ with income distribution $\{\alpha_t\}_{t=1}^{T}$. If each consumer's demand function is locally monotone at $(\bar{p}, \bar{I}_t)$, where $\bar{I}_t = \alpha_t(\bar{p} \cdot \omega)$, then the equilibrium correspondence $\mathcal{C}$ for the distribution economy $\mathcal{E}_\alpha$[3] is locally monotone in a neighborhood $\mathcal{P} \times \mathcal{U}$ of $(\bar{p}, \omega)$, i.e., if $(p', \omega') \in (\mathcal{P} \times \mathcal{U}) \cap \mathcal{C}$ then*

$$(p' - p) \cdot (\omega' - \omega) < 0.$$

This conclusion of locally monotone comparative statics implies in particular that if the aggregate supply of a good increases, all else held constant, then its equilibrium price will fall, at least locally.

The main conclusion of the paper is now an immediate consequence of the results of section 2 and the properties of locally monotone demand functions in distribution

[2] To see this, note that if $F(\bar{p}) = 0$ and F is locally monotone at $\bar{p}$, then $(p - \bar{p}) \cdot F(p) < 0$ for p sufficiently close to $\bar{p}$. In particular, $F(p) \neq 0$ for such p.

[3] Here the income distribution is assumed to be constant as aggregate endowment changes, so the equilibrium correspondence $\mathcal{C}$ for a distribution economy $\mathcal{E}_\alpha$ is the set of pairs (p, ω) such that p is an equilibrium price for the economy in which consumer t has utility U_t and endowment $\alpha_t \omega$.

economies: local uniqueness, local stability, and local monotone comparative statics are not refutable given a finite set of observations of prices, income levels, and aggregate consumption.[4]

Proof of Theorem 2: Let $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$, $r = 1, \ldots, N$, be a finite set of observations which can be rationalized in an economy in which each consumer has a smooth, strictly quasiconcave, monotone utility function. Then conditions (a), (b), and (c) of Lemma 2 must hold. By Theorem 1, there exist smooth, strictly quasiconcave, monotone utility functions rationalizing the data such that p^r is an equilibrium price vector for the distribution economy $\mathcal{E}_{\alpha^r}$ and such that the market excess demand function for $\mathcal{E}_{\alpha^r}$ is locally monotone at p^r for each $r = 1, \ldots, N$. Thus p^r is a locally unique equilibrium in the economy $\mathcal{E}_{\alpha^r}$. By Malinvaud's results, p^r is locally stable under tâtonnement, and the equilibrium correspondence in the distribution economy $\mathcal{E}_{\alpha^r}$ is locally monotone at (p^r, ω^r) for each $r = 1, \ldots, N$. □

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[4] Note that this result is still valid if in addition individual demands or net trades are observed.

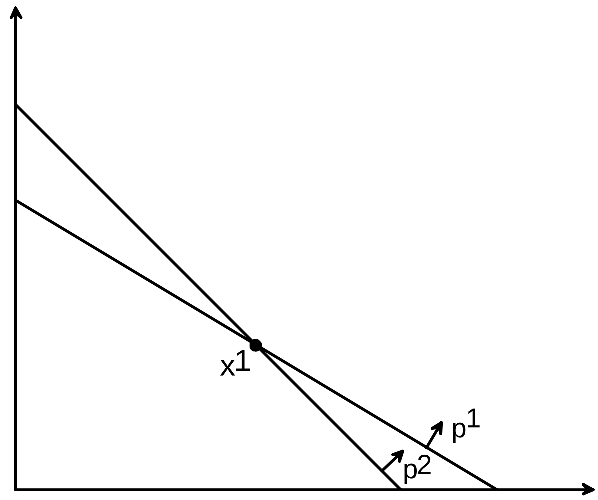

Figure 1

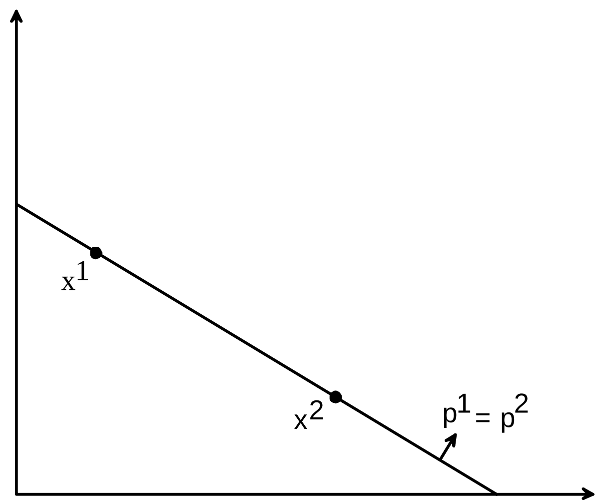

Figure 2