MATRICES WITH IDENTICAL SETS OF NEIGHBORS

Imre Bárány
Mathematical Institute of the Hungarian Academy of Sciences
P. O. Box 127, Budapest, 1364 Hungary

and

Herbert E. Scarf
Cowles Foundation, Yale University
New Haven, Connecticut 06520 USA

Abstract

Given a generic m by n matrix A, a lattice point h in $\mathbb{Z}^n$ is a neighbor of the origin if the body $\{x : Ax \leq b\}$, with $b_i = \max\{0, a_i h\}$, $i = 1, ..., m$, contains no lattice point other than 0 and h. The set of neighbors, $N(A)$, is finite and 0-symmetric. We show that if A' is another matrix of the same size with the property that sign $a_i h$ = sign $a'_i h$ for every i and every $h \in N(A)$, then A' has precisely the same set of neighbors as A. The collection of such matrices is a polyhedral cone, described by a finite set of linear inequalities, each such inequality corresponding to a generator of one of the cones $C_i = \text{pos}\{h \in N(A) : a_i h < 0\}$. Computational experience shows that C_i has "few" generators. We demonstrate this in the first nontrivial case $n = 3$, $m = 4$.

Key words: Test Sets for Discrete Programming. Sensitivity to the Production Matrix.

1 Introduction

Test sets for integer programming were introduced by Graver (1975) and Scarf (1986). They provide a way of telling if a feasible solution $z \in \mathbb{Z}^n$ is optimal or not by checking, for each h in the test set, whether $z + h$ is feasible and yields an improved value of the objective function.

The test set of Scarf, the set of neighbors of the origin, is associated with a matrix A of size m by n, and is applied to the class of problems of the form

$$\begin{aligned} \min \quad & a_1 z \\ \text{subject to} \quad & a_i z \leq b_i \ (i = 2, ..., m), \ z \in \mathbb{Z}^n \end{aligned} \tag{1.1}$$

in which a single row of A becomes the objective, and the remaining rows are used, with arbitrary b_i, to form the constraints.

For each lattice point $h \in \mathbb{Z}^n$, the smallest body of the form

$$K_b = \{x \in R^n \ : \ Ax \leq b\} \tag{1.2}$$

containing 0 and h is given by $b_i = \max\{0, a_i h\}$, for $i = 1, 2, ..., m$. We designate this body by $\langle 0, h \rangle$. The lattice point $h \in \mathbb{Z}^n$ $(h \neq 0)$ is defined to be a *neighbor of the origin* if $\langle 0, h \rangle$ contains no lattice points in its *interior*. The collection of such neighbors is denoted by $N(A)$. Note that in this definition the special role of a_i as the objective function has disappeared.

In the next section we introduce various conditions on A to ensure that $N(A)$ is a test set for the integer programs (1.1), or that $N(A)$ is nonempty and finite. Finiteness of $N(A)$ is proved in quantitative form (Theorem 3). Our main result (Theorem 1) characterizes matrices with identical sets of neighbors. It turns out that this collection of matrices $\mathcal{C}(A)$ is a polyhedral set determined by the cones

$$C_i = \text{pos}\{h \in N(A) \ : \ a_i h < 0\} \tag{1.3}$$

where A is a generic (cf. Section 2) matrix. $\mathcal{C}(A)$ has a product structure since the rows of the matrices in it vary in the interior of C_i^*, the polar of C_i, independently of each other.

Computational experience and some theoretical results (cf. Remark in Section 2) indicate that C_i has "few" generators. We demonstrate this (Theorem 2) in the first nontrivial case $n = 3$,

$m = 4$. The proof is based on properties of the neighbors and of 3-dimensional lattices. It uses geometry of numbers and is basically elementary.

2 Results

We assume throughout that the rank of A is n. Notice first that $N(A)$ is symmetric about the origin. This follows from $\langle 0, h\rangle - h = \langle 0, -h\rangle$.

Next, we need to formulate various conditions on the matrix A. A convenient way to do so is to consider the dual feasible region

$$D(A) = \{y \in R^m \,:\, yA = 0,\ y \geq 0\}.$$

The first condition we need is

(A1) There is $y \in D(A)$ with $y_i > 0$ $(\forall i)$.

This is equivalent to saying that K_b is bounded for every b, or that $0 \in \text{int conv}\{a_1, ..., a_m\}$. We will show (Claim 1 in Section 3) that (A1) implies that $N(A)$ is nonempty and, further, that it is a test set for the integer programs (1.1).

Condition (A1) implies that there exists a non-zero vector in $D(A)$ with $n+1$ or fewer positive components. Our next condition, a weak form of non-degeneracy of A, says

(A2) every non-zero $y \in D(A)$ has at least $n + 1$ positive components,

which is the same as saying that 0 is not in the convex hull of any n rows of A. We will show in Theorem 3 that, under (A1) and (A2), $N(A)$ is finite in a quantitative form.

Finiteness of $N(A)$ was proved in White (1983) and in Bárány, et al. (1995) under the stronger condition "all n by n minors of A are nonsingular."

In general, the set of neighbors need not form a minimal test set for the integer programs (1.1); a proper subset of $N(A)$ may also be a test set. The reason for this ambiguity is that we may have two bodies $\langle 0, h\rangle$ and $\langle 0, h'\rangle$, with distinct lattice points h and h', which are identical, free of interior lattice points, but with h' on the boundary of the first body and h on the boundary of the second. In this case, removal of either one of these points h or h' results in a smaller test set. As we shall see, this is more a problem of exposition than substance, aside from a lower dimensional set of matrices.

The matrix A is called *generic* if it satisfies conditions (A1) and (A2) and

(A3) $a_i h \neq 0$ for every i and every $h \in N(A)$.

For a generic matrix A, $N(A)$ is the unique minimal test set for (1.1). Notice that generic matrices form a dense set in the collection of matrices satisfying (A1) and (A2): any such matrix with algebraically independent entries is automatically generic.

Now let A be a generic matrix and $\mathcal{C}(A)$ the collection of matrices A' satisfying, for every i and every $h \in N(A)$

$$\text{sign } a_i' h = \text{sign } a_i h \ . \tag{2.1}$$

As we shall see the closure of $\mathcal{C}(A)$ is a polyhedral cone. This follows from

THEOREM 1: *Let A be a generic matrix and $A' \in \mathcal{C}(A)$. Then A' is also generic and has precisely the same set of neighbors as A. Moreover, dual feasible bases of A and of A' coincide.*

This, of course, shows that $\mathcal{C}(A) = \mathcal{C}(A')$. Theorem 1 says, in other words, that elements of $\mathcal{C}(A)$ are characterized by conditions (cf. (2.1))

$$a_i' \in \text{int } C_i^* \ , \quad i = 1, ..., m$$

where C_i^* is the polar of the cone C_i defined in (1.3). Thus $\mathcal{C}(A)$ has a product structure: any choice $a_i' \in \text{int } C_i^*$ ($i = 1, ..., m$, the a_i' are chosen independently!) gives rise to a generic matrix $A' = [a_1', ..., a_m']^T \in \mathcal{C}(A)$.

Write now G_i for the set of generators of the cone C_i. Each G_i is finite and

$$C_i^* = \{x \, : \, gx \leq 0, \ g \in G_i\}$$

is a (minimal) polyhedral description of C_i^* and of $\mathcal{C}(A)$. The simpler the structure of the G_i, the simpler this polyhedral description becomes.

We have investigated the structure of $N(A)$ on several examples, mainly in dimension 3, 4, and 5. The computational experiments provided beautiful pictures and insightful examples, and showed structural properties of the neighbors. The experiments led to the conjecture that the cones $\mathcal{C}_i$ have “few” generators. We prove this in the first nontrivial case.

THEOREM 2: *If A is a generic 4 by 3 matrix, then C_i has (i) either three generators and they form a basis of $\mathbb{Z}^3$, (ii) or four generators, and some three of them form a basis of $\mathbb{Z}^3$.*

Before proceeding to the proofs some remarks are in place here.

REMARK 1: Most frequently, test sets are considered when the corresponding matrix A is integral (Lovász (1989), Sturmfels and Thomas (1994), and others). These matrices often lie on the boundary of the decomposition (given by Theorem 1) of the set of matrices satisfying (A1). For matrices on the boundary of a cell $\mathcal{C}$ the set of neighbors need not be a minimal test set.

REMARK 2: In the 4 by 3 case the number of generators of C_i, $|G_i|$, is bounded independently of A (according to Theorem 2). It is unlikely though that, in general, $|G_i|$ is bounded by a function of n and m alone. However, as A. Barvinok (1995) pointed out, a deep result of R. Kannan (1990) shows that $|G_i|$ is polynomial in the size of A. We mention further that, in the 4 by 3 case, in every computational example we had with 4 generators, the generators formed a parallelogram.

REMARK 3: The cones C_i play a role in another question as well. Sturmfels and Thomas (1994) considered integer programs of the form $\min\{cx \,:\, Ax \leq b,\ x \in \mathbb{Z}^n\}$ with c and b varying while A is a fixed national (or integral) matrix. They show that there is a fan, i.e., a subdivision of R^n into cones $K_1, ..., K_k$ with nice intersection properties, such that for every $b \in R^m$ and every c_i, $c_i' \in \text{int } K_i$, the integer programs

$$\min\{c_i x \,:\, Ax \leq b,\ x \in \mathbb{Z}^n\} \quad \text{and}$$

$$\min\{c_i' x \,:\, Ax \leq b,\ x \in \mathbb{Z}^n\}$$

have the same solution. It can be shown (using the results of this paper) that for any particular $c_i \in \text{int } K_i$, K_i is the polar of $\text{pos}\{h \in N(A_i) \,:\, c_i h < 0\}$ where $A_i = [c_i, a_1, ..., a_m]^T$.

REMARK 4: There is yet another case where the cones C_i come up. Given a generic $m \times n$ matrix A and $b \in R^m$ the set K_b of the form (1.3) is a *maximal lattice free convex body* if $\mathbb{Z}^n \cap \text{int } K_b = \phi$ but $\mathbb{Z}^n \cap \text{int } K = \phi$ for every convex body K properly containing K_b. Every facet of K_b contains exactly one lattice point in its relative interior. Associating this set of lattice points with the maximal lattice free convex body K_b gives rise to a simplicial complex $\mathcal{K}(A)$ depending only on A (see Bárány, et al. (1994) and Bářany, et al. (1995) for the precise definition). The proof of Theorem 1 shows that for $A' \in \mathcal{C}(A)$, the simplicial conplexes $\mathcal{K}(A)$ and $\mathcal{K}(A')$ coincide.

3 $N(A)$ is Nonempty and Finite

We show first that, under condition (A1), $N(A)$ is nonempty in the following stronger form.

CLAIM 1: If A satisfies (A1), then every set K_b with $0 \in K_b$ and $|\mathbb{Z}^n \cap K_b| \geq 2$ contains a neighbor of A.

PROOF: Suppose $0,\, z \in \mathbb{Z}^n \cap K_b$, $z \neq 0$. We construct a (finite) sequence $z = z_0,\, z_1,\, ...,\, z_\ell$ so that $z_i \in \text{int}\langle 0,\, z_{i-1}\rangle$, $\langle 0,\, z_i\rangle \subset \langle 0,\, z_{i-1}\rangle$ $(i = 1,\, ...,\, \ell)$ and $z_\ell \in N(A)$.

Assume z_i has been constructed. If $\mathbb{Z}^n \cap \text{int}\langle 0,\, z_i\rangle = \phi$, set $\ell = i$ and stop. Otherwise pick any $z_{i+1} \in \mathbb{Z}^n \cap \text{int}\langle 0,\, z_i\rangle$ and continue. The algorithm stops since, in view of (A1), K_b is bounded and $z_0,\, z_1,\, ...,\, z_\ell$ all belong to $\langle 0,\, z_0\rangle \subset Z_b$. □

The Claim implies that $N(A) \neq \phi$ and, further, that $N(A)$ is a test set for the integer programs (1.1). Now we turn to the proof of finiteness of $N(A)$.

As $N(A)$ does not change if a_i is multiplied by a positive number we may and do assume that $\|a_i\| = 1$ for all i. Define

$$d = \min\{|\det B| \,:\, B \text{ is a nonsingular } n \times n \text{ minor of } A\}\ . \tag{3.1}$$

THEOREM 3: *If A satisfies* (A1) *and* (A2), *then for every $h \in N(A)$*

$$\|h\| \leq \frac{n^2}{d}\ . \tag{3.2}$$

PROOF: Fix $h \in N(A)$, $\langle 0, h\rangle$ is bounded (by (A1)) and $\text{int}\langle 0, h\rangle \neq \phi$ because of (A2). Consider the ball B inscribed in $\langle 0, h\rangle$ that has the largest radius ρ, let its center be c.

$\mathbb{Z}^n \cap B = \phi$ implies, via a simple induction, that $\rho \leq \frac{1}{2}\sqrt{n}$.

Write I for the set of indices $i \in \{1, ..., m\}$ for which the hyperplane $\{x : a_i x = b_i\}$ is tangent to B. (Here $b_i = \max\{0, a_i h\}$.) For $i \in I$ the equation of this hyperplane can be written as

$$a_i(x - c) = \rho \ . \tag{3.3}$$

The corresponding inequalities represent the "active" constraints on the largest inscribed ball. The simple necessary condition for the maximality of ρ is $0 \in \text{conv}\{a_i : i \in I\}$. Then condition (A2) implies $0 \in \text{int conv}\{a_i : i \in I\}$ which shows, in turn, that the polyhedron

$$P = \{x : a_i x \leq b_i, \ i \in I\}$$

is bounded (and, further, that B is unique but we won't need this). Clearly $\langle 0, h\rangle \subset P$.

A vertex v, of P, is the solution of n equations of the form (3.3). Write M for the matrix whose rows are the a_i of these n equations. Further, let M^j be the matrix obtained from M by replacing its jth column by the all–one column. We get for the jth component of $v - c$

$$(v - c)_j = \rho \frac{\det M^j}{\det M}.$$

The denominator here is nonzero since otherwise the corresponding equations do not determine a vertex. Expanding the numerator along the all-one column and using $\|a_i\| \leq 1$ we get $|(v - c)_i| \leq \rho n/d$. By (3.1)

$$\|v - c\| \leq \rho n\sqrt{n}/d \leq n^2/2d \ .$$

But $\text{diam}\langle 0, h\rangle \leq \text{diam } P \leq n^2/d$ because the diameter of P occurs between two of its vertices.

□

4 Proof of Theorem 1

We start the argument by taking A' to be identical with A in rows 2, ..., m and differing only in row 1. By assumption sign $a'_1 h =$ sign $a_1 h$ for every $h \in N(A)$.

Claim 2: $N(A') \subset N(A)$.

Proof: Let $h' \in N(A')$. There is no loss in generality in assuming that $a_i h' \leq 0$ since if this were not true we could select the neighbor $-h' \in N(A')$.

Assume h' is not a neighbor of A. Then by Claim 1 of the previous section there is an $h \in N(A)$ with $h \in \text{int}\langle 0, h' \rangle_A$, so that

$$a_1 h < \max\{0,\, a_1 h'\} = 0 \ ,$$

$$a_i h < \max\{0,\, a_i h'\} = \max\{0,\, a_i' h'\} \ , \quad i = 2, ..., m \ .$$

We show now that $h \in \text{int}\langle 0, h' \rangle_{A'}$ contradicting the assumption that $h' \in N(A')$.

We certainly have $a_i' h = a_i h < \max\{0,\, a_i' h'\}$ for $i = 2, ..., m$. In order to demonstrate $a_i' h < \max\{0,\, a_i' h'\}$ it suffices to show that $a_i' h < 0$. But since $h \in N(A)$ we have sign $a_i' h =$ sign $a_i h < 0$. □

Write now $A(t) = tA + (1-t)A'$ and $a_1(t) = ta_1 + (1-t)a_1'$. We use a homotopy argument for

Lemma 1: *$A(t)$ is generic for every $t \in [0,1]$.*

Proof: Set

$$t^* = \min\{t \geq 0 \,:\, A(t) \text{ is not generic}\} \ .$$

where the existence of the minimum and $t^* > 0$ are easily justified. Assume, by way of contradiction, that $t^* \leq 1$. Clearly sign $a_1(t)h =$ sign $a_1 h$ for every $h \in N(A)$ and every $t \in [0,1]$. Thus $A(t)$ satisfies condition (A3) for every $t \in [0,1]$. Claim 2 implies, further, $N(A(t)) \subset N(A)$ for every $t \in [0, t^*)$.

We can reformulate conditions (A1) and (A2) for $A(t)$ as

(A1′) $0 \in \text{int conv}\{a_1(t),\, a_2,\, ...,\, a_m\}$,

(A2′) $0 \notin \text{conv}\{\text{any } n \text{ of them}\}$.

These conditions are true for $t \in [0, t^*)$ but one of them fails at t^*. If (A1′) fails, then 0 appears on the boundary of $\text{conv}\{a_1(t^*), a_2, ..., a_m\}$. By Caratheodory's theorem, 0 is in the relative interior of the convex hull of some of these vectors, including, of course, $a_1(t^*)$. Renaming these vectors suitably we get

$$0 \in \text{relint conv}\{a_1(t^*), a_2, ..., a_k\} \tag{4.1}$$

where $k \le n$ and we assume, further, that $a_2, ..., a_k$ are linearly independent.

If (A2′) fails at t^*, then 0 is in the convex hull of some n or fewer of the rows of $A(t^*)$. We conclude again, that (4.1) holds with $k \le n$ and $a_2, ..., a_k$ linearly independent.

Claim 3: There are $n + 1 - k$ rows of $A(t)$ which we can take to be $a_{k+1}, ..., a_{n+1}$ so that for all $t \in [0, t^*)$

$$0 \in \text{int conv}\{a_1(t), a_2, ..., a_{n+1}\} \ . \tag{4.2}$$

Proof: Define $L = \text{lin}\{a_2, ..., a_k\} = \text{pos}\{a_1(t^*), a_2, ..., a_k\}$ and let $\bar{x}$ denote the orthogonal projection of $x \in R^n$ onto $L^\perp$, the orthogonal complement of L. Set $Q(t) = \text{conv}\{\bar{a}_1(t), \bar{a}_{k+1}, ..., \bar{a}_m\}$. (A1) implies

$$0 \in \text{relint } Q(t) \text{ for } t \in [0, t^*) \ .$$

The halfline $\{-\lambda \bar{a}_1(t) \ : \ \lambda \ge 0\}$ intersects the boundary of $Q(t)$ (which is a convex polytope in $L^\perp$) at $-\lambda(t)\bar{a}_1(t)$. This point belongs to a facet $F(t)$ of $Q(t)$. Since $\bar{a}_1(t)$ is not on this facet and since $\bar{a}_1(t)$ changes linearly with t, $F(t)$ is constant on an interval $[t', t^*)$. By Caratheodory's theorem there are linearly independent vertices of $F(t)$, which we take to be $\bar{a}_{k+1} ..., \bar{a}_p$, such that there are $-\lambda(t)\bar{a}_1(t) \in \text{conv}\{\bar{a}_{k+1}, ..., \bar{a}_p\}$ implying

$$-\bar{a}_1(t) = \sum_{i=k+1}^{p} \alpha_i(t)\bar{a}_i \tag{4.3}$$

with $\alpha_i(t)$ continuous on $[t', t^*]$, positive on $[t', t^*)$, and 0 at t^*. The linear independence of $\bar{a}_{k+1}, ..., \bar{a}_p$ shows $p \le n + 1$.

Lifting (4.3) back to R^n we get

$$-a_1(t) = \ell(t) + \sum_{k+1}^{p} \alpha_i(t) a_i$$

where $\ell(t) \in L$ so that $\ell(t) = \sum_2^k \alpha_i(t)a_i$ with uniquely determined and continuous (since $\ell(t)$ is continuous) coefficients $\alpha_i(t)$. We then have

$$0 = a_1(t) + \sum_2^p \alpha_i(t)a_i \ . \tag{4.4}$$

Here $\alpha_i(t) > 0$ for $i > k$, and $\alpha_i(t) > 0$ for $i = 2, ..., k$ on $[t'', t^*)$ as well since $\alpha_i(t^*) > 0$ as follows from (4.1).

(4.4) shows $0 \in$ relint conv$\{a_1(t), ..., a_p\}$ when $t \in [t'', t^*)$. By (A2′) $p = n + 1$ and $0 \in$ int conv$\{a_1(t), ..., a_{p+1}\}$. By (A2′), again, this holds for all $t \in [0, t^*)$. □

It follows from Claim 3 and (A1′) that the cone

$$C(t) = \{x \in R^n \ : \ a_1(t)x < 0, \ a_2x < 0, ..., a_nx < 0\}$$

is simplicial and nonempty. Then

$$\min\{a_{n+1}z \ : \ z \in C(t) \cap \mathbb{Z}^n\}$$

is reached at some $h(t) \in C(t) \cap \mathbb{Z}^n$. Since $h(t)$ is a neighbor for the matrix $[a_1(t), a_2, ..., a_{n+1}]^T$, it is a neighbor for $A(t)$ as well. By Claim 2, $h(t) \in N(A)$. As $N(A)$ is finite, there is a sequence $t_\mu \to t^*$ (as $\mu \to \infty$) so that $h(t_\mu) = h \in N(A)$ for all μ. Thus $a_1(t^*)h < 0$, $a_2h < 0$, ..., $a_kh < 0$ showing that the hyperplane $\{x \ : \ hx = 0\}$ strictly separates 0 from $\{a_1(t^*), a_2, ..., a_k\}$. This contradicts (4.1) and finishes the proof of Lemma 1. □

Thus A' is generic and sign $a_i'h =$ sign a_ih for every i and every $h \in N(A')$ since $N(A') \subset N(A)$ by Claim 2. Claim 2 applies again with the roles of A and A' interchanged showing $N(A) = N(A')$.

To finish the proof of Theorem 1 we repeat the same argument for every row in A. Finally, it follows easily from this proof that all dual feasible bases of A remain the same during the homotopy. □

5 Few Generators

From now on we work with the 4×3 case. The arguments of the next two sections provide a proof of Theorem 2.

Shallcross (1992) has given a complete characterization of the neighbors in this case. Although we do not use this characterization explicitly, it provides considerable insight. Claims 1 and 2 below can be found in Shallcross (1992) as well.

With a slight change of notation let a_0, a_1, a_2, a_3 be the rows of A. We assume again that A is generic. Define H_i^0, H_i^+, H_i^- as the set of $x \in R^3$ with $a_i x = 0,\ > 0,\ < 0$ respectively.

We are interested in the neighbors $N = \{h \in N(A) : a_0 h < 0\}$. They lie in cones of the type $H_0^- \cap H_1^+ \cap H_2^+ \cap H_3^-$ which we denote by C_{12}: the index shows which of the H_i go with + superscript. By condition (A1) $H_0^- \cap H_1^- \cap H_2^- \cap H_3^- = \emptyset$. So the cones in question are $C_1, C_2, C_3, C_{12}, C_{23}, C_{31}$, and C_{123}.

Observe that the cones C_1, C_2, C_3, and C_{123} contain exactly one neighbor, to be denoted by s_1, s_2, s_3, and s_0, respectively. To see this note that, for instance s_2 is the unique solution to the integer program

$$\min\{a_2 x \ : \ a_i x < 0,\ i = 0,\ 1,\ 3,\ x \in \mathbb{Z}^3\}\ .$$

Since multiplying a_i by a positive number does not change the neighbors we may assume that $a_i s_i = 1, (i = 0,\ 1,\ 2,\ 3)$. Set

$$Q = \{x \in R^3 : |a_i x| \leq 1, i = 0, 1, 2, 3\}\ .$$

Claim 1: $N \subset Q$.

Proof: Assume $h \in N$ but $h \notin Q$, $a_0 h > 1$, say. As h is a neighbor, there is no integer other than 0 and h satisfying $a_i x \leq \max\{a_i h,\ 0\}$ for all i. But s_0 satisfies all these inequalities since $a_0 s_0 = 1 < a_0 h$ and $a_i s_0 < 0$ when $i \leq 1,\ 2,\ 3$. □

Recall now the definition of $C = \text{pos}\ N$ and write $D = C \cup (-C)$. We know from Theorem 2 that a_0 can be moved without changing $N(A)$ as long as H_0 does not meet C.

Claim 2: $Q \backslash D$ contains no lattice point.

Proof: Assume to the contrary that there is a point $z \in \mathbb{Z}^3 \cap Q \backslash D$. Move a_0 along $a_0(t) = a_0 + ta$ until $H_0(t)$ passes through the first such lattice point z. This happens at $t = t_0$, say. Since z is not a neighbor, it is in one of the cones C_{12}, C_{23}, or C_{31}, say C_{12}. But as $H_0(t)$ passes through z, it will be in the cone $H_0^+(t) \cap H_1^+ \cap H_2^+ \cap H_3^-$, which contains the unique neighbor $-s_3$. So $z = -s_3$, a contradiction. □

Claim 3: If u and v are generators of C, then $u - v \notin Q$.

Proof: If $u - v \in Q$ then, by Claim 2, $u - v$ is either in C or in $-C$. Assuming $u - v \in C$, $u \in v + C$, so $u = v + c$ for some $c \in C$. But then u is not a generator of C. □

Now if $u, v \in C \cap Q$ belong to the same cone C_{12}, C_{23}, or C_{31}, then automatically $u - v \in Q$. This shows that C can have at most seven generators, one in each of the cones C_i, C_{ij}, C_{123}. The trivial observation $s_0 \in \text{pos}\{s_1, s_2, s_3\}$ implies that C has at most six generators. The next claim takes this number down to four.

Claim 4: If s_1 and s_2 are generators of C, then C has no generator in C_{12}.

Proof: Assume $h \in N \cap C_{12}$ is such a generator. We will show that $a_i(s_1 + s_2) \leq \max\{a_i h, 0\}$ for $i = 0, 1, 2, 3$ contradicting $h \in N$. First, for $i = 0$ or $i = 3$

$$a_i(s_1 + s_2) < 0 = \max\{a_i h, 0\} .$$

By Claim 3, $s_1 - h \notin Q$. Now $|a_3(s_1 - h)| < 1$ clearly, and $a_i(s_1 - h) = 1 - a_1 h \in (0, 1)$ since $h \in Q$. Further, a_0 can be moved without changing N so that H_0 almost contains s_1 and h. This follows from Theorem 2 and the fact that s_1 and h are consecutive generators of C. Then $a_0(s_1 - h)$ is between -1 and 1. Consequently, $a_2(s_1 - h) < -1$. So we get $a_2(s_1 + s_2 - h) = a_2 s_2 + a_2(s_1 - h) < 0$, i.e.,

$$a_2(s_1 + s_2) < a_2 h = \max\{a_2 h, 0\} .$$

One proves $a_1(s_1 + s_2) < a_1 h = \max\{a_1 h, 0\}$ the same way. □

Figure 1 about here

Figure 1 presents the remaining six cases in the plane $a_0 x = 1$; the three lines are the traces of the planes H_1, H_2, H_3.

6 The Structure of the Generators

Claim 1: If u, v are generators of C, then u, v form a basis of the lattice $\mathbb{Z}^3 \cap \text{lin}\{u, v\}$.

Proof: By Claim 2 of the previous section there is no integer in the triangle $[0, u, -v]$ other than its vertices. Consequently $[0, u, -v, u-v]$ is a lattice parallelogram. $\square$

Here and in what follows we write $[a, b, c, d]$ for the convex hull of $a, b, c, d \in R^3$. We say that $[a, b, c, d]$ is *special* if it contains no lattice point other than a, b, c, d. The notation and terminology is extended to triangles and segments as well.

Claim 2: If u, v, w are consecutive generators of C, then $[0, u, v, -w]$ and $[0, -u, v, w]$ are special simplices.

Proof: This is true because of the previous claim and because the simplices in question are contained in $Q \backslash D$. $\square$

Lemma 3: *If* $0,\ a,\ b,\ c \in \mathbb{Z}^3$ *are not coplanar and the simplices* $[0,\ a,\ a+b,\ a+c]$, $[0,\ a+c,\ b+c,\ c]$, *and* $[a,\ c,\ a+c,\ a+b+c]$ *are special, then so are* $[a+b+c,\ b+c,\ c,\ b]$, $[a+b+c,\ b,\ a,\ a+b]$, *and* $[b+c,\ a+b,\ b,\ 0]$. *Moreover, all lattice points in* $T = \{\alpha a + \beta b + \gamma c : 0 < \alpha,\ \beta,\ \gamma < 1\}$ *are of the form* $\mu(a+b+c)$ *for some* $\mu \in (0,1)$.

Proof: The first statement follows simply by reflection through $\frac{1}{2}(a+b+c)$. The second needs more meditation.

Obviously, a and b generate the lattice $\mathbb{Z}^3 \cap \text{lin}\{a, b\}$. Then we can pick $z \in T \cap \mathbb{Z}^3$ so that a, b, z form a basis of $\mathbb{Z}^3$. Thus

$$c = \lambda_1 a + \lambda_2 b + \lambda_3 z$$

with λ_i an integer. In fact $\lambda_3 > 1$ since $\lambda_3 = 1$ would mean that a, b, c form a basis of $\mathbb{Z}^3$ and then $\mathbb{Z}^3 \cap \text{int}\, T = \emptyset$. Since $z \in T$ and $\lambda_3 z = c - \lambda_1 a - \lambda_2 b$, $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$.

Clearly $z \in \text{pos}\{a, b, c\}$ and the conditions concerning special simplices imply $z \in \text{pos}\{a +$

$b,\ b+c,\ c+a\}$. Then, with $\mu_i > 0$ one has

$$\lambda_3 z = c - \lambda_1 a - \lambda_2 b = \mu_1(a+b) + \mu_2(b+c) + \mu_3(c+a)\ .$$

The solution is $\mu_2 = \frac{1}{2}(1+\lambda_1-\lambda_2)$ and $\mu_3 = \frac{1}{2}(1-\lambda_1+\lambda_2)$ which is possible if and only if $\lambda_1 = \lambda_2$. Then $\mu_2 = \mu_3 = \frac{1}{2}$. Thus $\lambda_3 z = c - \lambda_1(a+b)$.

In the plane spanned by $a+b$ and c the triangle $[0,\ c,\ c+\frac{1}{2}(a+b)]$ is special as a consequence of the specialty of $[0,\ a+c,\ b+c,\ c]$ and of $[a+b+c,\ b,\ a,\ a+b]$, see Figure 2.

Figure 2 about here

Assuming $\lambda_1 \neq -1$ we have $\lambda_1 \leq -2$. The halfline $\{\lambda z : \lambda \geq 0\}$ intersects the parallelograms $T_1 = [0,\ \frac{1}{2}(a+b),\ \frac{1}{2}(a+b)+c,\ a+b+c]$ and $T_2 = \frac{1}{2}(a+b)+T_1$ in segments of the same length (because $\lambda_3 \geq 2$), and the first intersection contains the segment $[0,z]$. Then the second intersection contains an integer point as well. But T_2 is special. This contradiction shows that $\lambda_1 = -1$ and so $\lambda_3 z = c+a+b$. $\square$

We will use the Lemma 3 in the form of

Corollary: *Let C have three generators $a,\ b,\ c$ with $|\det(a,b,c)| = \lambda > 1$. If $T = \{\alpha a + \beta b + \gamma c : 0 < \alpha,\ \beta,\ \gamma < 1\}$ then $T \cap \mathbb{Z}^3 = \{\frac{k}{\lambda}(a+b+c)$ for $k = 1,\ ...,\ \lambda - 1\}$.*

Write G for the set of generators of C. We are to check the cases separately.

Case 1. $G = \{s_1, s_2, s_3\}$. If $s_1 + s_2 \notin Q$, then $a_3(s_1+s_2) < -1$ must hold since $a_0 s_1$ and $a_0 s_2$ can be taken almost equal to zero. So if $s_1 + s_2 \notin Q$ then $a_3(s_1+s_2+s_3) < 0$. Similarly, $s_2 + s_3 \notin Q$ and $s_3 + s_1 \notin Q$, respectively, imply $a_1(s_1+s_2+s_3) < 0$ and $a_2(s_1+s_2+s_3) < 0$. Since $a_0(s_1+s_2+s_3) < 0$ automatically, and $a_i(s_1+s_2+s_3) < 0$ for $i = 0,1,2,3$ contradicts (A1) we must have either $s_1+s_2 \in Q$ or $s_2+s_3 \in Q$ or $s_3+s_1 \in Q$. Assume, say, $s_1+s_2 \in Q$. Then the interior of the segment $[-s_3, s_1+s_2]$ lies in $Q\backslash D$ so the segment is special. But then $s_3 + [-s_3, s_1+s_2] = [0, s_1+s_2+s_3]$, is also special and the Corollary implies $\det(s_1,s_2,s_3) = \pm 1$.

Case 2. $G = \{s_1,\ s_2,\ h_{23}\}$. Then $s_1 + h_{23} \in Q$ and the segment $[-s_2,\ s_1+h_{23}] \in Q\backslash D$. The same argument as above shows that $\det(s_1, s_2, h_{23}) = \pm 1$.

Case 3. $G = \{s_1, h_{12}, h_{23}\}$. Again $s_1 + h_{23} \in Q$ and the segment $[-h_{12}, s_1 + h_{23}] \in Q \backslash D$, and we repeat the above argument.

Case 4. $G = \{h_{12}, h_{23}, h_{31}\}$. We are done again if $h_{12} + h_{23} \in Q$. If none of $h_{12} + h_{23}$, $h_{23} + h_{31}$, and $h_{31} + h_{12}$ is in Q, then $h_{12} + h_{23} + h_{31} \in C_{123}$ as one can easily check. Let $z = \frac{1}{\lambda}(h_{12} + h_{23} + h_{31})$ be the first integral point on the diagonal of T, where, of course, $\lambda \in \mathbb{Z}$ and assume $\lambda \geq 2$. Then

$$1 = a_0 s_0 \leq |a_0 z| = \frac{1}{\lambda}|a_0(h_{12} + h_{23} + h_{31})| \ .$$

But since a_0 can be moved so that $a_0 h_{12}$ and $a_0 h_{31}$ are almost zero, we get

$$1 \leq \frac{1}{\lambda}|a_0 h_{23}| \leq \frac{1}{\lambda} a_0 s_0 = \frac{1}{\lambda} \ .$$

Case 5: If C has four generators a, b, c, d in this consecutive order, then Claim 2 applies to consecutive generators d, a, b and a, b, c and b, c, d and d, c, a showing that $[0, -d, a, b]$, $[0, a, b, -c]$, $[0, b, -c, -d]$, and $[0, -c, -d, a]$ are special. This implies that $[0, a, b, -c, -d]$ is also special. By a theorem of Scarf (1986), these five points must lie on two consecutive lattice hyperplanes, H_1, H_2, say. If four of them lie in one of the hyperplanes, then they have to be $a, b, -c, -d$ as otherwise three of the generators would lie in a hyperplane through the origin. But then a, b, c, d are the vertices of a (special) parallelogram and any three of them form a basis of $\mathbb{Z}^3$. We may assume now that three of the points lie in H_1, and the other two in H_2. If $0 \in H_1$, then the two generators in H_1 with each generator from H_2 form a basis. Finally, if $0 \in H_2$, then the three generators in H_1 form a basis of $\mathbb{Z}^3$. This completes the proof of Theorem 2. □

ACKNOWLEDGEMENT

The research of the first author is partially supported by Hungarian National Science Foundation Grant No. 4296 and 016937. The research of both authors is supported by the program in Discrete Mathematics of the Cowles Foundation at Yale University.

REFERENCES

Bárány, I., R. Howe, and H.E.Scarf (1994). The complex of maximal lattice free simplices, *Math. Programming*, **66**, 273–281.

Bárány, I., H.E.Scarf, and D. Shallcross (1995). The topological structure of maximal lattice free convex bodies: the general case, *Fifth IPCO*, Kopenhagen.

Barvinok, A. (1995). Personal communication.

Danzer, L., B. Grünbaum, and V. Klee (1963). Helly's theorem and its relatives. *Proc. Symp. Pure Math.*, vol. VIII, *Convexity*, AMS, Providence, RI.

Graver, J.E. (1975). On the foundations of linear and integral programming, I, *Math. Programming*, **8**, 207–266.

Kannan, R. (1990). Test sets for integer programs with $\forall, \exists$ sentences, *DIMACS series*, vol. 1, AMS, 39–47.

Lovász, L. (1989). Geometry of numbers and integer programming. M. Iri and K. Tanabe, eds., *Mathematical Programming: Recent Developments and Applications*, Norwell, MA: Kluwer Acad., 177–210.

Scarf, H.E. (1986). Neighborhood systems for production sets with indivisibilities, *Econometrica*, **54**, 507–532.

Shallcross, D. (1992). Neighbors of the origin for four by three matrices, *Math. of O.R.*, **17**, 608–614.

Sturmfels, B. and R. Thomas (1994). Variation of cost function in integer programming, manuscript, Cornell University, Ithaca.

White, P. (1983). Discrete activity analysis, Ph.D Thesis, Yale University, Department of Economics.