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LINEAR COMPLEMENTARITY AND THE AVERAGE VOLUME OF SIMPLICIAL CONES

by

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§0. Smale [S] has recently shown how to estimate the average number of pivot steps in Lemke's algorithm for the linear complementarity problem (LCP) in terms of the "volumes" of certain cones. In this paper we discuss the notion of average volume of cones, and give two applications to the LCP.

§1. Let $X \subseteq \mathbb{R}^n$ be a set. If $\lambda \in \mathbb{R}$, then define λX , the λ -dilate of X by

$$\lambda X = \{\lambda x : x \in X\} .$$

If $\lambda X = X$, for all $\lambda > 0$, we say X is conical.

We define a measure on conical sets in the following way. Let

$$S^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i^2 = 1\}$$

be the unit sphere in \mathbb{R}^n . The sphere is invariant under rotations of \mathbb{R}^n and carries a unique measure ν which is invariant under rotations and such that $\nu(S^{n-1})$, the total measure of the sphere, equals 1.

If $X \subseteq \mathbb{R}^n$ is conical, we define

$$(1.1) \quad \nu(X) = \nu(X \cap S^{n-1}) .$$

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We call $v(X)$ the solid angle of X , or the conical volume of X , or usually simply the volume of X .

Given a set $Y \subseteq \mathbb{R}^n$, let $C(Y)$ denote the convex cone generated by Y . That is, $C(Y)$ is the set of linear combinations of elements of Y with non-negative coefficients. If Y contains exactly n non-zero elements, we call $C(Y)$ a simplicial cone. Strictly speaking, we should reserve that terminology for the case when the cone is non-degenerate in the sense that the elements of Y span \mathbb{R}^n (equivalently, are linearly independent; equivalently, are a basis) but it will be convenient in this discussion to allow degenerate cones also.

Suppose $Y = \{y^1, y^2, \dots, y^n\}$. Choose scalars $\lambda_i > 0$, and set $Y' = \{\lambda_1 y^1, \dots, \lambda_n y^n\}$. Obviously $C(Y') = C(Y)$. Hence in generating cones from sets of vectors, we are free to scale the vectors as we wish. A non-degenerate simplicial cone will thus be generated by a unique set of vectors of unit length in the Euclidean space. We will always take a simplicial cone to be so generated. In other words, we consider a simplicial cone in \mathbb{R}^n to be a point in $(S^{n-1})^n$, the space of n -tuples of unit vectors.

By a random simplicial cone, we mean a cone generated by n unit vectors chosen independently at random in S^{n-1} according to the probability distribution ν . In other words, it is a point in $(S^{n-1})^n$, chosen at random according to ν^n , the n -fold direct product of ν with itself. Somewhat more generally, let $Y_0 = \{y^1, \dots, y^k\}$ be a k -tuple of unit vectors in \mathbb{R}^n . Then a random simplicial cone based on $C(Y_0)$ is any n -tuple in $(S^{n-1})^n$ whose first k elements are y^1, y^2, \dots, y^k , and whose last $n-k$ elements are chosen independently at random from S^{n-1} according to the probability distribution ν .

With these definitions, the volume of a random simplicial cone, or a random simplicial cone based on $C(Y_0)$, becomes a random variable on the appropriate product of spheres. Our main result describes the mean value of this random variable--the average volume of random cones.

If $Y_0 = \{y^1, \dots, y^k\}$ is a k -tuple of unit vectors, then it spans typically a k -dimensional subspace. Let us identify R^k with the subspace of R^n consisting of n -tuples whose coordinates beyond the k^{th} are all zero. By a rotation, we can move Y_0 into R^k . So we will simply assume that $Y_0 \subseteq R^k$. Inside R^k , the cone $C(Y_0)$ is a simplicial cone, and it has a (k -dimensional) conical volume. For clarity in what follows, we will write this as $v^{(k)}(C(Y_0))$, and if $Y \subseteq (S^{n-1})^n$ defines a simplicial cone in R^n , its conical volume will be written $v^{(n)}(C(Y))$.

Theorem 1.1. Let $Y_0 = \{y^1, \dots, y^k\} \subseteq R^k \subseteq R^n$ be a k -tuple of unit vectors. Let Y be a random simplicial cone based on $C(Y_0)$. Then

$$(1.2) \quad \int_{(S^{n-1})^{n-k}} v^{(n)}(C(Y)) (dv)^{n-k} = 2^{-(n-k)} v^{(k)}(C(Y_0)) .$$

That is, the average random simplicial cone based on $C(Y_0)$ has volume $2^{-(n-k)} v^{(k)}(C(Y_0))$. In particular the volume of the average random simplicial cone in R^n is 2^{-n} (which is the volume of an orthant).

Proof. Let $X \subseteq R^k$ be an arbitrary conical set. Let $Y_1 = \{y^{k+1}, \dots, y^n\}$ be an $(n-k)$ -tuple of unit vectors in R^n . Consider the set

$$X + C(Y_1) = \{x + z : x \in X, z \in C(Y_1)\} .$$

If $X = C(Y_0)$, then clearly $X + C(Y_1) = C(Y_0 \cup Y_1)$. Consider the

average over $(S^{n-1})^{n-k}$ of the sets $X + C(Y)$. This is the integral

$$(1.3) \quad A(X) = \int_{(S^{n-1})^{n-k}} v^{(n)}(X + C(Y_1)) (dv)^{n-k}$$

We observe that the quantity $A(X)$ has two key properties:

i) $A(X)$ is invariant under rotation.

For if we move X by a rotation r of \mathbb{R}^k , we can extend r to a rotation \tilde{r} of all \mathbb{R}^n by letting it leave the orthogonal complement of \mathbb{R}^k (the space of vectors with their first k coordinates all zero) pointwise fixed. Clearly

$$\begin{aligned} v^{(n)}(\tilde{r}(X + C(Y_1))) &= v^{(n)}(\tilde{r}(X) + \tilde{r}(C(Y_1))) \\ &= v^{(n)}(r(X) + C(\tilde{r}(Y_1))) . \end{aligned}$$

Since the map $Y_1 \rightarrow \tilde{r}(Y_1)$ of $(S^{n-1})^{n-k}$ to itself clearly preserves the measure v^{n-k} , property i) follows.

ii) If X_1 and X_2 are disjoint (except for the origin) conical sets in \mathbb{R}^k , then $A(X_1 \cup X_2) = A(X_1) + A(X_2)$.

For if Y_1 generates a subspace Z of \mathbb{R}^n which is complementary to \mathbb{R}^k (i.e. such that \mathbb{R}^n is the direct sum $\mathbb{R}^k \oplus Z$), then it is easy to see that $X_1 + C(Y_1)$ and $X_2 + C(Y_1)$ are disjoint (except for the origin). Hence

$$v^{(n)}((X_1 \cup X_2) + C(Y_1)) = v^{(n)}(X_1 + C(Y_1)) + v^{(n)}(X_2 + C(Y_1)) .$$

Since the space Z generated by Y_1 will be complementary to \mathbb{R}^k outside a subset of measure zero (actually a closed subvariety), property ii) follows.

From properties i) and ii) of $A(X)$ we see it defines a rotationally invariant measure on conical sets of \mathbb{R}^k . Hence it must coincide with the volume measure $v^{(k)}$ up to multiples. In other words formula (1.2) is true except perhaps $2^{-(n-k)}$ should be replaced by another constant. That $2^{-(n-k)}$ is the correct constant may be seen as follows.

Evidently, the appropriate constant in formula (1.2) is $A(\mathbb{R}^k)$. Consider a sequence $\epsilon = (\epsilon_{k+1}, \dots, \epsilon_n)$ where each ϵ_j is ± 1 . Given an $(n-k)$ -tuple $Y_1 = \{y^{k+1}, \dots, y^n\}$ of unit vectors in \mathbb{R}^n , set

$$Y_1(\epsilon) = \{\epsilon_{k+1}y^{k+1}, \dots, \epsilon_n y^n\}.$$

If the span Z of Y_1 is complementary to \mathbb{R}^k , then it is easy to see that the interiors of the sets

$$\mathbb{R}^k + C(Y_1(\epsilon))$$

where ϵ ranges over all possible $(n-k)$ -tuples of ± 1 's, are disjoint.

Furthermore, the union of all these sets is \mathbb{R}^n . Hence

$$(1.4) \quad \sum_{\epsilon} v^{(n)}(\mathbb{R}^k + C(Y_1(\epsilon))) = 1.$$

Since the map $Y_1 \rightarrow Y_1(\epsilon)$ clearly preserves the measure on $(S^{n-1})^{n-k}$, we see that

$$(1.5) \quad A(\mathbb{R}^k) = \int_{(S^{n-1})^{n-k}} v^{(n)}(\mathbb{R}^k + C(Y_1(\epsilon))) (dv)^{n-k}$$

for any ϵ . Summing the relations (1.5) over all ϵ , and using equation (1.4), which holds on a set of full measure in $(S^{n-1})^{n-k}$, we conclude that $2^{(n-k)} A(\mathbb{R}^k) = 1$, as desired. This proves Theorem 1.1.

§2. We will now give two applications of Theorem 1.1 to the linear complementarity problem. Let z be in \mathbb{R}^n , and let M be an $n \times n$ matrix. Then $\text{LCP}(M, z)$, the linear complementarity problem associated to M and z , is the problem:

Find $x, y \in \mathbb{R}^n$ such that

$$(2.1) \quad \text{i) } x - My = z$$

$$\text{ii) } x \geq 0, \quad y \geq 0$$

$$\text{iii) For each } i, \quad 1 \leq i \leq n, \quad \text{either } x_i = 0 \quad \text{or } y_i = 0.$$

Here the $x_i \in \mathbb{R}$ are the coordinates of x , and $x \geq 0$ means $x_i \geq 0$ for each i .

In [ES] (see also [HS]) it is shown that solving $\text{LCP}(M, z)$ is equivalent to inverting a certain piecewise linear map T_M . We will briefly recall the construction of T_M . Let

$$\mathbf{N} = \{1, 2, \dots, n\}$$

be the set of the first n positive integers. For a set $A \subseteq \mathbf{N}$, let Q_A be the orthant in \mathbb{R}^n consisting of vectors x such that $x_i \leq 0$ if $i \in A$, and $x_i \geq 0$ otherwise. Thus if ϕ is the empty set, then $Q_\phi = \mathbb{R}^{n+}$, the positive orthant; and there are 2^n orthants in all. Define the matrix M_A by the rule

$$(2.2) \quad \text{i) If } i \in A, \quad \text{the } i^{\text{th}} \text{ column of } M_A \text{ is the } i^{\text{th}} \text{ column of } M,$$

$$\text{ii) If } i \notin A, \quad \text{the } i^{\text{th}} \text{ column of } M_A \text{ is the } i^{\text{th}} \text{ column of } I, \quad \text{the identity matrix.}$$

Define a piecewise linear map

$$T_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

by the recipe

$$(2.3) \quad \text{If } x \in Q_A, \text{ then } T_M(x) = M_A(x) .$$

It is easy to check that solving $LCP(M, z)$ amounts to finding a point in $T_M^{-1}(z)$.

The first topic concerning the LCP to which we will apply Theorem 1.1 is the number of solutions there are to $LCP(M, z)$. First we consider the average over z , then over M .

Let $\#(T_M^{-1}(z))$ be the number of solutions to $LCP(M, z)$. If M_A is non-singular, then $\#(T_M^{-1}(z) \cap Q_A)$ is at most 1, and the set of z for which $\#(T_M^{-1}(z) \cap Q_A)$ is non-zero is precisely the set of z in the simplicial cone $M_A(Q_A)$. In other words, the portion of z such that $T_M^{-1}(z)$ intersects Q_A is $v^{(n)}(M_A(Q_A))$. If z is not in the image under T_M of any coordinate hyperplane, or any orthant Q_A on which M_A is singular, then we evidently have

$$(2.4) \quad \#(T_M^{-1}(z)) = \sum_{A \subseteq \underline{N}} \#(T_M^{-1}(z) \cap Q_A) .$$

Since the conditions for formula (2.4) to make sense hold on a set of full measure, we reach the following conclusion.

Lemma 2.1. The average number of solutions to $LCP(M,z)$ as z varies in S^{n-1} is

$$(2.5) \quad \int_{S^{n-1}} \#(T_M^{-1}(z)) dv = \sum_{A \in \underline{CN}} \int_{S^{n-1}} \#(T_M^{-1}(z) \cap Q_A) \\ = \sum_{A \in \underline{CN}} v^{(n)}(M_A(Q_A)) .$$

Now we wish to average expression (2.5) over M . To do this we must settle on some measure on matrices. Since the space of $n \times n$ matrices is identifiable to R^m with $m = n^2$, one might propose to use the measure on conical sets in R^m as defined in formula (1.1) above. This would be a close analogue to what Smale does in [S].

However, we shall use a different measure on conical sets because of the following considerations. Let D be a positive diagonal matrix. Then the problems $LCP(M,z)$ and $LCP(MD,z)$ are essentially equivalent; that is, they are easily transformable one into the other. Specifically, if the vectors x, y solve $LCP(M,z)$, then the vectors $x, D^{-1}y$ solve $LCP(MD,z)$. Furthermore the pivot steps for Lemke's algorithm applied to $LCP(M,z)$ and to $LCP(MD,z)$ will be exactly the same. The point is that for each orthant Q_A , the cones $M_A(Q_A)$ and $(MD)_A(Q_A)$ are exactly the same. For let d_i be the diagonal entries of D , and let M^i be the columns of M . Then

$$(2.6) \quad (MD)^i = d_i M^i .$$

Thus changing M to MD simply amounts to rescaling the generators of the cones $M_A(Q_A)$, and each cone as a whole is left unchanged.

These facts suggest that to compute an average over M of the

quantity in formula (2.5), one should use a measure invariant under the transformations $M \rightarrow MD$. The measure coming from \mathbb{R}^m ($m = n^2$) does not have this property. The measure we will define and use does.

Let \mathbf{M}^n be the space of $n \times n$ matrices. Let $X \subseteq \mathbf{M}^n$ be a subset. Given a matrix A , set

$$XA = \{MA : M \in X\} .$$

If $XD = X$ for all positive diagonal D , we will call X multi-conical. Here is a convenient way to construct multi-conical sets. For $1 \leq i \leq n$, let $Y_i \subseteq \mathbb{R}^n$ be a conical set. Put

$$(2.7) \quad \text{Col}(Y_1, Y_2, \dots, Y_n) = \{M \in \mathbf{M}^n : M^i \in Y_i\} .$$

The multi-conical sets form an algebra of sets; a union, intersection or complement of multi-conical sets is also multi-conical. It is not hard to convince oneself that every open multi-conical set is a union of sets of the form (2.7). Thus we may define a measure v on multi-conical sets by requiring

$$(2.8) \quad v(\text{Col}(Y_1, Y_2, \dots, Y_n)) = \prod_i v(Y_i)$$

We will work with the measure v defined by formula (2.8). An alternative description of it is as follows. Given a multi-conical set X of matrices, consider the subset X_1 of X consisting of matrices all of whose columns have unit length. The set X_1 can be regarded as a subset of $(S^{n-1})^n$, and the multi-conical volume of X is clearly just the measure of X_1 with respect to the product measure v^n on $(S^{n-1})^n$. That is, the analogue of formula (1.1) holds. One may thus

think intuitively of the measure ν as being appropriate when one regards a matrix M as being a collection of n columns which vary completely independently of each other.

Having defined the measure with respect to which we will compute average behavior, we can formulate the first application of Theorem 1.1 to the LCP.

Theorem 2.2. With the measure ν as defined above, we have

$$(2.9) \quad \int_{\mathbf{M}^n} \left(\int_{S^{n-1}} \#(T_M^{-1}(z)) d\nu(z) \right) d\nu(M) = 1 .$$

In words, the average, in the above defined sense, over M and z of the number of solutions to $LCP(M,z)$ is 1.

Proof. For the inner integral in (2.9) we substitute the third expression in formula (2.5), and we compute the integral for each term in the sum separately. The typical term

$$(2.10) \quad \int_{\mathbf{M}^n} \nu^{(n)}(M_A(Q_A)) d\nu(M)$$

may be evaluated by Theorem 1.1. We have

$$M_A(Q_A) = C(\{M^i : i \in A\} \cup \{e_j : j \in \mathbf{N}-A\})$$

where e_j is the j^{th} column of the identity matrix. Moreover, from the definition of the measure ν we see we may take the M^i to be unit vectors uniformly and independently distributed in S^{n-1} . Since the volume of the cone spanned by the standard basis vectors e_j , $j \in \mathbf{N}-A$,

is $2^{-(n-\ell)}$ where $\ell = \#(A)$, Theorem 1.1 tells us that the value of the integral (2.10) is $2^{-\ell}2^{-(n-\ell)} = 2^{-n}$. Summation over all orthants then establishes formula (2.9).

Example. We illustrate formula (2.9) in two dimensions. We parametrize the class of the LCP defined by M by the angular variables θ_1 and θ_2 , for $-\pi \leq \theta_i \leq \pi$, where θ_1 is the angle, measured counter clockwise, between e_1 and M^2 , while θ_2 is the angle, measured clockwise, between e_2 and M^1 . See Figure 1.

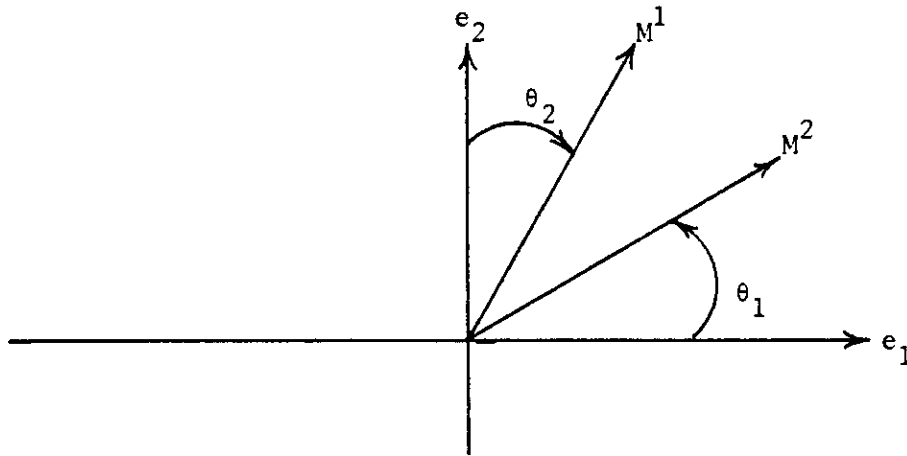


FIGURE 2.1

The value of expression (2.5) in various parts of the (θ_1, θ_2) rectangle is plotted in Figure 2. The reader may verify for himself that the average value is 1.

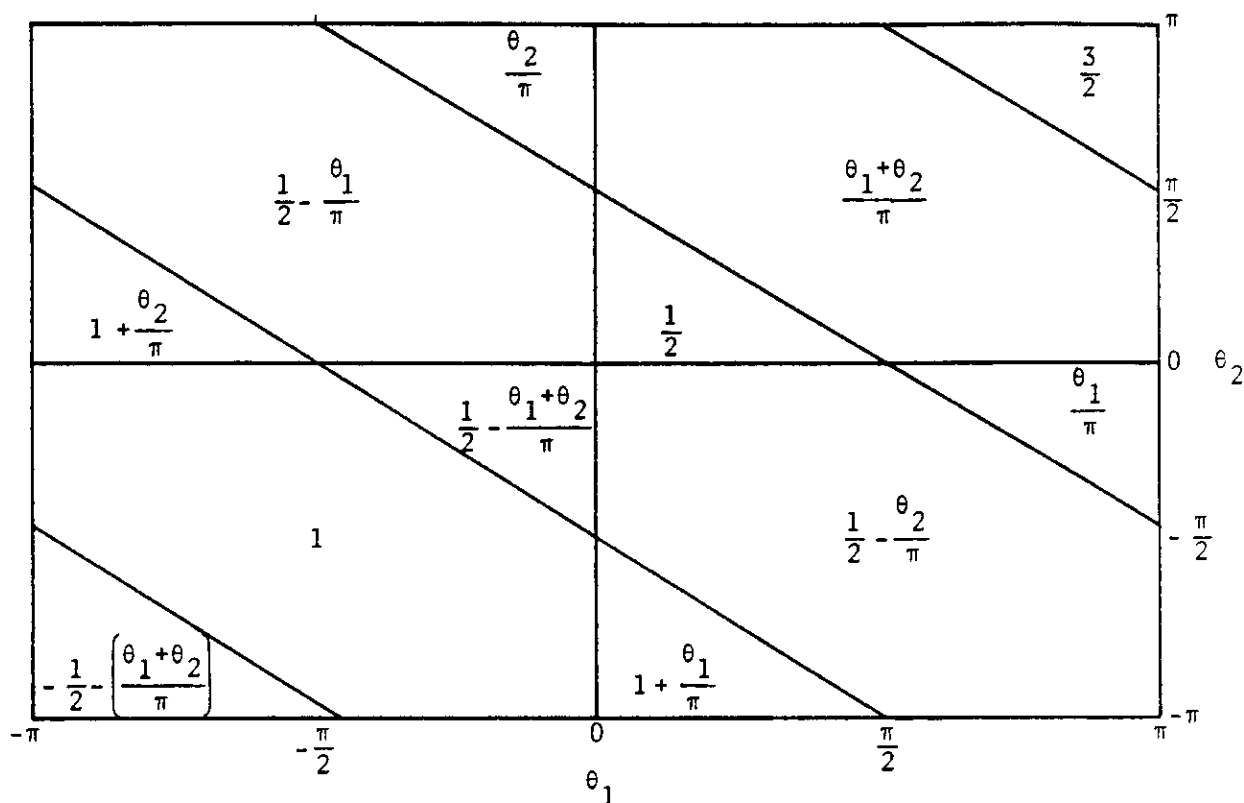


FIGURE 2.2

Theorem 2.1 allows us to partially countervail the somewhat pessimistic results of [HS] and [H] concerning the complexity of the LCP. To explain this, we first briefly review the relevant facts from [HS] and [H].

Let T_M be the piecewise linear map constructed by recipe (2.3) from the $n \times n$ matrix M . As explained in [HS], under the non-degeneracy assumption

(ND) For each subset A , the matrix M_A of recipe (2.2) is non-singular

(which assumption holds for a subset of M of full measure), the map

$T_M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ restricts to define a map

$$T_M : \mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^n - \{0\} .$$

This allows one to define a certain integer, the degree of T_M as a map

of $\mathbb{R}^n - \{0\}$ to itself. This is written $\deg T_M$. From standard results in topology one knows that

$$(2.11) \quad \#(T_M^{-1}(z)) \geq |\deg T_M| \quad \text{for all } z \in \mathbb{R}^n - \{0\} .$$

Also in [HS] it is shown by example that $\deg T_M$ can grow exponentially in n . Furthermore, in [H] it is shown that the inequality (2.11) can be strict for all z . Precisely, call a matrix M k-fold superfluous if

$$(2.12) \quad \#(T_M^{-1}(z)) \geq |\deg T_M| + 2k \quad \text{for all } z \in \mathbb{R}^n - \{0\} .$$

In [H] it is shown, again by example, that the foldness-of-superfluity of a matrix can grow exponentially in n .

Theorem 2.1, on the other hand, allows us give a quantitative bound on this type of unseemly behavior. Let $M(d,k)$ be the set of $n \times n$ matrices M satisfying (ND) and such that $\deg T_M = d$ and M is (exactly) k -fold superfluous. Clearly $M(d,k)$ is a multi-conical set, and so the multi-conical volume $v(M(d,k))$ is defined. If we combine formula (2.9) with inequality (2.12) we obtain the following result.

Corollary 2.3. With the sets $M(d,k)$ as just defined, we have

$$(2.13) \quad \sum_{d,k} (|d| + 2k)v(M(d,k)) < 1 .$$

In particular, Corollary 2.3 shows that the sets of those matrices for which d or k is very large are very small.

Corollary 2.3 also has some bearing on the question of Q -matrices (cf. [C]), those matrices M for which T_M is surjective, or

equivalently, for which $\#(T_M^{-1}(z)) \geq 1$ for all z . Observe that $M(0,0)$ is precisely the set of non-Q-matrices which also satisfy (ND). Also $M(\pm 1,0)$ are the sets of degree ± 1 matrices for which $\#(T_M^{-1}(z)) = 1$ for at least 1 non-zero z . Since the sets $M(d,k)$ form a partition of all matrices satisfying (ND), we clearly have

$$(2.14) \quad \sum_{d,k} v(M(d,k)) = 1 .$$

Combining equations (2.14) with inequality (2.13) we find

Corollary 2.4. The sets $M(0,0)$ and $M(\pm 1,0)$ are relatively large in the sense that

$$(2.15) \quad \begin{aligned} \text{i)} \quad v(M(0,0)) &> \sum_{|d|+2k \geq 2} v(M(d,k)) \\ \text{ii)} \quad v(M(0,0)) + v(M(1,0)) + v(M(-1,0)) &\geq \frac{1}{2} . \end{aligned}$$

We turn now to the second application of Theorem 1.1 to the LCP. This is an expression for Smale's estimate [S] for the average number of pivot steps to termination of Lemke's algorithm [S], [CD] for the LCP.

We review Smale's Main Formula. As above, the vectors e_i are the columns of the identity matrix and the vectors M^i are the columns of the matrix M . We also define the vector

$$E = (1, 1, \dots, 1) .$$

Recall that N is the set of the first n positive integers.

Smale's Formula [S]: The average over z of the number of pivot steps to termination of Lemke's algorithm for $LCP(M,z)$ is bounded by

$$(2.16) \quad S(M) = \sum_{A,B} v^{(n)}(C(-E, \{e_i : i \in A\}, \{M^j : j \in B\}))$$

where the sum is over sets $A, B \subseteq N$ with $A \cap B = \phi$ and $\#(A) + \#(B) = n-1$.

We observe that each term in the sum for $S(M)$ is invariant if M is replaced by MD where D is a positive diagonal matrix. Therefore it makes sense to integrate the sum in equation (2.16) with respect to the multi-conical measure v defined above in order to obtain an average value for $S(M)$. Theorem 1.1 can be used to evaluate the resulting integrals. We find

$$(2.17) \quad \int_{M^n} S(M) dv(M) = \sum_{k=1}^n 2^{-(n-k)} \binom{n-k+1}{\#(A)=k-1} \sum v^{(k)}(C(-E, \{e_i : i \in A\})) .$$

The volumes of the k -dimensional cones $C(-E, \{e_i : i \in A\})$ are clearly independent of which set $A \subseteq N$ of cardinality $k-1$ is involved, and is thus a function only of n and k . Let us denote this function by $v(n,k)$. Then we may further simplify (2.17) to

$$(2.18) \quad \int_{M^n} S(M) dv(M) = 2^n \sum_{j=0}^{n-1} \binom{n-1}{j} 2^{-(n-j)} v(n,j+1) .$$

Thus the behavior for large n of Smale's estimate depends on the behavior of the $v(n,k)$. Although the cones whose volumes are the $v(n,k)$ are, as cones go, quite simple minded, the precise values for the $v(n,k)$ seem to be hard to come by. One can compute that for $n = 2$ the value of quantity (2.18) is $5/4$, while for $n = 3$, it is $2 + \frac{3}{2\pi} \arcsin(3/\sqrt{3})$, which is already transcendental. Thus one probably will have to be satisfied with estimates on the $v(n,k)$, and on the average of $S(M)$. Since the coefficients of the $v(n,j+1)$ on the right hand side of (2.18) are very large (they grow exponentially in n), the dependence of this quantity on the $v(n,j+1)$ is quite delicate. We observe that

$$(2.19) \quad \text{i) } v(n,n) = \frac{1}{n}(1 - 2^{-n})$$

ii) $v(n,k)$ is decreasing in n for $n \geq k$

iii) $v(n,k) \underset{\sim}{\sim} 2^{-k}$ for fixed k and large n .

If $v(n,k)$ decreases to 2^{-k} quite rapidly as $n \rightarrow \infty$, then either side of formula (2.18) will grow slowly with n , but if the $v(n,k)$ decrease slowly as n increases, then the quantity in (2.18) could grow exponentially with n .

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