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FAILURE OF THE ALTERNATION THEOREM IN RATIONAL

APPROXIMATION OVER  $C_0[-\infty, \infty]$

by

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IN RATIONAL APPROXIMATION OVER  $C_0[-\infty, \infty]$

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Let  $C_0[-\infty, \infty]$  denote the class of all continuous real valued positive functions on  $(-\infty, \infty)$  that vanish at  $\pm\infty$ . Define the class of rational approximants

$$R_{nm}(x) = s(x) \frac{b_0 x^n + b_1 x^{n-1} + \dots + b_n}{a_0 x^m + a_1 x^{m-1} + \dots + a_m} = s(x) \frac{B(x)}{A(x)}$$

where  $s(x)$  is a given function in  $C_0[-\infty, \infty]$ ,  $m$  and  $n$  are even integers and  $A(x) > 0$  for  $x \in (-\infty, \infty)$ . We wish to approximate functions  $f \in C_0[-\infty, \infty]$  by rational fractions of the form  $R_{nm}(x)$  using the Chebyshev norm  $\|f\| = \sup_{x \in (-\infty, \infty)} |f(x)|$ .

A practical example of Chebyshev approximation in this context arises when  $f$  represents a probability density supported on the entire real axis. Then  $s$  may be a primitive function belonging to the same class as  $f$  or simply a convenient density such as the normal

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$(2\pi)^{-1/2} \exp\{-x^2/2\}$  . Some recent applications in this setting have been studied by the author elsewhere [4] and some early numerical approximations to the normal density in a related setting were given by Hastings [3].

The classical theory of Chebyshev approximation does not strictly apply in this context. In particular, Achieser [1] requires that  $s(x)$  behave like the reciprocal of a polynomial of even degree  $k = n-m$  as  $|x| \rightarrow \infty$  . This requirement can be conveniently relaxed in the proofs of existence and uniqueness, which go through with only minor modifications. But the requirement is essential for the validity of the alternation theorem characterizing the best approximant in [1]. This is confirmed by the following example which shows that in cases of superior degeneracy in the denominator the standard alternation condition is no longer necessary. A similar problem was discovered by Brink and Taylor [2] in the context of Chebyshev approximation of oscillating decay functions by reciprocals of polynomials over  $[0, \infty)$  .

Let  $f = [\pi(1+x^2)]^{-1} + t(x)$  for  $x \in (-\infty, \infty)$  where  $t(x)$  is a piecewise linear and continuous function which produces spikes in  $f$  at  $x = -1, 0, +1$  . More precisely, define  $t(x) = (h/\eta)(x+1+\eta)$  for  $x \in [-1-\eta, -1]$  ,  $h - (h/\eta)(x+1)$  for  $x \in [-1, -1+\eta]$  ,  $-(h/\epsilon)(x+\epsilon)$  for  $x \in [-\epsilon, 0]$  ,  $-h + (h/\epsilon)x$  for  $x \in [0, \epsilon]$  ,  $(h/\eta)(x-1+\eta)$  for  $x \in [1-\eta, 1]$  ,  $h - (h/\eta)(x-1)$  for  $x \in [1, 1+\eta]$  and  $t(x) = 0$  elsewhere on  $(-\infty, \infty)$  . In this definition  $h$  and  $\eta$  are taken to be small positive numbers with  $h < \pi^{-1}$  and  $\eta = \epsilon/2$  so that  $f \in C_0[-\infty, \infty]$  and the area under  $f$  is normalized at unity. Then  $f$  represents a Cauchy density with minor perturbations in the locality of the points  $\{-1, 0, +1\}$  . Now consider rational approximants to  $f$  in the class  $R_{nm}(x)$  with

$s(x) = [\pi(1+x^2)]^{-1}$ . Take  $n = 0$ ,  $m = 2$  and first consider

$$R_{02}(x) = \left[ \frac{1}{\pi(1+x^2)} \right] \frac{b_0}{1+a_0x^2}, \quad x \in (-\infty, \infty).$$

The best approximant to  $f$  in  $R_{02}(x)$  is obtained with  $b_0 = 1$ ,  $a_0 = 0$ , giving the same best approximant as for the class  $R_{00}(x)$ . The error on the approximant is  $e(x) = t(x)$ ,  $\|e(x)\| = h$ , and an alternant occurs at the point  $\{-1, 0, +1\}$ . To see that  $R_{00}(x) = s(x)$  is indeed the best approximant we observe that the error can be reduced at  $x = 0$  only by setting  $b_0 < 1$ . Since we need to approximate  $f$  over  $[-\infty, \infty]$  we also require  $a_0 \geq 0$  so that reducing the error at  $x = 0$  must always increase the error at  $x = \pm 1$ . The same result holds for the general member of the class in which  $n = 0$ ,  $m = 2$ :

$$R_{02}(x) = \left[ \frac{1}{\pi(1+x^2)} \right] \frac{b_0}{1+a_1x+a_0x^2}, \quad x \in (-\infty, \infty).$$

In order to reduce the error on the approximant  $R_{00}(x) = s(x)$  we need to set  $b_0 < 1$ . As before  $a_0$  must be non-negative and this means that the modulus of the error on  $R_{02}(x)$  must be greater than  $h$  at either  $-1$  or  $+1$  depending on the sign of  $a_1$ . Note that the number of alternations here is 3 which is less than the number, viz.  $n+m+2-d = 4$  (where the "defect"  $d = 0$ ), than would have been required if the classical theory [1] applied. Some further modification of the example shows that, in fact, less than 3 alternations are a required characteristic of the best approximant. We may, for instance, redefine  $f$  by setting  $t(x) = 0$  over  $x \in [-\infty, -\epsilon)$ , maintaining its earlier definition elsewhere. We find that  $R_{00}(x) = s(x)$  is still the best approximant in the class

$R_{02}(x)$  and the number of alternations is now reduced to 2.

In both cases, the difficulty in the application of classical theory is removed if we confine our attention to a finite interval of approximation (which relaxes the sign requirement on the coefficient  $a_0$ ) or if we utilize a coefficient function for which  $s(x)x^{n-m}$  has a finite non zero limit (which allows  $\infty$  to become a point of alternation). Each of these alternatives has shortcomings in problems of density function approximation, where we are often required to consider infinite intervals as the proper domain of approximation and where the most interesting candidates for  $s(x)$  will usually themselves be in  $C_0[-\infty, \infty]$ .

Failure of the classical theory of alternation arises in the above examples when the denominator of the best approximant is the more degenerate. The proof in [1] pp. 55-56 shows that this is not possible when the requirement that  $s(x)x^{n-m}$  tend to a finite non-zero limit at infinity is imposed. In other cases (viz. non degeneracy or when the numerator is the more degenerate in the best approximant) the standard alternation result and, indeed, the proof in [1] still apply. When the denominator is the more degenerate the construction in the proof of [1] needs modification. Thus, working in the notation of [1] on pp. 55-56, we may find polynomials  $\phi(x)$  and  $\psi(x)$  of degrees  $m-\mu+[v]_e$  and  $n-\nu+[v]_e$  respectively, where  $[v]_e$  denotes the largest even integer less than or equal to  $\nu$ , for which  $\phi(x) = A(x)\psi(x) - B(x)\phi(x)$ . The remainder of the proof now holds with  $\Omega(x)$  selected to be a positive polynomial of degree  $[v]_e$  and  $N' \leq n+m-\mu-\nu+[v]_e+1$ . This proves the following necessary condition in the case of a best approximant whose denominator is the more defective: *the number of consecutive points in the interval  $[-\infty, \infty]$  at which the error takes on its maximum value with alternate changes of sign is at least  $n+m-\mu-\nu+[v]_e+2$* . The author has not

been able to prove that this condition is sufficient in the denominator more defective case or to find a counterexample.

#### REFERENCES

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