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STOCHASTIC GAMES II: THE MINMAX THEOREM

by

Curt Alfred Monash

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1. INTRODUCTION

A two-person, zero-sum stochastic game consists of a (finite) set S of states; each state S is a (finite) matrix game. The entries of these matrices consist of

- 1) a payoff (from the column-chooser, B , to the row-chooser, A) and
- 2) a lottery on S , determining which state will be played next.

Shapley [1953] introduced this concept, studying stochastic games which terminate with probability 1 after finitely many steps; equivalently, these games could be thought of as infinite in duration, but with a non-zero discount rate. In this case the min-max theorem is straightforward (Shapley [1953], Monash [1979, 1981]). Gillette [1957] studied stochastic games with zero stop probabilities, establishing the min-max theorem in a couple of special cases. In these cases, the optimal strategies are stationary (i.e., dependent only upon the current state, rather than the history); thus the game "should" go into a Markov chain. The payoff can be defined either as the Cesaro limit $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N d_i$ or the Abel limit $\lim_{r \rightarrow 0} r \sum_{i=1}^{\infty} d_i (1-r)^{i-1}$, where d_i = the payoff on the i^{th} play, since, with best play, these limits exist and are equal (compare Royden [1963]).

In The Big Match, Blackwell and Ferguson [1968] considered a more difficult example. Although this game still has a value, it cannot be guaranteed by stationary strategies; furthermore, no strategy is better than ϵ -optimal. Extending these methods, Bewley and Kohlberg [1976] showed that the Cesaro limit of the values of the N -stage games exists, and equals the Abel limit of the values of the r -discounted games;

furthermore, no strategy for either player can guarantee an average pay-off (in any sense) better than this number v_∞ . Thus v_∞ is the only candidate for min-max value. Finally, the min-max theorem for stochastic games was proved by Monash [1979] and independently by Mertens and Neyman [1980]. This paper is a revision of Monash [1979].

2. DEFINITIONS

Without loss of generality, a stochastic game can be described by finite sets S , A , B , C and measurable functions

$$d : S \times B \times B \times C \rightarrow [-\bar{M}, \bar{M}] ,$$

$$s : S \times A \times B \times C \rightarrow S , \text{ and}$$

$$q : [0,1] \rightarrow C$$

such that:

- 1) S is the state space;
- 2) Player A (resp. B) chooses a move from his choice set A (resp. B);
- 3) s , composed with q , reproduces the lottery in each entry of each state matrix; and
- 4) d is the payoff function.

A state $s^* \in S$ is absorbing if $s(s^*, a, b, c) = s^*$ for all a , b , c and $d(s^*, a, b, c) = v(s^*)$, a constant. $S^* \subset S$ is the set of absorbing states $S_\infty = S - S^*$.

A play of the game is just a sequence $s_0, a_1, b_1, c_1, s_1, a_2, b_2, c_2, s_2, \dots$, where $s_i = s(s_{i-1}, a_i, b_i, c_i)$, for all i ; let $d_i = d(s_{i-1}, a_i, b_i, c_i)$, the payoff on the i^{th} turn. Writing $t_i = (s_{i-1}, a_i, b_i, c_i) \in T = S \times A \times B \times C$, we denote a play by

$t = (t_1, t_2, t_3, \dots)$; thus

$$\begin{aligned} T^\infty &= S \times A \times B \times C \times S \times A \times B \times C \times S \times \dots \\ &= \{\text{all possible plays}\} . \end{aligned}$$

The subsequence (t_1, \dots, t_n) is denoted by $t(n)$; we use this notation even if we are thinking of this subsequence as belonging to many different possible plays.

Strategies for A will always be denoted by σ , and strategies for B by τ . These strategies will always be of the form

$$\text{Prob}(a \in A \text{ (resp. } b \in B) \text{ on turn } k) = \text{function}(t_1, \dots, t_{k-1}) .$$

Thus, by the Kolmogorov Extension Theorem (see Kolmogorov [1950] or Monash [1981]), a pair (σ, τ) determines a probability measure $\mu(\sigma, \tau)$ on T^∞ . Unless otherwise noted, all expectations below are with respect to this measure. Let

$$T^* = \{t \in T : s_i \in S^* \text{ for some } i\} ,$$

and T_∞ the complement. In the next section we write $P(*) = \mu(T^*)$.

Following Bewley and Kohlberg [1976] or Monash [1981], recall that for all $s \in S$, for all $r \in (0,1)$, $V_s(r)$ = the value of the r -discount game, starting in s , satisfies

$$V_s(r) = \text{val}(\exp(d(s,a,b,c) + (1-r) \int_{s \in S} P(\bar{s}) V_{\bar{s}}(r)) , \quad (2.1s)$$

where $P(\bar{s})$ = the probability that $d(s,a,b,c) = \bar{s}$, and val is the ordinary min-max value. For some $\tilde{r} > 0$, all the $V_s(r)$ are algebraic, as are the optimal strategies in the games (2.1s). Thus, on $(0, \tilde{r})$,

$$\begin{aligned}
 V_s(r) &= V_\infty(s) \cdot r^{-1} + () r^{-1+\frac{1}{n}} + \dots \\
 &= V_\infty(s) \cdot u^{-n} + () u^{-n+1} + \dots,
 \end{aligned}$$

where $u = r^{1/n}$. Let $0 \leq \tilde{u} \leq \tilde{r}^{1/n}$; on $(0, \tilde{u})$, we write $W_s(u) = V_s(u^n)$, so that $\lim_{u \rightarrow 0^+} u^n W_s(u) = \lim_{r \rightarrow 0^+} r V_s(r) = v_\infty(s)$.

In Sections 4 through 6, we assume $v_\infty(s) = 0$ for all $s \in S_\infty$.

In that case we have $\lim_{u \rightarrow 0^+} u^{n-1} W_s(u) < \infty$ for all s ; thus, writing

$$\bar{W}(u) = \max_{s \in S_\infty} |W_s(u)|, \text{ we have } \lim_{u \rightarrow 0^+} u^{n-1} \bar{W}(u) < \infty, \text{ also.}$$

3. STATEMENT OF THEOREM

Our main result is

Theorem I: For any starting state $s_0 \in S$, for any $\epsilon > 0$, there exists a strategy σ for A such that, for any strategy τ for B,

$$\liminf_{N \rightarrow \infty} \exp \left(\frac{1}{N} \sum_{i=1}^N d_i \right) > v_\infty(s) - \epsilon.$$

Theorem I clearly follows from the following two propositions:

Proposition 3.1: Suppose, for all $s \notin S^*$, $v_\infty(s) = 0$. Then the conclusion of Theorem I holds.

Proposition 3.2: Proposition 3.1 \implies Theorem I.

In this section, we prove Proposition 3.2; the remainder of the paper is devoted to Proposition 3.1.

The proof of Proposition 3.2 depends upon

Lemma 3.3: Let G be a stochastic game, with state set S . Let H be another stochastic game, identical to G except for the following modification: Replace a single state $x \in S$ by an absorbing state y such that $v(y) = v_\infty(x)$. Then, for all $s \in S$,

$$v_{\infty, H}(s) = v_{\infty, G}(s) ,$$

where $v_{\infty, G}(s)$ (resp. $v_{\infty, H}(s)$) is simply $v_\infty(s)$ in the game G (resp. H).

Proof: Let $V_{G,s}(r)$ (resp. $V_{H,s}(r)$) be $V_s(r)$ in the game (resp. H). Define

$$\hat{V}(r) = r^{-1} \cdot v_\infty(x) - V_{G,x}(r)$$

$$\bar{V}(r) = \min_{s \in S - \{x\}} (V_{H,s}(r) - V_{G,s}(r)) .$$

Then, for any $s \in S - \{x\}$, (2.1s) gives

$$\begin{aligned} V_{H,s}(r) &= \text{val}_C(\text{Exp}(d(s,a,b,c))) + (1-r) \sum_{\bar{s} \in S - \{y\}} P(\bar{s}) \cdot V_{H,\bar{s}}(r) \\ &\quad + r^{-1} \cdot (1-r)P(y) \cdot v(y) \end{aligned}$$

$$\begin{aligned} &\geq \text{val}_C(\text{Exp}(d(s,a,b,c))) + (1-r) \sum_{\bar{s} \in S} P(\bar{s}) \cdot V_{G,\bar{s}}(r) \\ &\quad + (1-r) \min_{P \in [0,1]} ((1-P) \cdot \bar{V}(r) + P \cdot \hat{V}(r)) , \end{aligned}$$

where P corresponds to $P(x : a, b, s)$,

$$= V_{G,s}(r) + (1-r)((1-P^*) \cdot \bar{V}(r) + P^* \cdot \hat{V}(r)) ,$$

for some $P^* \in [0,1]$.

Picking \tilde{s} now so that

$$V_{H,s}^{\vee}(r) - V_{G,s}^{\vee}(r) = \bar{V}(r) \quad \text{in some interval } [0, \tilde{r}) ,$$

$$\begin{aligned} \bar{V}(r) &= V_{H,s}^{\vee}(r) - V_{G,s}^{\vee}(r) \\ &\geq (1-r)((1-P^*)\bar{V}(r) + P^*\check{V}(r)) \end{aligned}$$

$$r\bar{V}(r) \geq (1-r) \cdot P^* \cdot (\check{V}(r) - \bar{V}(r)) .$$

So either $\bar{V}(r) \geq 0$, or $\check{V}(r) - \bar{V}(r) < 0$. In either case,

$$\begin{aligned} &\min_{s \in S - \{x\}} (v_{\infty, H}(s) - v_{\infty, G}(s)) \\ &= \min_{s \in S - \{x\}} \lim_{r \rightarrow 0^+} (rV_{H,s}^{\vee}(r) - rV_{G,s}^{\vee}(r)) \\ &= \lim_{r \rightarrow 0^+} r \min_{s \in S - \{x\}} (V_{H,s}^{\vee}(r) - V_{G,s}^{\vee}(r)) \\ &= \lim_{r \rightarrow 0^+} r\bar{V}(r) \\ &\geq 0 \quad \text{or} \quad \geq \lim_{r \rightarrow 0^+} r\check{V}(r) = 0 . \end{aligned}$$

So, for all $s \in S - \{x\}$, $v_{\infty, H}(s) - v_{\infty, G}(s) \geq 0$; that is,

$$v_{\infty, H}(s) \geq v_{\infty, G}(s) .$$

But, by symmetry (i.e., interchanging the names A and B),

$$-v_{\infty, H}(s) \geq -v_{\infty, G}(s) .$$

Hence $v_{\infty, H}(s) = v_{\infty, G}(s)$, for all $s \in S - \{x\}$.

Since Lemma 3.3 is clearly true for state x , we are done.

□

Proof of Proposition 3.2:

We now proceed by induction on $|S_\infty|$, the number of non-absorbing states.

$|S_\infty| = 0$. Trivially true.

So assume for $|S_\infty| - 1$, and prove for $|S_\infty|$.

To every state s , associate a number $\alpha(s)$ such that

$$v_\infty(s) - \alpha(s) = \sup_{\sigma} \inf_{\tau} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \exp(d_i) .$$

By the Bewley-Kohlberg result, $\alpha(s) \geq 0$ for all $s \in S$.

Want to show: $\alpha(s) = 0$ for all s . If so, done.

So suppose otherwise.

Definition: We will call a strategy σ , starting in state s , ϵ -optimal (for s) if

$$\inf_{\tau} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \exp(d_i) \geq v_\infty(s) - \epsilon .$$

Case 1: There exist states s_1, s_2 such that $\alpha(s_1) > \alpha(s_2) \geq 0$.

Let $\epsilon = \frac{1}{3}(\alpha(s_1) - \alpha(s_2))$.

Consider the modified game H , where s_2 is replaced by an absorbing state y such that $v(y) = v_\infty(s_2)$. Then H , by induction, has an ϵ -optimal strategy, for any initial state.

Consider, then, the following strategy, for the game G starting in state s_1 : Play the ϵ -optimal strategy for H , until "absorbed" in "y"; this is meaningful because G and H are identical outside

of state s_2 . Once in s_2 , play an $(\alpha(s_2) + \epsilon)$ -optimal strategy, which exists by the definition of $\alpha(s_2)$. Then this strategy is clearly

$$(\alpha(s_2) + 2\epsilon)\text{-optimal for } s_1 .$$

But

$$\alpha(s_2) + 2\epsilon = \frac{2}{3}\alpha(s_1) + \frac{1}{3}\alpha(s_2) < \alpha(s_1) ,$$

contradicting the definition of $\alpha(s_1)$.

Hence the only possibility is:

Case 2: There exists $\bar{\alpha} > 0$ such that, for all $s \in S_\infty$, $\alpha(s) = \bar{\alpha}$.

Now, let $v_0 = \min_{s \in S_\infty} v_\infty(s)$.

Let $S_0 \subseteq S_\infty$ be $\{s \in S_\infty : v_\infty(s) = v_0\}$; let \hat{S} be the complement.

Case 2a: \hat{S} is non-empty. Then let $v_1 = \min_{s \in \hat{S}} (v_\infty(s))$; $v_1 > v_0$. Let

$$\beta = \frac{v_1 - v_0}{2(v_1 + M)} \leq \frac{1}{2} .$$

By repeated applications of Lemma 3.3, replace the states in \hat{S} by absorbing states with the same v_∞ . Then the states in S_0 still have value v_0 . Assuming Proposition 3.1, this new game has an ϵ -optimal strategy, where

$$\epsilon = \min \left\{ \frac{v_1 - v_0}{4}, \frac{\beta \bar{\alpha}}{4} \right\} .$$

Play this strategy until "absorbed," and an $(\bar{\alpha} + \epsilon)$ -optimal strategy thereafter (unless the "absorption" is genuine). Fixing (any) τ , we have two cases:

Case 1: Expected value if "absorbed" $\geq \frac{v_0 + v_1}{2}$ or $P(*) = 0$. Then

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \exp(d_i) &\geq P(*) \left[\frac{v_0 + v_1}{2} - (\bar{\alpha} + \epsilon) \right] + (1 - P(*))(v_0 - \epsilon) \\ &= v_0 - \bar{\alpha} + P(*) \left[\frac{v_1 - v_0}{2} - \epsilon \right] + (1 - P*)(\bar{\alpha} - \epsilon) \\ &\geq v_0 - \bar{\alpha} + \min \left[\frac{v_1 - v_0}{4}, \frac{7\bar{\alpha}}{8} \right]; \end{aligned}$$

since τ was arbitrary, this contradicts the definition of $\bar{\alpha}$.

Case 2: Expected value if "absorbed" $< \frac{v_0 + v_1}{2}$ and $P(*) > 0$.

Let $\gamma = \frac{\text{prob}(\text{genuine absorption})}{P(*)}$. Then

$$\begin{aligned} \frac{v_0 + v_1}{2} &> \text{Expected value if "absorbed"} \\ &\geq \gamma(-M) + (1 - \gamma) \cdot v_1; \quad \text{i.e.,} \end{aligned}$$

$$v_1 - \frac{v_1 - v_0}{2} > v_1 - \gamma(v_1 + M)$$

$$\gamma > \frac{(v_1 - v_0)}{2(v_1 + M)} = \beta.$$

Hence

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \exp(d_i) &\geq v_0 - \epsilon - P(*) (1 - \gamma) (\bar{\alpha} + \epsilon) \\ &\geq v_0 - \epsilon - (1 - \beta) (\bar{\alpha} + \epsilon) \\ &> v_0 - \bar{\alpha} + (\beta \bar{\alpha} - 2\epsilon); \end{aligned}$$

since $\epsilon \leq \frac{\beta \bar{\alpha}}{4}$, this is a contradiction.

Case 2b: $\hat{S} = \phi$.

Deducting v_0 from all payoffs, this is exactly the case of Proposition 3.1. Hence there exists an $\frac{\alpha}{2}$ -optimal strategy, for our final contradiction.

So $\alpha(s) = 0$ for all $s \in S^*$.

But this is exactly what we wanted to prove.

□

4. PRELIMINARY COMPUTATIONS

For the rest of this paper, we will assume $v_\infty(s) = 0$ for all $s \in S_\infty$. We will always choose A's strategy σ to be $\text{Prob}(a) = f(a,s,u)$ = the optimal (stationary) strategy in the u^n -discount game, for some u , in the current state s . Without loss of generality (see Monash [1979] or [1981]), B's strategy τ is pure: $b_k = \text{function}(t(k-1))$.

Let us now focus on one move of the game. Fix $s \in S_\infty$, $u \in (0, \hat{u})$, and $b \in B$, with A playing strategy $\{f(a,s,u)\}$. Let $P_*(u) = \text{Prob}(\delta(s,a,b,c) \in S^*)$, given the probability distributions $f(a,s,u)$ on A and q on C. In Sections 5 and 6, if a play t is understood along with a sequence of u 's, we will let

$$P_*(i) = \begin{cases} P_*(u) & \text{on the } i^{\text{th}} \text{ turn, if } s_{i-1} \in S_\infty \\ 0 & \text{, if } s_{i-1} \in S^* . \end{cases}$$

Meanwhile, let $\bar{s} = \delta(s,a,b,c)$.

We distinguish three cases:

- 1) $P_*(u) \equiv 0$ on $(0, \hat{u})$;
- 2) Not 1, and

$$\lim_{u \rightarrow 0^+} \exp(v_\infty(\bar{s}) : \bar{s} \in S^*) = 0 ;$$

3) Not 1 or 2.

We further distinguish between:

A. Either Case 1, or $\text{order}(P_*(u)) \geq n$;

B. Not Case 1, and $\text{order}(P_*(u)) \leq n-1$.

Observe that $P_*(u)$ is a rational function of u , and thus has finitely many zeroes; without loss of generality, none of them occur on $(0, \hat{u})$. Define $\delta(u)$ as follows (where we suppress the dependence upon s and b):

If Case A, then

$$\delta(u) = -\exp(v_\infty(\bar{s}) : \bar{s} \in S^*) \cdot P_*(u) \cdot u^{-n} ;$$

if Case B, then

$$\delta(u) = -\exp(v_\infty(\bar{s}) : \bar{s} \in S^*) \cdot P_*(u) + (1 - (1-u)^n)(1-P_*(u))W_S(u) \cdot u^{-n} .$$

The point of this definition may be found in the following propositions

(where we write $\exp(d : S_\infty)$ for $\exp(d : \bar{s} \in S_\infty)$, and so forth):

Proposition 4.1: $\exp(d : S_\infty)(1 - P_*(u)) \geq \delta(u) - \exp(W_S^- - W_S(u) : S_\infty)(1 - P_*(u)) + \eta(u)$, for $u \in (0, \hat{u})$, where $\lim_{u \rightarrow 0^+} \eta(u) = 0$.

and

Proposition 4.2:

1. If Case 1 (above), then $\delta(u) = 0$ and $P_*(u) \cdot \exp(v_\infty(\bar{s}) : S^*) = 0$;
2. If Case 2, then

$$|\exp(v_\infty(\bar{s}) : S^*)| < o(u^0)$$

and

$$\left| \frac{\delta(u) \cdot u^n}{P_*(u)} \right| < o(u^0) .$$

3. If Case 3,

$$\frac{-P_*(u) \cdot \exp(v_\infty(\bar{s}) : S^*) \cdot u^{-n}}{\delta(u)} = 1 + o(u^0) .$$

From Equation (2.1s), we have

$$\begin{aligned} W_S(u) \leq & (1 - P_*(u)) \cdot (\exp(d : S_\infty) + (1 - u^n) \cdot \exp(W_S^-(u) : S_\infty)) \\ & + P_*(u) \cdot \exp(v_\infty(\bar{s}) : S^*) u^{-n} . \end{aligned} \quad (4.3)$$

Proof of Proposition 4.1:

Rearranging (4.3), we have

$$\begin{aligned} (1 - P_*(u)) \exp(d : S_\infty) & \geq (1 - (1 - P_*(u))(1 - u^n)) W_S(u) \\ & - (1 - P_*(u))(1 - u^n) \exp(W_S^-(u) - W_S(u) : S_\infty) \\ & - P_*(u) \exp(v_\infty(\bar{s}) : S^*) \end{aligned} \quad (4.4)$$

If Case A holds, then

$$\begin{aligned} (4.4) = & \delta(u) - (1 - P_*(u)) \exp(W_S^-(u) - W_S(u) : S_\infty) \\ & + (P_*(u) + u^n - u^n P_*(u)) W_S(u) + u^n (1 - P_*(u)) \\ & \cdot \exp(W_S^-(u) - W_S(u) : S_\infty) . \end{aligned} \quad (4.5)$$

If Case B. holds, then (4.4) equals

$$\begin{aligned} \delta(u) &= (1 - P_*(u)) \cdot \exp(W_{\bar{s}}(u) - W_s(u) : S_{\infty}) \\ &+ u^n (1 - P_*(u)) \cdot \exp(W_{\bar{s}}(u) - W_s(u) : S_{\infty}) . \end{aligned} \quad (4.6)$$

Let $\bar{P} > 0$ be such that $|P_*(u)| \leq \bar{P}u^n$ whenever Case A holds. Writing

$$\eta(u) = -(\bar{P}+4)u^n \bar{W}(u) ,$$

and observing that

$$(4.5) \geq \delta(u) - (1 - P_*(u)) \cdot \exp(W_{\bar{s}} - W_s(u) : S_{\infty}) + \eta(u) ,$$

$$(4.6) \geq \delta(u) - (1 - P_*(u)) \cdot \exp(W_{\bar{s}} - W_s(u) : S_{\infty}) + \eta(u)$$

and $\lim_{u \rightarrow 0^+} \eta(u) = 0$,

we are done. □

Proof of Proposition 4.2:

1. Suppose Case 1 holds: $P_*(u) \equiv 0$. Then so does Case A, and

$$\begin{aligned} \delta(u) &= -\exp(v_{\infty}(\bar{s}) : \bar{s} \in S^*) \cdot P_*(u) \cdot u^{-n} \\ &= 0 \text{ for all } u , \end{aligned}$$

and so done.

2. Suppose, then, Case 2 holds. Since $\exp(v_{\infty}(\bar{s}) : S^*)$ is a power series in u , with limit 0 as $u \rightarrow 0$, it is indeed $o(u^0)$. Now, if Case A, then

$$\left| \frac{\delta(u) \cdot u^n}{P_*(u)} \right| = |-\exp(v_\infty(\bar{s}) : S^*)|$$

$$< o(u^0) ;$$

while, if Case B, then

$$\left| \frac{\delta(u) \cdot u^n}{P_*(u)} \right| = \left| -\exp(v_\infty(\bar{s}) : S^*) + \frac{P_*(u) \cdot W_S(u) \cdot u^n}{P_*(u)} + \text{higher order terms} \right|$$

$$< o(u^0) + |u^n W_S(u)| + \text{higher order terms}$$

$$< o(u^0) .$$

3. Suppose Case 3 holds. If Case A, then

$$\frac{-P_*(u) \cdot \exp(v_\infty(\bar{s}) : S^*) u^{-n}}{\delta(u)} \equiv 1 , \text{ by definition.}$$

So suppose Case B: order $P_*(u) \leq n-1$. It is clearly enough to check

$$\left| \frac{(P_*(u) + u^n - u^n P_*(u)) W_S(u)}{-P_*(u) \cdot \exp(v_\infty(\bar{s}) : S^*) u^{-n}} \right| < o(u^0) .$$

Then order $(P_*(u) + u^n - u^n P_*(u)) = \text{order}(P_*(u))$

$$\text{order}(\exp(v_\infty(\bar{s}) : S^*)) = 0 ,$$

and so the order of the left-hand-side is

$$\geq \text{order}(P_*(u)) + \text{order}(W_S(u)) - \text{order}(P_*(u)) - 0 - \text{order}(u^{-n})$$

$$= \text{order}(u^n W_S(u))$$

$$\geq 1 .$$

Hence done.

□

5. THE ABSORBING CASE

Recall that a fixed strategy pair (σ, τ) induces a probability measure $\mu(\sigma, \tau)$ on T^∞ , the space of all possible plays. If $s_0 \in S^*$, Proposition 3.1 is trivial; thus it follows immediately from

Proposition 5.1: For any starting state $s \in S_\infty$, for any $\varepsilon > 0$, there exists a strategy σ for A such that

$$\inf_{\tau} \liminf_{N \rightarrow \infty} \int_{T^\infty} \frac{1}{N} \sum_{i=1}^N d_i d\mu(\sigma, \tau) > -(6\tilde{M} + 3)\varepsilon .$$

Proof of Proposition 5.1:

As remarked earlier, the strategy σ will be the form $\text{Prob}(a) = f(a, s, u)$, the optimal strategy in the u^n -discount game, for u cleverly chosen. Specifically, writing u_N for the u prevailing on the $N+1^{\text{st}}$ move, we set $u_N = u_0(1 - \frac{1}{2}\varepsilon)^{v(N)}$, for u_0 sufficiently small and $v(N)$ a non-negative integer depending upon the history of the first $N-1$ moves.

Write $q = 1 - \frac{1}{2}\varepsilon$. Recalling Proposition 4.2, choose $R > 0$ and \tilde{u} sufficiently small so that each $\sigma(u^0)$ is $< Ru$. Assume $\varepsilon < 1$.

Then $u_0 \in (0, \tilde{u}) \subseteq (0, 1)$ must satisfy the following four conditions:

1. For every $u \in (0, u_0]$, $\eta(0) > -\varepsilon$
2. For every $u \in (0, u_0]$, $\bar{W}(u) < \frac{\varepsilon}{4} \cdot u^{-n+\frac{1}{2}}$
3. $Ru_0 < \varepsilon$
4. $(1+\varepsilon)^3 \cdot \frac{u_0^{1/2}}{1-q^{1/2}} < \varepsilon$.

To define $v(N)$, we first define a set of benchmarks \tilde{m} on $1, 0, 1, 2, \dots$ by:

$$\tilde{m}(-1) = -\infty$$

$$\tilde{m}(0) = 0$$

$$\tilde{m}(i) = \tilde{m}(i-1) + (u_0 q^{i-1})^{-n+\frac{1}{2}}, \text{ for } i = 1, 2, 3, \dots .$$

Next, define sequences $\bar{m}_0, \bar{m}_1, \bar{m}_2, \dots$ and $\mathcal{L} = (\ell_0 = 0, \ell_1, \ell_2, \dots)$, \mathcal{L} increasing, in conjunction with the sequences u_0, u_1, u_2, \dots and $v(0), v(1), v(2), \dots$ by:

- 1) $\bar{m}_0 = 0$
- 2) $v(0) = 0$
- 3) $u_N = u_0 q^{v(N)}$ for $N = 1, 2, 3, \dots$
- 4) If $\bar{m}_{N-1} + \delta_N(u_{N-1}) > \tilde{m}(v(N-1) + 1)$, then $v(N) = v(N-1) + 1$ and $N \in \mathcal{L}$; if $\bar{m}_{N-1} + \delta_N(u_{N-1}) < \tilde{m}(v(N-1) - 1)$, then $v(N) = v(N-1) - 1$ and $N \in \mathcal{L}$; otherwise $v(N) = v(N-1)$ and $N \notin \mathcal{L}$.
- 5) If $N \notin \mathcal{L}$, then $\bar{m}_N = \bar{m}_{N-1} + \delta_N(u_{N-1})$.
- 6) If $N = \ell_i \in \mathcal{L}$, then $\bar{m}_N = \bar{m}_{N-1} + \delta_N(u_{N-1}) + W_{s_{\ell_{i-1}}}(u_{N-1}) - W_{s_{\ell_i}}(u_{N-1})$.

Fix σ as above, and any (pure) τ . Proposition 5.1 follows instantly (by redefining ε) from:

Proposition 5.2: $\lim_{N \rightarrow \infty} \int_{T^*} \frac{1}{N} \sum_{i=1}^N d_i d\mu \geq -\varepsilon$

and

Proposition 5.3: $\liminf_{N \rightarrow \infty} \int_{T_\infty} \frac{1}{N} \sum_{i=1}^N d_i d\mu > -\varepsilon$.

We now prove Proposition 5.2, deferring Proposition 5.3 to the next section.

Proof of Proposition 5.2:

Let $T_k = \{t = (t_1, t_2, \dots) \in T^* : s(t_k) \in S^* \text{ but } s(t_{k-1}) \notin S^*\}$;
 thus $T^* = T_1 \cup T_2 \cup T_3 \cup \dots$.

$$\begin{aligned} \text{So} \quad & \lim_{N \rightarrow \infty} \int_{T^*} \frac{1}{N} \sum_{i=1}^N d_i d\mu \\ &= \lim_{N \rightarrow \infty} \sum_{k=1}^{\infty} \int_{T_k} \frac{1}{N} \sum_{i=1}^N d_i d\mu \\ &= \sum_{k=1}^{\infty} \lim_{N \rightarrow \infty} \int_{T_k} \frac{1}{N} \sum_{i=1}^N d_i d\mu , \end{aligned}$$

by the Lebesgue Dominated Convergence Theorem (Royden [1963]),

$$\begin{aligned} &= \sum_{k=1}^{\infty} \int_{T_k} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N d_i d\mu \\ &= \sum_{k=1}^{\infty} \int_{T^{\infty}} P_*(k) \cdot \exp(v_{\infty}(s_t) : t \in T_k) d\mu . \end{aligned} \tag{5.4}$$

The following is a special case of Proposition 4.1 of Monash [1981]
 (identifying $Z_i^* = s^{-1}(S^*)$ for all i).

Proposition 5.5: There exists a probability measure $\tilde{\mu}$ on T such
 that, for all N , for all $f_N : T^{N-1} \rightarrow \mathbb{R}$ such that $s(t_{N-1}) \in S^*$
 implies $f_N(t_1, \dots, t_{N-1}) = 0$,

$$\int_{T^{\infty}} f_N(t^{(N-1)}) d\tilde{\mu}(H) = \int_{T^{\infty}} f_N(t^{(N-1)}) \cdot \prod_{i=1}^{N-1} (1 - P_*(i)) d\tilde{\mu}(t)$$

Now, assume temporarily,

Proposition 5.6: For all N , for all t ,

$$\sum_{k=1}^N (P_*(k) \cdot \exp(v_\infty(s_k) : t \in T_k) \cdot \prod_{i=1}^{k-1} (1 - P_*(i))) > -\epsilon$$

Let
$$f_k(t(k-1)) = \begin{cases} P_*(k) \cdot \exp(v_\infty(s) : t \in T_k) & \text{if } \delta(t_{k-1}) \notin S^* \\ 0 & \text{if } \delta(t_{k-1}) \in S^* . \end{cases}$$

Then f_k satisfies the hypothesis of Proposition 5.5. Thus, for all N ,

$$\begin{aligned} & \sum_{k=1}^N P_*(k) \cdot \exp(v_\infty(s_k) : t \in T_k) \\ &= \sum_{k=1}^N \int_{T^\infty} f_k(t(k-1)) d\mu \\ &= \sum_{k=1}^N \int_{T^\infty} f_k(t(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P^*(i)) d\tilde{\mu} \\ &= \int_{T^\infty} \sum_{k=1}^N P_*(k) \cdot \exp(v_\infty(s_k) : t \in T_k) \cdot \prod_{i=1}^{k-1} (1 - P^*(i)) d\tilde{\mu} \\ &> \int_{T^\infty} (-\epsilon) d\tilde{\mu}, \text{ by Proposition 5.6,} \\ &= -\epsilon ; \end{aligned}$$

as these are the partial sums of equation (5.4), this establishes Proposition 5.1.

So we pass to the

Proof of Proposition 5.6:

Fix ℓ and N . Recalling Proposition 4.2, we make the simplifying assumption that Cases 1 or 3 hold everywhere (for fullest detail see Monash [1979]); thus, for $k = 1, \dots, N$,

$$\begin{aligned} & \text{either } \delta(u_{k-1}) = P_*(k) \cdot \exp(v_\infty(s_k) : T_k) = 0 \\ & \text{or } \left| \frac{-P_*(k) \cdot \exp(v_\infty(s_k) : T_k) \cdot u_{k-1}^{-n}}{\delta(u_{k-1})} \right| \in (1 - Ru_{k-1}, 1 + Ru_{k-1}) \\ & \qquad \subseteq (1 - \varepsilon, 1 + \varepsilon) . \end{aligned} \tag{5.8}$$

Writing $F_k = P_*(k) \exp(v_\infty(s_k) : T_k) \cdot \prod_{i=1}^{k-1} (1 - P_*(i))$, our task is to bound $\sum_{k=1}^N F(k)$ below. We spread out this sum as the integral of a step function by defining $A(z)$ on $[0, N)$: $A(z) = F([z] + 1)$, where $[z]$ is

the usual greatest integer function. Thus $\int_0^N A(z) dz = \sum_{k=1}^N F_k$.

First, observe that, for $\ell_j \in \mathcal{L}$,

$$\frac{\bar{m}_{\ell_{j+1}} - \bar{m}_{\ell_j}}{\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})} = 1 + \frac{W_{s_{\ell_j}}(u_{\ell_j}) - W_{s_{\ell_{j+1}}}(u_{\ell_j})}{\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})} \in (1 - \varepsilon, 1 + \varepsilon) ,$$

since $|W_{s_{\ell_j}}(u_{\ell_j}) - W_{s_{\ell_{j+1}}}(u_{\ell_j})| \leq 2\bar{W}(u_{\ell_j}) < \frac{\varepsilon}{2} u_{\ell_j}^{-n+\frac{1}{2}}$,

while $|\sum_{k=\ell_j+1}^{\ell_{j+1}} \delta_k(u_{\ell_j})| \approx u_{\ell_j}^{-n+\frac{1}{2}}$, by definition.

Thus we can define a function $m(z)$ on $[0, N]$ such that

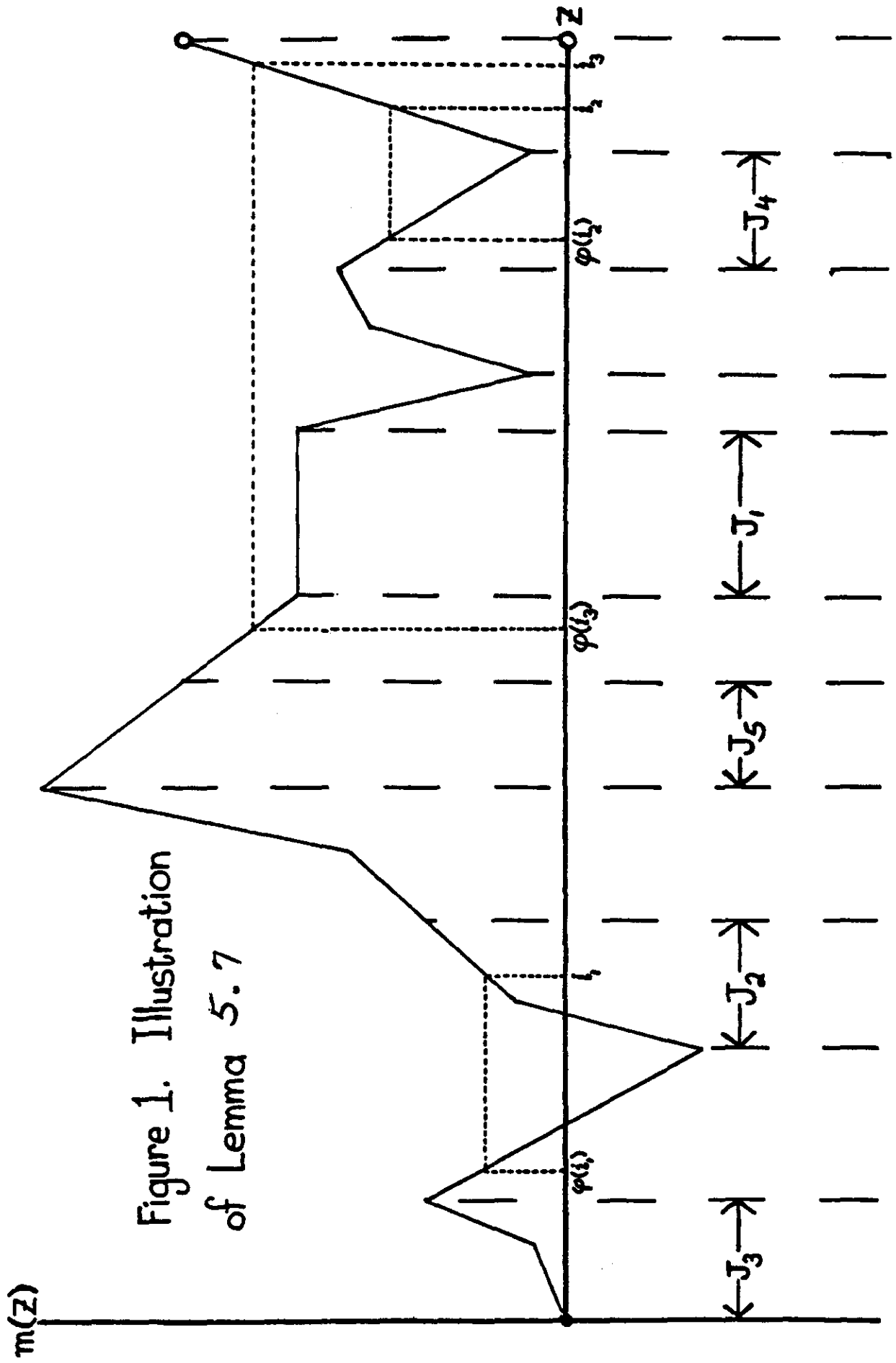


Figure 1. Illustration of Lemma 5.7

- 1) m is linear on $[k, k+1]$, for $k = 0, 1, \dots, N-1$.
- 2) $m(\ell_j) = \bar{m}_{\ell_j}$, for $\ell_j \in \mathcal{L}$.
- 3) $\frac{m(k+1) - m(k)}{\delta_{k+1}(u_k)} \in (1-\varepsilon, 1+\varepsilon)$, for $k = 0, 1, \dots, N-1$. (5.9)

We now want a finite, increasing sequence $J = \{0 = j_0, j_1, j_2, \dots, N\}$, containing every integer $0, 1, \dots, N$; J should be partitioned into five sets, with the following properties:

- 1) For $j_1 \in J_1$, m is constant on $[j_i, j_{i+1}]$.
- 2) For $j_1 \in J_2$ or J_3 , m is increasing on $[j_i, j_{i+1}]$.
- 3) For $j_1 \in J_4$ or J_5 , m is decreasing on $[j_i, j_{i+1}]$.
- 4) There exists a bijection $\phi : J_2 \rightarrow J_4$ such that if $\phi(j_h) = j_i$,
 - 1) $i < h$
 - 2) $m(j_h) = m(j_{i+1})$
 - 3) $m(j_{h+1}) = m(j_i)$.
- 5) For any $j_h, j_1 \in J_3$ with $h < i$, $m(j_h) < m(j_i)$.

Let $J_i = \bigcup_{j_h \in J_i} [j_h, j_{h+1})$, $i = 1, 2, 3, 4, 5$.

- 6) The sets J_i cover $[0, N)$.

Lemma 5.7. There exists such a sequence J .

Proof (See Figure 1):

We will prove this lemma by induction on the number H of maxima attained by m on the interval $[0, N]$ (this number is $\leq N+1$, and hence finite).

Clearly it will be enough to construct the sets J_1, \dots, J_5 .

$H = 1$: Then m is either monotonic non-increasing or monotonic non-decreasing. In the first case, let J_5 be the set on which m is

strictly decreasing, and J_1 the balance; in the second let J_3 be the set where m is strictly increasing, and J_1 the balance.

Inductive step: Assume true for H .

So suppose the maxima occur at y_1, \dots, y_{H+1} , and the minima at $(x_0), x_1, x_2, \dots, x_H, (x_{H+1})$, so that $x_{i-1} < y_i < x_i$, for $i = 1, \dots, H+1$ (x_0 or x_{H+1} may not exist). Apply induction to $[0, x_H]$ (recall that m changes direction only at integral arguments), and construct a tentative J_1, \dots, J_5 . Then, for every point j in $[x_H, y_{H+1}]$, either there exists $i \in J_5 \subset [0, x_H]$ such that $m(i) \geq m(j)$, or else not. In the first case, put j into J_2 and move i into J_4 ; in the second, put j into J_3 . Finally, put $[y_{H+1}, N)$ into J_5 .

It is clear that this is the desired partition.

□

Our result is now clear (again redefining ϵ) from the following four lemmas:

Lemma 5.10a: $\int_{J_1} A(z) dz = 0$.

Lemma 5.10b: $\int_{J_5} A(z) dz \geq 0$.

Lemma 5.10c: $\int_{J_3} A(z) dz > -\epsilon$.

Lemma 5.10d: $\int_{J_2 \cup J_4} \geq -6\hat{M}\epsilon$.

Lemma 5.10a is obvious, since $A(z) \equiv 0$ on J_1 by construction.

Similarly, $A(z)$ is increasing on J_5 , and so Lemma 5.10b holds.

Proof of Lemma 5.10c:

$$\begin{aligned}
\int_{j_3} A(z) dz &= \sum_{j_1 \in J_3} A([j_1]) \cdot (j_{i+1} - j_i) \\
&\geq -(1+\epsilon)^2 \sum_{j_1 \in J_3} u_{[j_1]}^n (m(j_{i+1}) - m(j_i)) \cdot \prod_{i=1}^{[j_1]} (1 - P_*(i)) , \\
&\text{by (5.8) and (5.9),} \\
&\geq -(1+\epsilon)^2 \cdot (1+\epsilon) \sum_{\ell=0}^{\infty} (u_0 q^\ell)^n (\hat{m}(\ell+1) - \hat{m}(\ell)) \\
&= -(1+\epsilon)^3 \sum_{\ell=0}^{\infty} (u_0 q^\ell)^{1/2} \\
&= -(1+\epsilon)^3 \frac{u_0^{1/2}}{1 - q^{1/2}} \\
&> -\epsilon .
\end{aligned}$$

Proof of 5.10d: Let $\gamma : J_4 \rightarrow J_2$ be ϕ^{-1} .

$$\begin{aligned}
\int_{J_2 \cup J_4} A(z) dz &= \sum_{j_1 \in J_2} A([j_1]) \cdot (j_{i+1} - j_i) + \sum_{j_1 \in J_4} A([j_1]) \cdot (j_{i+1} - j_i) \\
&\geq - \sum_{j_1 \in J_2} (1+\epsilon)^2 (m(j_{i+1}) - m(j_i)) \cdot u_{[j_1]}^n \prod_{i=1}^{[j_1]} (1 - P_*(i)) \\
&\quad - \sum_{j_1 \in J_4} (1-\epsilon)^2 (m(j_{i+1}) - m(j_i)) \cdot u_{[j_1]}^n \prod_{i=1}^{[j_1]} (1 - P_*(i)) \\
&\geq \sum_{j_1 \in J_4} ((1-\epsilon)^2 u_{[j_1]}^n - (1+\epsilon)^2 u_{[\gamma(j_1)]}^n) (m(j_{i+1}) - m(j_i)) \prod_{i=1}^{[j_1]} (1 - P_*(i)) \\
&\hspace{15em} (5.11)
\end{aligned}$$

by the defining properties of ϕ .

Now, $u_{[\gamma(j_i)]} = q^\lambda u_{[j_i]}$, where $\lambda = -1, 0, 1$ (to see this, observe that if $\hat{m}(i) \leq \bar{m}_{N-1} \leq \hat{m}(i+1)$, $v(N)$ must = i or $i+1$). Thus

$$\begin{aligned} & \geq ((1-\varepsilon)^2 - q^{-n}(1+\varepsilon)^2) \cdot \sum_{j_i \in J_4} u_{[j_i]}^n (m(j_{i+1}) - m(j_i)) \cdot \prod_{i=1}^{[j_i]} (1 - P_*(i)) \\ & \geq -5\varepsilon \sum_{k=1}^{\infty} \hat{M}(1+\varepsilon)^2 P_*(k) \prod_{i=1}^{k-1} (1 - P_*(i)) \end{aligned}$$

by the properties of m and the fact that $|v_\infty(s^*)| \leq \hat{M}$ for all $s^* \in S^*$,

$$\geq -6\hat{M}\varepsilon.$$

This completes the proof of Lemma 5.10d, hence of Proposition 5.6, and hence of Proposition 5.2. \square

6. THE NON-ABSORBING CASE

We now prove

Proposition 5.3:

$$\liminf_{T \rightarrow \infty} \int_{T_\infty} \frac{1}{N} \sum_{i=1}^N d_i d\mu > -\varepsilon.$$

Proof:

Lemma 6.1:

Let $\{e_{ik}\}_{k \rightarrow \infty}$ converge uniformly.

$$\text{Let } E = \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (\lim_{k \rightarrow \infty} e_{ik})$$

$$\text{Then } E = \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N e_{jj}.$$

Proof: Easy.

□

Identifying e_{ik} as $\int_{(T_1 \cup \dots \cup T_k)^c} d_i d\mu$, we observe that

$$\left| \int_{T_\infty} d_i d\mu - \int_{(T_1 \cup \dots \cup T_k)^c} d_i d\mu \right| \leq \hat{M} \sum_{k+1}^{\infty} \mu(T_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Thus Lemma 6.1 gives

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \int_{T_\infty} d_i d\mu = \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \int_{(T_1 \cup \dots \cup T_i)^c} d_i d\mu.$$

Let $\omega_i(u) = W_{s_i}(u) - W_{s_{i-1}}(u)$. Then, by Proposition 4.1,

$$\begin{aligned} \int_{(T_1 \cup \dots \cup T_i)^c} d_i d\mu &\geq \int_{(T_1 \cup \dots \cup T_i)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1}) + \eta(u_{i-1})) du \\ &\geq \int_{(T_1 \cup \dots \cup T_i)^c} (\delta_i(u_{i-1}) - \omega_i(u_{i-1})) du - \varepsilon \cdot \mu((T_1 \cup \dots \cup T_c)^c). \end{aligned}$$

Of course, also,

$$\int_{(T_1 \cup \dots \cup T_i)^c} d_i d\mu \geq -\hat{M} \mu((T_1 \cup \dots \cup T_c)^c).$$

Thus, setting

$$f_k(t(k-1)) = \begin{cases} \max(-\hat{M}, \delta_k(u_{k-1}) - \omega_k(u_{k-1})) & \text{if } s(t_{k-1}) \notin S^* \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\begin{aligned} \int_{(T_1 \cup \dots \cup T_i)^c} f_i d\mu &\geq \int_{(T_1 \cup \dots \cup T_i)^c} f_i(t(i-1)) d\mu - \epsilon \\ &= \int_{T^\infty} f_i(t(i-1)) d\mu - \epsilon . \end{aligned}$$

Hence, to establish Proposition 5.3, it is

Enough to show:

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \int_{T^\infty} f_k(t(k-1)) d\mu \geq 0 . \quad (6.2)$$

Applying Proposition 5.5, for each N ,

$$\int_{T^\infty} \sum_{k=1}^N f_k(t(k-1)) d\mu = \int_{T^\infty} \sum_{k=1}^N f_k \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) d\mu .$$

Thus

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \int_{T^\infty} f_k d\mu &= \liminf_{N \rightarrow \infty} \int_{T^\infty} \frac{1}{N} \sum_{k=1}^N f_k \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) d\mu \\ &\geq \int_{T^\infty} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k \prod_{i=1}^{k-1} (1 - P_*(i)) d\mu , \end{aligned}$$

by Fatou's Lemma (Royden [1963]).

So, if we establish

Lemma 6.3: For all $t \in T^\infty$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(t(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) \geq 0 ,$$

we are done.

Proof: Suppose we know

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(\mathcal{t}(k-1)) \cdot \prod_{i=1}^k (1 - P_*(i)) \geq 0 .$$

Then either there exists N such that $k > N$ implies that $1 - P_*(k) > \frac{1}{2}$, in which case we are done immediately, or else

$$\begin{aligned} & \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k(\mathcal{t}(k-1)) \cdot \prod_{i=1}^{k-1} (1 - P_*(i)) \\ & \geq \liminf_{N \rightarrow \infty} \left(- \frac{1}{N} \sum_{k=1}^N \prod_{i=1}^{k-1} (1 - P_*(i)) \right) = 0 . \end{aligned}$$

But we can in fact show the stronger

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f'_k(\mathcal{t}(k-1)) \cdot \prod_{i=1}^k (1 - P(\text{abs} : \mathcal{t}, i-1)) \geq 0 , \quad (6.3')$$

where $f'_k = \delta_k - \omega_k \leq f_k$.

Letting $P_i = P_*(i)$, for all i , we have

$$\begin{aligned} \sum_{k=1}^N f'_k \cdot \prod_{i=1}^k (1 - P_i) &= \sum_{k=1}^N f'_k \cdot \prod_{i=1}^N (1 - P_i) \\ &+ \sum_{j=1}^{N-1} \left(\sum_{k=1}^j f'_k \prod_{i=1}^j (1 - P_i) \cdot P_j \right) . \end{aligned} \quad (6.4)$$

Lemma 6.5: There exists a number Ω_0 such that for all \mathcal{t} , for all

N , for all N_0 , $\sum_{k=1}^{N_0} f'_k \leq 0$ implies that

$$\sum_{k=N_0+1}^N f'_k \cdot \prod_{i=N_0+1}^N (1 - P_*(i)) > -\Omega_0 .$$

(In particular, this conclusion holds when $N_0 = 0$.)

Proof: $\sum_{k=1}^{N_0} f'_k \leq 0$ implies that $u_{N_0} = u_0$ or u_1 . Write
 $\sum_{k=N_0+1}^N f'_k = -M - 2\bar{W}(u_1)$; assume

$$M \geq 0 \text{ (else } \sum_{k=N_0+1}^N f'_k \cdot \prod_{i=N_0+1}^N (1 - P_i) \geq -2\bar{W}(u_1) \text{)} .$$

Then

$$\begin{aligned} \sum_{k=N_0+1}^N \delta_k &\leq \sum_{k=N_0+1}^N (\delta_k - \omega_k) + 2\bar{W}(u_1) \\ &= -M - 2\bar{W}(u_1) + 2\bar{W}(u_1) \\ &= -M . \end{aligned} \tag{6.6}$$

Recalling Proposition 4.2, and noting that

$$|v_\infty(s)| \leq M \text{ for all } s \in S ,$$

(6.6) implies that

$$\begin{aligned} \sum_{k=N_0+1}^N P_*(k) &\geq \frac{(1-\varepsilon) \cdot Mu_1^n}{\hat{M}} \\ &> \frac{Mu_1^n}{2\hat{M}} ; \end{aligned}$$

hence

$$\begin{aligned} \sum_{k=N_0+1}^{N-1} (1 - P_*(k)) &\leq \prod_{k=N_0+1}^{N-1} (1 - P_*(k)) \\ &\leq e^{-\frac{Mu_1^n}{2\hat{M}}} , \end{aligned}$$

by a well-known inequality (which can be derived immediately from the observation $\ln(1-P) < -P$ for $0 < P < 1$). Thus

$$\sum_{k=N_0+1}^N f'_k \prod_{i=N_0+1}^N (1-P_i) > (-M - 2\bar{W}(u_1)) e^{-\frac{Mu_1^n}{2M}} \\ - \frac{2M}{u_1^n} \cdot \frac{1}{e} - 2\bar{W}(u_1) .$$

So if we set $\Omega_0 = 3Mu_1^{-n}$, we are done.

Returning now to the proof of (6.3'), we distinguish two cases:

Case 1: $\sum_{k=1}^{\infty} P_k < \infty$.

Then

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1-P_i) \\ \geq \liminf_{N \rightarrow \infty} \frac{1}{N} (-\Omega_0 + \sum_{j=1}^{N-1} (-\Omega_0) \cdot P_{j+1}) ,$$

by (6.4) and Lemma 6.5,

$$\geq \liminf_{N \rightarrow \infty} \left(-\frac{\Omega_0}{N} \right) \cdot \left(1 + \sum_{k=2}^{\infty} P_k \right) \\ = 0 .$$

Case 2: $\sum_{k=1}^{\infty} P_k = \infty$.

Case 2a: There exists N_1 such that $N > N_1$ implies $\sum_{k=1}^N f_k > 0$.

Then

$$\begin{aligned}
 & \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \\
 = & \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) + \sum_{j=1}^{N_1} \left(\sum_{k=1}^j f_k \cdot \prod_{i=1}^j (1 - P_i) \cdot P_{j+1} \right) \\
 & + \sum_{j=N_1+1}^{N-1} \left(\sum_{k=1}^j f_k \cdot \prod_{i=1}^k (1 - P_i) \cdot P_{j+1} \right) \\
 \geq & -\Omega_0 + \sum_{j=1}^{N_1} (-\Omega_0) \cdot P_{j+1} + \text{positive terms;}
 \end{aligned}$$

hence

$$\begin{aligned}
 & \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \\
 \geq & \liminf_{N \rightarrow \infty} \frac{\text{constant}}{N} = 0.
 \end{aligned}$$

Case 2b: There exists no such N_1 . Then for arbitrary $\epsilon^* > 0$, there exists N^* such that

$$\begin{aligned}
 & \sum_{i=1}^{N^*} (1 - P_i) < \epsilon^* \quad \text{and} \\
 & \sum_{k=1}^{N^*} f_k \leq 0.
 \end{aligned}$$

Let $N > N^*$.

Then

$$\begin{aligned}
& \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \\
&= \sum_{k=1}^{N^*} f_k \cdot \prod_{i=1}^k (1 - P_i) + \sum_{k=N^*+1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \cdot \prod_{i=1}^N (1 - P_i) \\
&\geq -\Omega_0 - \sum_{j=1}^{N^*-1} \Omega_0 \cdot P_{j+1} - \epsilon^* - \Omega_0 - \sum_{j=N^*+1}^{N-1} \Omega_0 \cdot P_{j+1}
\end{aligned}$$

by Lemma 6.5 and a slight extension of (6.4),

$$\geq -\text{constant} - N\epsilon^*\Omega_0 ;$$

hence
$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \geq -\Omega_0 \epsilon^* .$$

But ϵ^* was arbitrary:

hence
$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_k \cdot \prod_{i=1}^k (1 - P_i) \geq 0 .$$

This completes the proof of equation (6.3'), hence of Lemma 6.3, hence of Proposition 5.3, hence of Proposition 5.1, hence of Proposition 3.1, and hence of Theorem I.

Q.E.D. \square

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