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EQUILIBRIA OF A TWO-PERSON NON-ZEROSUM NOISY GAME OF TIMING

by

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Abstract: Necessary and sufficient conditions are obtained for the existence of an equilibrium point (as well as for the existence of a dominating equilibrium point) in a two-person non-zerosum game of timing.

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1. Introduction

Two toothpaste manufacturers are competing for a larger share of the dentifrice market. Each is in the process of developing a new and better toothpaste. The longer one company waits to introduce its new toothpaste, the better its chances are of successfully capturing a share of the market, if its product hits the stores first. (This is assuming that the toothpaste is being technologically improved as time goes on.) Alternatively, if a company waits too long to introduce its product, then it might be too late to successfully capture any of the market. (Everyone might already be quite happy brushing with the other company's toothpaste introduced just last week!) Essentially, the problem for each company is one of choosing a time at which to introduce their particular brand of toothpaste to the public.

Two researchers are working independently on a particular problem. When to publish one's results is a big question. By publishing one's results first, one has some advantage over the other. Alternatively, by waiting until later, one can capitalize on weaknesses in the other's results.

The above examples illustrate some characteristics of a 2-person noisy game of timing which may or may not be zerosum. Mathematically, a 2-person noisy game of timing has the following structure. The player set is $\{P_1, P_2\}$. The pure strategy set for P_i consists of all choices of times of action in $[0,1]$, the closed unit interval. The strategy set for P_i then consists of all cumulative distribution functions on the closed unit interval. Let the strategy set for P_i be denoted by $\mathcal{F}$. Thus $F \in \mathcal{F}$ if F is a right-continuous, non-negative, non-decreasing real-valued function defined on the real line $\mathbb{R}$ such that $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t \geq 1$. Let the degenerate distribution with a jump of 1 at a point $T \in [0,1]$ be denoted by δ_T . Thus

$$\delta_T(t) = \begin{cases} 0 & \text{for} \quad t < T \\ 1 & \text{for} \quad t \geq T \end{cases}$$

or, alternatively, we may write $\delta_T(T) - \delta_T(T-) = 1$.

The payoff to P_i , if each P_k acts according to a pure strategy δ_{t_k} , $t_k \in [0,1]$ for $k = 1, 2$, is denoted by $K_i(\delta_{t_i}, \delta_{t_j})$ and is equal to

$$K_i(\delta_{t_i}, \delta_{t_j}) = \begin{cases} L_i(t_i) & \text{if} \quad t_i < t_j \\ \phi_i(t_i) & \text{if} \quad t_i = t_j \\ M_i(t_j) & \text{if} \quad t_i > t_j \end{cases}$$

where L_i , ϕ_i and M_i are real-valued functions defined on $[0,1]$. Thus P_i receives (i) $L_i(t_i)$, if P_i acts first at time t_i , (ii) $\phi_i(t_i)$, if both P_i and P_j act simultaneously at time t_i , or (iii) $M_i(t_j)$, if P_j acts first at time t_j . The above game is zerosum if $K_1 + K_2 = 0$ at all times.

If P_1 and P_2 choose mixed strategies F_1 and F_2 in $\mathcal{F}$, then the payoff to P_i , denoted by $K_i(F_i, F_j)$, is equal to the Lebesgue-Stieltjes integral of the kernel $K_i(\delta_{t_i}, \delta_{t_j})$ with respect to the measures F_1 and F_2 , i.e.

$$K_i(F_i, F_j) = \int_{[0,1]} K_i(\delta_t, F_j)\,dF_i(t)$$

$$= \int_{[0,1]} \{ \int_{[0,t)} M_i(s)\,dF_j(s) + \phi_i(t)\alpha_j(t) + L_i(t)(1 - F_j(t)) \}\,dF_i(t)$$

where $\alpha_j(t) = F_j(t) - F_j(t-)$ is the *size of the jump* at t of F_j.

The above 2-person game of timing will be denoted by $(\mathcal{F}, K_1, K_2)$.

A strategy pair (F_1, F_2) is an equilibrium point (hereafter denoted by EP) of $(\mathcal{F}, K_1, K_2)$ if and only if $K_i(F_i, F_j) \geq K_i(F, F_j)$ for all $F \in \mathcal{F}$, $i = 1, 2$, $\{i,j\} = \{1,2\}$. An equivalent definition is that a strategy pair (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$ if and only if $K_i(F_i, F_j) \geq K_i(\delta_T, F_j)$ for all $T \in [0,1]$. They are equivalent since $F \in \mathcal{F}$ is a right-continuous, non-negative, non-decreasing function on $[0,1]$ such that $F(1) = 1$.

The early literature concentrates on EP's of 2-person zerosum games of timing with various restrictions on the kernels of each player. See [Blackwell, 1948], [Blackwell, 1949], [Glicksberg, 1950], [Blackwell and Girshick, 1954], [Karlin, 1959], [Fox and Kimeldorf, 1969], [Owen, 1976].

Sūdžiūtė initiated the study of non-zerosum silent games of timing in [1969]. In a silent game of timing, L_i, ϕ_i and M_i are functions of both t_i and t_j (signifying that each player does not know if the other has acted or not). More recently, Kilgour, [1973], has obtained sufficient conditions for the existence of an EP in a 2-person non-zerosum noisy game of timing (with differentiability conditions on the kernel which imply conditions (i), (ii) below).

This paper is concerned with obtaining necessary and sufficient conditions for the existence of an EP in the (not necessarily zerosum) 2-person noisy game of timing in which P_i's kernel satisfies the following for $i = 1, 2$:

Let a_i maximize $\text{Min}\{L_i(t), M_i(t)\}$ in $[0,1]$.

(i) L_i, ϕ_i and M_i are continuous real-valued functions on $[0,1]$ such that L_i is a strictly increasing function while M_i is a strictly decreasing function.

(ii) *either* $\lim_{t \to a_i} [L_i(t) - L_i(a_i)]/[L_i(t) - M_i(t)]$ exists and is strictly positive (hereafter, this condition will be known as Condition I), *or* $a_i = 0$ and either $L_i(0) > M_i(0)$ (which implies that $\lim_{t \to a_i} [L_i(t) - L_i(a_i)]/[L_i(t) - M_i(t)] = 0$) or $L_i(0) = M_i(0)$ and $\exists \varepsilon > 0$ such that L_i is differentiable in $(0,\varepsilon]$ and $L_i'(t)/[L_i(t) - M_i(t)]$ is bounded for $t \in (0,\varepsilon]$ (hereafter, this condition will be known as Condition II).

Condition I is used solely in the "only if" part of Lemma 7A while Condition II is used solely in the "only if" part of Lemma 7B.

The first main result of this paper, Theorem 8, gives necessary and sufficient conditions for the existence of an EP in a game (F, K_1, K_2) which satisfies conditions (i) and (ii) above. (Hereafter, (F, K_1, K_2) will denote a game of timing (F, K_1, K_2) described in the Introduction which satisfies conditions (i) and (ii) above.)

A strategy pair (F_1, F_2) is a *dominating EP* of (F, K_1, K_2) if and only if (F_1, F_2) is an EP such that $K_i(F_i, F_j) \geq K_i(G_i, G_j)$ for $i = 1, 2$, $\{i,j\} = \{1,2\}$ for any EP (G_1, G_2) of (F, K_1, K_2) , i.e., a dominating EP is an EP at which the payoff to each player is larger than or equal to the payoff received at any other EP. A second result of this paper, Theorem 10, gives necessary and sufficient conditions for the existence of a dominating EP in (F, K_1, K_2) satisfying, in addition to (i), (ii) above, (iii) below:

(iii) $L_i(0) \leq M_i(0)$ for $i = 1, 2$.

2. Preliminary Notation and Definitions

Alternate proofs of Lemmas 2, 3 and 4 in Section 3 and of the "if" part of Lemma 7 in Section 4 can be found in [Kilgour, 1973] and [Kilgour, 1979]. For completeness, the author offers these proofs (some of which use Lemma 1 very efficiently). In order to begin, the following notation and definitions will be needed.

Let $\mathrm{Supp}(F)$ denote the *support* of $F \in \mathcal{F}$, i.e., $\mathrm{Supp}(F)$ is the complement of the set of all points which have a neighborhood on which F is constant.

Let J_i denote the *set of jump points* of $F_i \in \mathcal{F}$, i.e., $J_i = \{t \in \mathrm{Supp}(F_i) : F_i(t) - F_i(t-) > 0\}$.

Recall that $\alpha_i(t)$ denotes the size of the jump at a jump point of F_i, i.e., $\alpha_i(t) = F_i(t) - F_i(t-)$ for $t \in J_i$. If $t \notin J_i$, then $\alpha_i(t) = 0$.

3. Preliminary Lemmas

This section gives shape to the supports of strategy pairs which are possible EP's of $(\mathcal{F}, K_1, K_2)$. The first simple yet useful lemma (see the proofs of Lemmas 3, 4, 5 and 6) is an elaboration of Lemma 2.2.1 in [Karlin, 1959]. Basically, Lemma 1 states that, if (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$ and $T \in \mathrm{Supp}(F_i)$, then, either T contributes to $K_i(F_i, F_j)$ as much as the whole $\mathrm{Supp}(F_i)$ does or there exist points $S_n \in \mathrm{Supp}(F_i)$, $S_n \neq T$, S_n converging to T that do the job. Let $\exists$ denote "there exist"; $\forall$, "for each"; and let χ_U denote, for $U \subset [0,1]$, the function defined by

$$\chi_U(t) = \begin{cases} 0 & \text{if } t \notin U \\ 1 & \text{if } t \in U \end{cases} .$$

Given a strategy $F_i \in \mathcal{F}$ we define a new function: $H_i : [0,1] \times \mathcal{F} \to \mathbb{R}$ as follows

$$H_i(T, F_j) = \int_{[0,T)} M_i(t)dF_j(t) + \alpha_j(T)\phi_i(T)\chi_{(0,1]}(\alpha_i(T))$$
$$+ L_i(T)(1 - F_j(T)) .$$

Note the difference between this function H_i and the restriction of the payoff function K_i of Player i, $K_i : [0,1] \times \mathcal{F} \to \mathbb{R}$ defined previously as

$$K_i(\delta_T, F_j) = \int_{[0,T)} M_i(t)dF_j(t) + \alpha_j(T)\phi_i(T) + L_i(T)(1 - F_j(T)) .$$

These functions may differ whenever the first variable T belongs to $\mathcal{C}J_i \cap J_j$ (where $\mathcal{C}$ denotes the complement of J_i in $[0,1]$) since, in that case, $\phi_i(T)$ does not appear in the computation of $H_i(T, F_j)$ but does appear in the computation of $K_i(\delta_T, F_j)$.

The following facts are almost immediate

(A) $\qquad H_i(T, F_j) = K_i(\delta_T, F_j)$ whenever $T \in J_i \cup \mathcal{C}J_j$.

Since $\int_U K_i(\delta_T, F_j)dF_i(T) = \int_U(\int_{[0,T)} M_i(t)dF_j(t) + \alpha_j(T)\phi_i(T)$ $+ L_i(T)(1 - F_j(T)))dF_i(T) = \int_U(\int_{[0,T)} M_i(t)dF_j(t) + \alpha_j(T)\phi_i(T)\chi_{(0,1]}(\alpha_i(T))$ $+ L_i(T)(1 - F_j(T)))dF_i(T) = \int_U H_i(T, F_j)dF_i(T)$, it is true that

(B) $\qquad \int_U K_i(\delta_T, F_j)dF_i(T) = \int_U H_i(T, F_j)dF_i(T)$ for any closed set $U \subseteq [0,1]$.

Suppose that (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$. If ever $K_i(F_i, F_j) < H_i(T, F_j)$ on a set U of positive F_i measure, then one could define a new distribution G_i (by translating F_i and multiplying by a normalizing constant) on a closed set V of positive F_i measure such that $K_i(G_i, F_j) =$ (by (B)) $\int_V H_i(T, F_j)dG_i(T) > K_i(F_i, F_j)$. This would contradict the hypothesis that (F_1, F_2) is an EP of (F, K_1, K_2) . Thus, it is true that

(C) If (F_1, F_2) is an EP of (F, K_1, K_2) , then $K_i(F_i, F_j) \geq H_i(T, F_j)$ almost everywhere with respect to F_i .

By facts (B) and (C), it is true that $K_i(F_i, F_j) = \int_{[0,1]} H_i(T, F_j)dF_i(T)$ and $K_i(F_i, F_j) \geq H_i(T, F_j)$ almost everywhere with respect to F_i whenever (F_1, F_2) is an EP of (F, K_1, K_2) . Thus,

(D) $K_i(F_i, F_j) = H_i(T, F_j)$ almost everywhere with respect to F_i whenever (F_1, F_2) is an EP of (F, K_1, K_2) .

Supp (F_i) is a closed set in $[0,1]$ whose only possible isolated points must be jumps of the distribution F_i . Thus,

(E) If $T \in \mathrm{Supp}(F_i)$ and $T \notin J_i$, then $\exists$ a sequence $\{T_n\} \subset \mathrm{Supp}(F_i) \cap (T-\varepsilon, T)$ for some $\varepsilon > 0$ (and/or $\exists$ a sequence $\{T_n\} \subset \mathrm{Supp}(F_i) \cap (T, T+\varepsilon)$ for some $\varepsilon > 0$) such that T_n converges to T (to be denoted by $T_n \to T$).

Finally, Lemma 1 can be stated as follows:

Lemma 1. Suppose that (F_1, F_2) is an EP of (F, K_1, K_2) . If $T \in \mathrm{Supp}(F_i)$ for some $i = 1$ or 2, then

(1) If $T \in J_i$, then $K_i(F_i, F_j) = K_i(\delta_T, F_j) = H(T, F_j)$,

(2) If $\exists$ a sequence $\{S_n\} \subset \mathrm{Supp}(F_i) \cap (T - \varepsilon_1, T)$ for some $\varepsilon_1 > 0$ such that $S_n \to T$, then $\exists$ $\{T_n\} \subset \mathrm{Supp}(F_i) \cap (T - \varepsilon_1, T)$ such that $T_n \to T$ and such that $K_i(F_i, F_j) = H_i(T_n, F_j)$ $\forall n$,

(3) If $\exists$ a sequence $\{S_n\} \subset \mathrm{Supp}(F_i) \cap (T, T + \varepsilon_2)$ for some $\varepsilon_2 > 0$ such that $S_n \to T$, then $\exists$ $\{T_n\} \subset \mathrm{Supp}(F_i) \cap (T, T + \varepsilon_2)$ such that $T_n \to T$ and such that $K_i(F_i, F_j) = H_i(T_n, F_j)$ $\forall n$.

Proof. Let (F_1, F_2) be an EP of (F, K_1, K_2) and let $T \in \mathrm{Supp}(F_i)$ for some $i = 1$ or 2.

(1) is true by facts (A) and (D) since $\{T\}$ is a set of positive F_i measure if $T \in J_i$.

Suppose that $\exists$ $\{S_n\} \subset \mathrm{Supp}(F_i) \cap V$ where $V = (T - \varepsilon_1, T)$ or $V = (T, T + \varepsilon_2)$ for some $\varepsilon_1, \varepsilon_2 > 0$. Let $V_m = (T - (\varepsilon_1/m), T)$ or let $V_m = (T, T + (\varepsilon_2/m))$ depending on whether $V = (T - \varepsilon_1, T)$ or $V = (T, T + \varepsilon_2)$ respectively. For each m , $\exists n$ such that $S_n \in V_m$, so that $\int_{V_m} dF_i(t) > 0$ (since $S_n \in \mathrm{Supp}(F_i)$). Thus $\exists$ $T_m \in V_m$ such that $H_i(T_m, F_j) = K_i(F_i, F_j)$ by (D), i.e., $\exists$ $\{T_m\} \subset \mathrm{Supp}(F_i) \cap V$ such that $T_m \to T$ and $K_i(F_i, F_j) = H_i(T_m, F_j)$ $\forall m$.

■

Since the limit of a constant sequence exists, $\lim_{m\to\infty} H_i(T_m, F_j)$ ($\{T_m\} \subset \mathrm{Supp}(F_i) \cap V$ as in (2) or (3) of Lemma 1) exists and is equal to

$$K_i(F_i, F_j) = \lim_{m\to\infty} H_i(T_m, F_j) = \int_{[0,T)} M_i(t)dF_j(t)$$

$$+ \alpha_j(T)[L_i(T)\chi_{(T-\varepsilon_1,T)}(T_m) + M_i(T)\chi_{(T,T+\varepsilon_2)}(T_m)]$$

$$+ L_i(T)(1 - F_j(T))$$

by the Lebesgue Dominated Convergence Theorem and the fact that $\sum_m \alpha_j(T_m)$ exists implies that $\alpha_j(T_m) \to 0$ which implies that $\alpha_j(T_m)\phi_i(T_m)\chi_{[0,1]}(\alpha_i(T_m)) \to 0$ as $T_m \to T$ since ϕ_i is bounded.

Thus, one can conclude that, if (F_1, F_2) is an EP of (F, K_1, K_2) and $T \in \text{Supp}(F_i)$, $i = 1$ or 2, then (i) if $T \in J_1 \cap J_2$ and $\exists$ sequence $\{S_n\}$ satisfying (2) in Lemma 1, then $\phi_i(T) = L_i(T)$, (ii) if $T \in J_1 \cap J_2$ and $\exists$ sequence $\{S_n\}$ satisfying (3) in Lemma 1, then $\phi_i(T) = M_i(T)$ and (iii) if $T \notin J_j$ and $\exists$ sequences satisfying both (2) and (3) in Lemma 1, then $L_i(T) = M_i(T)$.

Lemma 2: The pure timing strategy pair (F_1, F_2) with $F_k = \delta_{T_k}$ for $k = 1, 2$, is an EP of (F, K_1, K_2) if and only if $T_1 = T_2 = T$ and for $i = 1, 2$

$$\phi_i(T) \geq \begin{cases} L_i(1) & \text{if } T = 1 \\ \text{Max}\{L_i(T), M_i(T)\} & \text{if } 0 < T < 1 \\ M_i(0) & \text{if } T = 0 \end{cases} .$$

Proof: If $T_i < T_j$, then by the definition of an EP, $L_i(T_i) = K_i(F_i, F_j)$ must be strictly larger than $K_i(\delta_t, F_j) = L_i(t)$ for each $t \in (T_i, T_j)$; but, this contradicts the assumption that L_i is an increasing function. Thus $T_1 = T_2 = T$. By the definition of an EP,

$$\phi_i(T) = K_i(F_i, F_j) \geq K_i(\delta_t, F_j) = \begin{cases} L_i(t) & \text{if} \quad t < T \\ \phi_i(T) & \text{if} \quad t = T \\ M_i(T) & \text{if} \quad t > T \end{cases}$$

for all $t \in [0,1]$. The Lemma now follows from the continuity and monotonicity of L_i . ∎

Lemma 3 (for an alternate proof, see [Kilgour, 1973], [Kilgour, 1979]) indicates that, if (F_1, F_2) is an EP of (F, K_1, K_2) , then the supports of F_1 and F_2 are identical until the probability of at least one player's having acted is one. A precise statement of this idea requires the following definitions.

Let $e(F) = \text{Max}\{t \in [0,1] : t \in \text{Supp}(F)\}$. Thus $e(F)$ is the earliest time of certain action corresponding to F .

Let $\text{Supp}(F,G) = \text{Supp}(F) \cap \text{Supp}(G)$, i.e., $\text{Supp}(F,G)$ denotes the *common support* of F and G .

Lemma 3: If (F_1, F_2) is an EP of (F, K_1, K_2) such that $e(F_i) \leq e(F_j)$, then $\text{Supp}(F_i) = \text{Supp}(F_j) \cap [0, e(F_i)]$.

Proof: Suppose that there exists a point $T \in \text{Supp}(F_i)$ such that $T \notin \text{Supp}(F_j)$. Since $\text{Supp}(F_j)$ is closed and $e(F_i) \leq e(F_j)$, there must exist points $t_1 < t_2$ such that $T \in [t_1, t_2)$, $[t_1, t_2] \cap \text{Supp}(F_j) = \emptyset$, and $F_j(t_2) < 1$. By Lemma 1 and facts (A) and (E), $\exists$ $S \in [t_1, t_2)$ such that

$$K_i(F_i, F_j) = K_i(\delta_S, F_j) =$$

$$= \int_{[0,S]} M_i(t)dF_j(t) + \int_{(S,t_2]} L_i(S)dF_j(t) + \int_{(t_2,1]} L_i(S)dF_j(t)$$

$$< \int_{[0,S]} M_i(t)dF_j(t) + \int_{(S,t_2]} M_i(t)dF_j(t) + \int_{(t_2,1]} L_i(t_2)dF_j(t)$$

$$= K_i(\delta_{t_2}, F_j) ,$$

since $F_j(t_2) - F_j(S) = 0$, $F_j(t_2) < 1$, and $L_i(S) < L_i(t_2)$. This contradicts the hypothesis that (F_1, F_2) is an EP. ■

Thus, $\text{Supp}(F_1, F_2) = \text{Supp}(F_i)$ whenever (F_1, F_2) is an EP of (F, K_1, K_2) and $e(F_i) \leq e(F_j)$. Hereafter, the term *initial support* of F_1 and F_2 naturally describes, and is synonymous with, the common support of F_1 and F_2 whenever (F_1, F_2) is an EP of (F, K_1, K_2). This result contrasts with one of Sūdžiūtė's results in [1969] which states that if (F_1, F_2) is an EP of a *silent* non-zerosum game of timing, then $\text{Supp}(F_1) \cap (0,1) = \text{Supp}(F_2) \cap (0,1)$. Lemmas 7A and 7B show that this is certainly not true for our *noisy* game of timing, (F, K_1, K_2) .

Lemma 4 (for an alternate proof, see [Kilgour, 1973], [Kilgour, 1979]) gives us more information about the possible behavior of an EP of (F, K_1, K_2) . Recall that a_i maximizes $\text{Min}\{L_i(t), M_i(t)\}$ for $t \in [0,1]$.

<u>Lemma 4</u>: Suppose that (F_1, F_2) is an EP of (F, K_1, K_2) such that $T \in \text{Supp}(F_1, F_2)$. If $T < e(F_j)$, a_i for $i \neq j$, then T is a common jump of the EP (F_1, F_2) , i.e., $T \in J_1 \cap J_2$.

<u>Proof</u>: Suppose, to the contrary, that (F_1, F_2) is an EP of (F, K_1, K_2) such that $\exists\ T \in \mathrm{Supp}(F_1, F_2)$ satisfying (i) $T < e(F_j)$, a_i, for $i \neq j$ and (ii) $T \not\in J_1 \cap J_2$. Since $T < e(F_j)$, a_i, $\exists\ \varepsilon \geq 0$ such that $\exists\ \varepsilon_1 > \varepsilon$ such that $T + \varepsilon_1 \not\in J_j$ and $T + \varepsilon_1 < e(F_j)$, a_i and such that $\exists$ a sequence $\{T_n\} \subset (T-\varepsilon, T+\varepsilon)$ satisfying the conclusions in (2) or (2) of Lemma 1 (such a sequence exists by Lemma 3 and fact (E)) so that

$$\begin{aligned} K_i(F_i, F_j) &= \lim_{n\to\infty} H_i(T_n, F_j) = \int_{[0,T)} M_i(t)dF_j(t) + \alpha_j(T)[M_i(T)\chi_{(T,T+\varepsilon)}(T_n) \\ &\quad + L_i(T)\chi_{(T-\varepsilon,T)}(T_n)] + L_i(T)(1 - F_j(T)) \\ &< \int_{[0,T)} M_i(t)dF_j(t) + \int_{[T,T+\varepsilon_1)} M_i(t)dF_j(t) \\ &\quad + \int_{(T+\varepsilon_1,1]} L_i(T+\varepsilon_1)dF_j(t) \\ &= K_i(\delta_{T+\varepsilon_1}, F_j) , \end{aligned}$$

since $T + \varepsilon_1 < e(F_j)$, a_i, and $T + \varepsilon_1 \not\in J_j$ i.e., since $F_j(T+\varepsilon_1) < 1$, $L_i(T) < L_i(T+\varepsilon_1)$, $M_i(t)$, for $t, T < a_i$ and $\alpha_j(T+\varepsilon_1) = 0$. This contradicts the definition of EP. ∎

An immediate corollary to Lemma 4 is the following: Suppose that (F_1, F_2) is an EP of (F, K_1, K_2) such that $T \in \mathrm{Supp}(F_1, F_2)$ and $T < e(F_i)$ for some i. If T is not a common jump, then $T \geq a_j$ for $j \neq i$.

4. Key Lemmas and First Main Theorem

The following lemma provides the key to both theorems. Lemma 5 tells us that the existence of an EP (F_1, F_2) of (F, K_1, K_2) with a common jump implies the existence of a pure EP of (F, K_1, K_2) . (If, in addition, $L_i(0) \leq M_i(0)$ for $i = 1, 2$, then it also implies the existence of a pure EP of (F, K_1, K_2) which dominates (F_1, F_2) .) The reason this information provides the key is that Section 3 already tells us a lot about the initial support of an EP of (F, K_1, K_2) which does not have a common jump. Section 3 also gives us necessary and sufficient conditions for the existence of a pure EP of (F, K_1, K_2) . Thus, after Lemma 5, it only remains to find necessary and sufficient conditions for the existence of an EP of (F, K_1, K_2) without any common jumps.

Lemma 5: Suppose that (F_1, F_2) is an EP of (F, K_1, K_2) . If $T \in J_1 \cap J_2$, then (δ_T, δ_T) is an EP of (F, K_1, K_2) .

Suppose further that $L_i(0) \leq M_i(0)$ for $i = 1, 2$. If $\exists\, T \in J_1 \cap J_2$, then $\exists$ a pure EP of (F, K_1, K_2) which dominates (F_1, F_2) .

Proof: Suppose that (F_1, F_2) is an EP of (F, K_1, K_2) such that $T \in J_1 \cap J_2$. For any sequence $\{S_n\} \subset [0,T)$ if $T > 0$ and any sequence $\{S_n\} \subset (T,1]$ if $T < 1$, if S_n converges to T, then

$$K_i(\delta_{S_n}, F_j) = \int_{[0,S_n)} M_i(t)dF(t) + \alpha_j(S_n)\phi_i(S_n) + \int_{(S_n,1]} L_i(S_n)dF_j(t) .$$

But, by the Lebesgue Convergence Theorem and the fact that $\alpha_j(S_n) \to 0$,

$$\lim_{n\to\infty} K_i(\delta_{S_n}, F_j) = \int_{[0,T)} M_i(t)dF_j(t) + M_i(T)\alpha_j(T)\chi_{[T,1)}(S_n)$$

$$+ L_i(T)\alpha_j(T)\chi_{[0,T)}(S_n) + \int_{(T,1]} L_i(T)dF_j(t)$$

$$> \int_{[0,T)} M_i(t)dF_j(t) + \phi_i(T)\alpha_j(T) + \int_{(T,1]} L_i(T)dF_j(t)$$

$$= K_i(F_i, F_j) \quad \text{(by Lemma 1)}$$

if either $\phi_i(T) < M_i(T)$ for $T < 1$ (choose $\{S_n\} \subset (T,1]$) or $\phi_i(T) < L_i(T)$ for $T > 0$ (choose $\{S_n\} \subset [0,T)$) since $\alpha_j(T) > 0$. This would contradict the hypothesis that (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$. Thus $\phi_i(T) \geq M_i(T)$ if $T < 1$ and $\phi_i(T) \geq L_i(T)$ if $T > 0$, i.e., (δ_T, δ_T) is an EP of $(\mathcal{F}, K_1, K_2)$ (by Lemma 2).

Further suppose that $L_i(0) \leq M_i(0)$ for $i = 1, 2$ and that $\exists$ a common jump of (F_1, F_2).

Let $T = \text{Inf}\{S \in J_1 \cap J_2\}$. (δ_s, δ_s) is an EP $\forall s \in J_1 \cap J_2$ (by above). Thus (δ_T, δ_T) is an EP of $(\mathcal{F}, K_1, K_2)$ by the continuity of L_i, ϕ_i and M_i. It remains to show that (δ_T, δ_T) dominates (F_1, F_2).

If $T = 0$, then $\phi_i(0) \geq M_i(0) \geq L_i(0)$ (by above and by assumption respectively) implies that

$$K_i(\delta_0, \delta_0) = \phi_i(0) \geq \begin{cases} \alpha_j(0)\phi_i(0) + (1-\alpha_j(0))L_i(0) = H_i(T, F_j) \\ \qquad \text{if } T \in J_1 \cap J_2 \\ \alpha_j(0)M_i(0) + (1-\alpha_j(0))L_i(0) = H_i(T, F_j) \\ \qquad \text{if } T_n \in J_1 \cap J_2 \text{ and } T_n \to T \end{cases}$$

$$= K_i(F_i, F_j)$$

(by Lemma 1).

If $0 < T \leq 1$, then $\phi_i(T) \geq L_i(T)$ since (δ_T, δ_T) is an EP. Thus, by Lemma 1,

$$K_i(F_i, F_j) = \begin{cases} H_i(T, F_j) & \text{if } T \in J_1 \cap J_2 \\ \lim_{n\to\infty} H_i(T_n, F_j) & \text{if } T_n \notin J_1 \cap J_2, \quad T_n \to T \end{cases}$$

$$= \begin{cases} \int_{[0,T)} M_i(t)dF_j(t) + \phi_i(T)\alpha_j(T) + L_i(T)(1 - F_j(T)) \\ \int_{[0,T)} M_i(t)dF_j(t) + M_i(T)\alpha_j(T) + L_i(T)(1 - F_j(T)) \end{cases}$$

$$\leq \phi_i(T) = K_i(\delta_T, \delta_T)$$

since $\int_{[0,T)} M_i(t)dF_j(t)$ is non-zero only if T does not begin the support of F_j in which case $\phi_i(T) \geq L_i(T) > M_i(t)$ for all $t \in \text{Supp}(F_j) \cap [0,T)$ (by Lemma 4, since T is the earliest possible common jump).

Thus, (δ_T, δ_T) is an EP of (F, K_1, K_2) which dominates (F_1, F_2). ■

And so, (F, K_1, K_2) has an EP with a common jump if and only if (F, K_1, K_2) has a pure EP (see Lemma 5) if and only if ϕ_i, $i = 1, 2$ are both large for some $T \in [0,1]$ (large in the sense of Lemma 2). Another consequence of Lemma 5 is that, if $L_i(0) \leq M_i(0)$ for $i = 1, 2$, then it is only necessary to search among pure EP's of (F, K_1, K_2) and EP's of (F, K_1, K_2) without any common jumps for the existence of a dominating EP of (F, K_1, K_2).

Lemma 2 already gives us necessary and sufficient conditions for the existence of a pure EP of (F, K_1, K_2). It remains to find necessary and sufficient conditions for the existence of an EP of (F, K_1, K_2) with no jumps in common. Lemmas 6, 7A and 7B provide us with exactly this information.

The next lemma rules out the possibility of an EP among a certain class of strategy pairs with no common jumps.

Lemma 6: Suppose that (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$. If $J_1 \cap J_2 = \emptyset$, then $\mathrm{Supp}(F_1, F_2) = \{e(F_j)\}$ where $e(F_j) \le e(F_i)$ $i \ne j$.

Proof: By Lemma 3, $\mathrm{Supp}(F_1, F_2) = \mathrm{Supp}(F_j)$. Suppose that $\mathrm{Supp}(F_j)$ contains more than one point, i.e., let T begin $\mathrm{Supp}(F_1, F_2)$ and suppose that $T < e(F_j) \le e(F_i)$, $\{i,j\} = \{1,2\}$.

Since $T \in \mathrm{Supp}(F_1, F_2)$ is not a common jump, $\exists$ a sequence $\{S_n\} \subset \mathrm{Supp}(F_1, F_2) \cap (T, T+\varepsilon)$ for some $\varepsilon > 0$ such that $S_n \to T$ (by fact (E) and Lemma 3). Thus, by Lemma 1, $\exists$ a sequence $\{T_n\}$ satisfying the conclusion in (3) of Lemma 1. Thus $K_k(F_k, F_\ell) = \lim_{n\to\infty} H_k(T_n, F_\ell)$ $= M_k(T)F_\ell(T) + L_k(T)(1 - F_\ell(T))$ for $k = 1, 2$, $\{k,\ell\} = \{1,2\}$.

Similarly, since $e(F_j) \in \mathrm{Supp}(F_1, F_2)$ is not a common jump, $\exists$ a sequence $\{T_n'\}$ satisfying the conclusion in (2) of Lemma 1. Let ℓ be such that $\alpha_\ell(e(F_j)) = 0$. Thus $K_k(F_k, F_\ell) = \lim_{n\to\infty} H_k(T_n', F_\ell) = \int_{[0,e(F_j)]} M_k(t)dF_\ell(t)$

But then a contradiction results since

$$K_k(F_k, F_\ell) = M_k(T)F_\ell(T) + L_k(T)(1 - F_\ell(T))$$

$$> \int_{[T,e(F_j)]} M_k(t)dF_\ell(t) = K_k(F_k, F_\ell)$$

since $T \ge a_\ell$ (by Lemma 4), $F_\ell(T) < 1$ and M_k is strictly decreasing.

■

Among the strategy pairs without any common jumps, the only remaining candidates for an EP are those for which the initial support, $\mathrm{Supp}(F_1, F_2)$ is a singleton set $\{T\}$ such that $T \notin J_1 \cap J_2$. The next lemma stipulates exactly under what conditions this type of EP can occur.

Let Q be the set of strategy pairs of (F, K_1, K_2) without any jumps in common but with $\mathrm{Supp}(F_1, F_2) = \mathrm{Supp}(F_i) = \{T\}$ where $e(F_i) \leq e(F_j)$, i.e.,

$$Q = \{(F_1, F_2) \in F \times F : \mathrm{Supp}(F_1, F_2) = \mathrm{Supp}(F_i) = \{T\}$$
$$\text{where } e(F_i) \leq e(F_j),\ F_j(T) = 0,\ \alpha_i(T) = 1\} .$$

Note that, in the above, $T < 1$.

For the proof of the "if" part of Lemma 7A, see [Kilgour, 1973], [Kilgour, 1979].

Recall that Condition I states that $\lim_{t \to a_i} [L(t) - L(a_i)]/[L(t) - M(t)]$ exists and is strictly positive.

Lemma 7A: $\exists\ (F_1, F_2) \in Q$ such that (F_1, F_2) is an EP of (F, K_1, K_2) (satisfying Condition I) if and only if $\exists\ T \in [0,1)$ such that $a_i < T \leq a_j$ and $M_j(T) \geq \phi_j'(T)$.

Proof: "if" Suppose that $\exists\ T \in [0,1)$ such that $L_i(T) > M_i(T)$, $M_j(T) \geq L_j(T)$, $\phi_j(T)$. Let $F_i(T) - F_i(T-) = 1$. Let F_j be any absolutely continuous distribution such that $F_j(T) = 0$ and

$$F_j(t) \geq \frac{L_i(t) - L_i(T)}{L_i(t) - M_i(T)} \quad \text{in} \quad (T,1]$$

(such an F_j exists since $L_i(t) > M_i(T)$ for $t \in [T,1]$). Then

$$K_j(F_j, F_i) = M_j(T) \geq \begin{cases} L_j(t) & \text{if } t < T \\ \phi_j(T) & \text{if } t = T \\ M_j(T) & \text{if } t > T \end{cases}$$

$$= K_j(\delta_t, F_j) \quad \forall\, t \in [0,1] ,$$

i.e., F_j is a best response for P_j against F_i .

Also $K_i(F_i, F_j) = L_i(T) \geq M_i(T)F_j(t) + L_i(t)(1 - F_j(t))$ (by assumption) $\geq \int_{[T,t)} M_i(s)dF_j(s) + L_i(t)(1 - F_j(t)) = K_i(\delta_t, F_j)$ $\forall\, t \in [T,1]$ since M_i is decreasing and $K_i(F_i, F_j) = L_i(T) > L_i(t) = K_i(\delta_t, F_j)$ $\forall\, t \in [0,T)$, i.e., F_i is a best response for P_i against F_j .

Thus, $(F_1, F_2) \in Q$ is an EP of (F, K_1, K_2) .

"only if" Suppose that $(F_1, F_2) \in Q$ is an EP of (F, K_1, K_2) . By definition of EP,

$$M_j(T) = K_j(F_i, F_j) \geq K_j(\delta_t, F_i) = \begin{cases} L_j(t) & \text{if } t < T \\ \phi_j(T) & \text{if } t = T \, . \\ M_j(T) & \text{if } t > T \end{cases}$$

Also, $T \in \text{Supp}(F_1, F_2)$, T not a common jump, $T < e(F_j)$ implies that $T \geq a_i$ (by Lemma 4). Thus $a_i \leq T \leq a_j$ and $M_j(T) \geq \phi_j(T)$.

It remains to show that T must be strictly larger than a_i . Suppose, to the contrary, that $L_i(T) = M_i(T)$. Then

$$L_i(T) = K_i(F_i, F_j) \geq K_i(\delta_t, F_j) = \int_{[T,t)} M_i(S)dF_j(S) + F_j(t)(1 - L_i(t))$$

$$\geq M_i(t)F_j(t) + F_j(t)(1 - L_i(t)) ,$$

for all but at most a countable set of $t \in J_j$, implies that

$F_j(t) \geq [L_i(t) - L_i(T)]/[L_i(t) - M_i(t)]$ which implies that

$\lim_{t \to T^+} F_j(t) \geq \lim_{t \to a_i} [L_i(t) - L_i(T)]/[L_i(t) - M_i(t)] > 0$ which contradicts the hypothesis that $F_j(T) = 0$ since F_j must be right-continuous. ■

The counterpart of Lemma 7A uses Condition II. Recall that Condition II states that $a_i = 0$ and *either* $L_i(0) > M_i(0)$ *or* $L_i(0) = M_i(0)$ and $\exists\ \varepsilon > 0$ such that L_i is differentiable in $(0, \varepsilon]$ and $L_i'(t)/[L_i(t) - M_i(t)]$ is bounded for $t \in (0, \varepsilon]$.

<u>Lemma 7B</u>: $\exists\ (F_1, F_2) \in \mathcal{Q}$ such that (F_1, F_2) is an EP of $(\mathcal{F}, K_1, K_2)$ (satisfying Condition II) if and only if $\exists\ T \in [0,1)$ such that $a_i \leq T \leq a_j$ and $M_j(T) \geq \phi_j(T)$.

<u>Proof</u>: "only if" $M_j(T) \geq L_j(T), \phi_j(T)$, as in Lemma 7A. Also $T < e(F_j)$ implies that $T \geq a_i$ (by Lemma 4). Thus $a_i \leq T \leq a_j$ and $M_j(T) \geq \phi_j(T)$.

"if" if either $0 < T \leq a_j$ and $M_j(T) \geq \phi_j(T)$ or $T = 0$ and $L_i(0) > M_i(0)$, $M_j(0) \geq \phi_j(0)$, then $(\delta_T, F_j) \in \mathcal{Q}$ as in the "if" part of Lemma 7A will be an EP of $(\mathcal{F}, K_1, K_2)$.

It remains to show that if $T = a_i = 0$ and $M_j(T) \geq \phi_j(T)$ then $\exists\ F_j \in \mathcal{F}$ such that $(\delta_T, F_j) \in \mathcal{Q}$ is an EP of $(\mathcal{F}, K_1, K_2)$.

Choose $F_j \in \mathcal{F}$ as follows. Let F_j be any absolutely continuous distribution such that $F_j(0) = 0$, $F_j(\varepsilon) = 1$ and such that $F_j'(t) > L_i'(t)/[L_i(t) - M_i(t)]$ in $(0, \varepsilon]$. $K_j(F_j, \delta_j) \geq K_j(F, \delta_0)$ for all $F \in \mathcal{F}$ by assumptions on M_j. It remains to show that δ_0 is best for P_i against F_j.

$$K_i(\delta_t, F_j) = \int_{[0,t)} M_i(S) dF_j(S) + L_i(t)(1 - F_j(t)) .$$

The derivative of $K_i(\delta_t, F_j)$ with respect to t is

$M_i(t)F_j'(t) + L_i'(t)(1 - F_j(t)) - L_i(t)F_j'(t) < 0$ by assumption on $F_j'(t)$ for all $t \in (0,\varepsilon]$. But $K_i(\delta_0, F_j) = L_i(0)$. Thus $K_i(\delta_t, F_j) < K_i(\delta_0, F_j)$ for all $t \in (0,\varepsilon]$. Also $K_i(\delta_t, F_j) = \int_{[0,e(F_j)]} M_i(s)dF_j(s) < L_i(0)$

for all $t \in (\varepsilon,1]$. Thus, δ_0 is best for P_i against F_j. Therefore $(\delta_0, F_j) \in \mathcal{Q}$ is an EP of (F, K_1, K_2). ∎

And so, by Lemmas 6, 7A and 7B, the only candidates for an EP of (F, K_1, K_2) without any common jumps are those strategy pairs whose initial common support is a singleton set $\{T\}$ such that $T \in [0,1)$, $M_j(T) \geq \phi_j(T)$ and $a_i < T \leq a_j$ if Condition I holds ($a_i = 0 \leq T \leq a_j$ if Condition II holds).

We are now ready to state necessary and sufficient conditions for the existence of an EP of (F, K_1, K_2).

Theorem 8: The game of timing, (F, K_1, K_2), has an EP if and only if there exists a point $T \in [0,1]$ such that

either (i) $T = 0$ and $\phi_i(0) \geq M_i(0)$ for $i = 1, 2$

or (ii) $0 < T < 1$ and $\phi_i(T) \geq \mathrm{Max}\{L_i(T), M_i(T)\}$ for $i = 1, 2$

or (iii) $T = 1$ and $\phi_i(1) \geq L_i(1)$ for $i = 1, 2$

or (iv) $a_i < T \leq a_j$ and $M_j(T) \geq \phi_j(T)$ for $i \neq j$

or (v) $0 = a_i = T \leq a_j$, $M_j(T) \geq \phi_j(T)$ for $i \neq j$ and Condition II holds (i.e., *either* $L_i(0) > M_i(0)$ *or* the derivative of L_i exists in some interval $(0,\varepsilon]$ for $\varepsilon > 0$ and $L_i'(t)/[L_i(t) - M_i(t)]$ is bounded in $(0,\varepsilon]$.

Proof: "if" If (i), (ii) or (iii) is true, then (δ_T, δ_T) is an EP of (F, K_1, K_2) by Lemma 2. If (iv) is true, then $\exists\ (F_1, F_2) \in Q$ which is an EP of (F, K_1, K_2) by the "if" parts of Lemmas 7A and 7B. If (v) is true, then $\exists\ (F_1, F_2) \in Q$ which is an EP of (F, K_1, K_2) by the "if" part of Lemma 7B.

"only if" Suppose that (F_1, F_2) is an EP of (F, K_1, K_2). If there exists a point $T \in J_1 \cap J_2$, then (δ_T, δ_T) is an EP (by Lemma 5) which implies, by Lemma 2, that one of (i), (ii) or (iii) is true. If $J_1 \cap J_2 = \emptyset$, then (iv) or (v) is true by Lemmas 6, 7A and 7B. ∎

5. Dominance Theorem

Let us assume that, in this section, the game of timing, (F, K_1, K_2) under consideration also satisfies condition (iii) in the Introduction, i.e., in addition to the continuity and monotonicity conditions, the kernel also satisfies the condition that $L_i(0) \leq M_i(0)$ for $i = 1, 2$. Thus far, this condition was assumed only in the second part of Lemma 5 which established that, if $L_i(0) \leq M_i(0)$ for $i = 1, 2$, then the existence of a common jump in an EP (F_1, F_2) of (F, K_1, K_2) implies the existence of a pure EP (δ_T, δ_T) of (F, K_1, K_2) which dominates (F_1, F_2). [If $L_i(0) > M_i(0)$ for some i, then this is not necessarily true, i.e., there may then exist an EP (F_1, F_2) with a jump in common even though no pure EP dominates (F_1, F_2).]

Theorem 10, in this section, will provide necessary and sufficient conditions for the existence of a dominating EP in (F, K_1, K_2). Before we proceed, we need some additional notation. Let

$$Q = \{S \in [0,1] : \exists \text{ EP } (F_1, F_2) \in Q \text{ with } \mathrm{Supp}(F_1, F_2) = \{S\}\} .$$

Thus, if Condition I holds, then

$$Q = \{S \in (0,1) : a_i < S \leq a_j,\ M_j(S) \geq \phi_j(S)\} ;$$

while, if Condition II holds, then

$$Q = \{S \in [0,1) : a_i = 0 \leq S \leq a_j,\ M_j(S) \geq \phi_j(S)\} .$$

Let

$$P = \{S \in [0,1] : (\delta_S, \delta_S) \text{ is an EP of } (F, K_1, K_2)\} .$$

Now, Theorem 8 can be restated as follows: An EP of (F, K_1, K_2) exists if and only if $P \cup Q \neq \emptyset$. The next lemma states that, if $\exists$ EP $\in Q$ which is a dominating EP of (F, K_1, K_2) , then Q is a singleton set.

<u>Lemma 9</u>: Suppose that Q contains more than one point. If $(F_1, F_2) \in Q$ is an EP of (F, K_1, K_2) , then (F_1, F_2) is not a dominant EP of (F, K_1, K_2) .

<u>Proof</u>: Let $\text{Supp}(F_1, F_2) = \{T\}$ for some $T \in Q$. The payoffs to P_i and P_j are $L_i(T)$, $M_j(T)$ respectively, for $i \neq j$ (see Lemmas 7A, 7B). There exists a point $S \neq T$, $S \in Q$ such that *either* both $L_i(T) < L_i(S)$ and $M_j(T) > M_j(S)$ *or* both $L_i(T) > L_i(S)$ and $M_j(T) < M_j(S)$. This is due to the monotonicity of L_i and M_j and to the assumption that Q contains more than one point. By Lemmas 7A, 7B there exists an EP (of (F, K_1, K_2)) $(G_1, G_2) \in Q$ such that $K_i(G_i, G_j) = L_i(S)$ and $K_j(G_j, G_i) = M_j(S)$. The EP (F_1, F_2) does not dominate the EP (G_1, G_2) . ∎

We are now prepared to state the conditions necessary and sufficient for the existence of a dominating EP of (F, K_1, K_2) .

Theorem 10: **A dominating EP of (F, K_1, K_2) exists if and only if** *either* (i) $\exists\, p \in P$ such that $\phi_i(p) \geq \phi_i(T)$ $\forall T \in P$, $i = 1, 2$ and $\phi_i(p) \geq L_i(T)$, $\phi_j(p) \geq M_j(T)$ for all $T \in Q$, $i \neq j$ (in which case, (δ_p, δ_p) is a dominating EP of (F, K_1, K_2))

or (ii) $Q = \{q\}$ and $L_i(q) \geq \phi_i(T)$, $M_j(q) \geq \phi_j(T)$ $\forall T \in P$, $i \neq j$ (in which case $\exists\, (F_1, F_2) \in Q$ which is a dominating EP of (F, K_1, K_2)).

Proof: "if" Suppose (i) or (ii) is true. Let (F_1, F_2) be the EP of (F, K_1, K_2) with respective payoffs to P_i , P_j , $i \neq j$, equal to $\phi_i(p)$, $\phi_j(p)$ if (i) is true (equal to $L_i(q)$, $M_j(q)$ if (ii) is true). Let (G_1, G_2) be any EP of (F, K_1, K_2) . If there exists a common jump in $\text{Supp}(G_1, G_2)$ then, by Lemma 5, there exists a pure EP of (F, K_1, K_2) which dominates (G_1, G_2) . But, by assumption, (F_1, F_2) dominates all pure EP's of (F, K_1, K_2) . Thus (F_1, F_2) dominates (G_1, G_2) . If there does not exist a common jump in $\text{Supp}(G_1, G_2)$ then, by Lemmas 7A and 7B, the payoffs to P_i and P_j from (G_1, G_2) are $L_i(T)$, $M_j(T)$ respectively, for some $T \in Q$. Thus (F_1, F_2) dominates (G_1, G_2) .

"only if" Suppose that (F_1, F_2) is a dominating EP of (F, K_1, K_2) (1) If $\exists\, p \in J_1 \cap J_2$, ((2) if $J_1 \cap J_2 = \emptyset$) then $K_i(F_i, F_j) = \phi_i(p)$ for $i = 1, 2$ by Lemma 5; since, otherwise, $K_i(F_i, F_j) < \phi_i(p)$ for $i = 1, 2$ for a pure EP (δ_p, δ_p) contradicts the dominancy of (F_1, F_2) (then $K_i(F_i, F_j) = L_i(q)$ and $K_j(F_j, F_i) = M_j(q)$ for $\{q\} = Q$ by Lemmas 6, 7A, 7B and 9). Thus (1) if $\exists\, p \in J_1 \cap J_2$ ((2) if $J_1 \cap J_2 = \emptyset$)

then (i)((ii)) must be true since $\forall\ T \in P\ \exists$ EP payoffs of $\phi_i(T)$ for P_i for $i = 1, 2$ and $\forall\ T \in Q\ \exists$ EP payoffs of $L_i(T)$, $M_j(T)$ for P_i , P_j respectively, $i \neq j$, by Theorem 8. ∎

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