COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

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COWLES FOUNDATION DISCUSSION PAPER No. 504

EQUITY, EFFICIENCY AND INCREASING RETURNS

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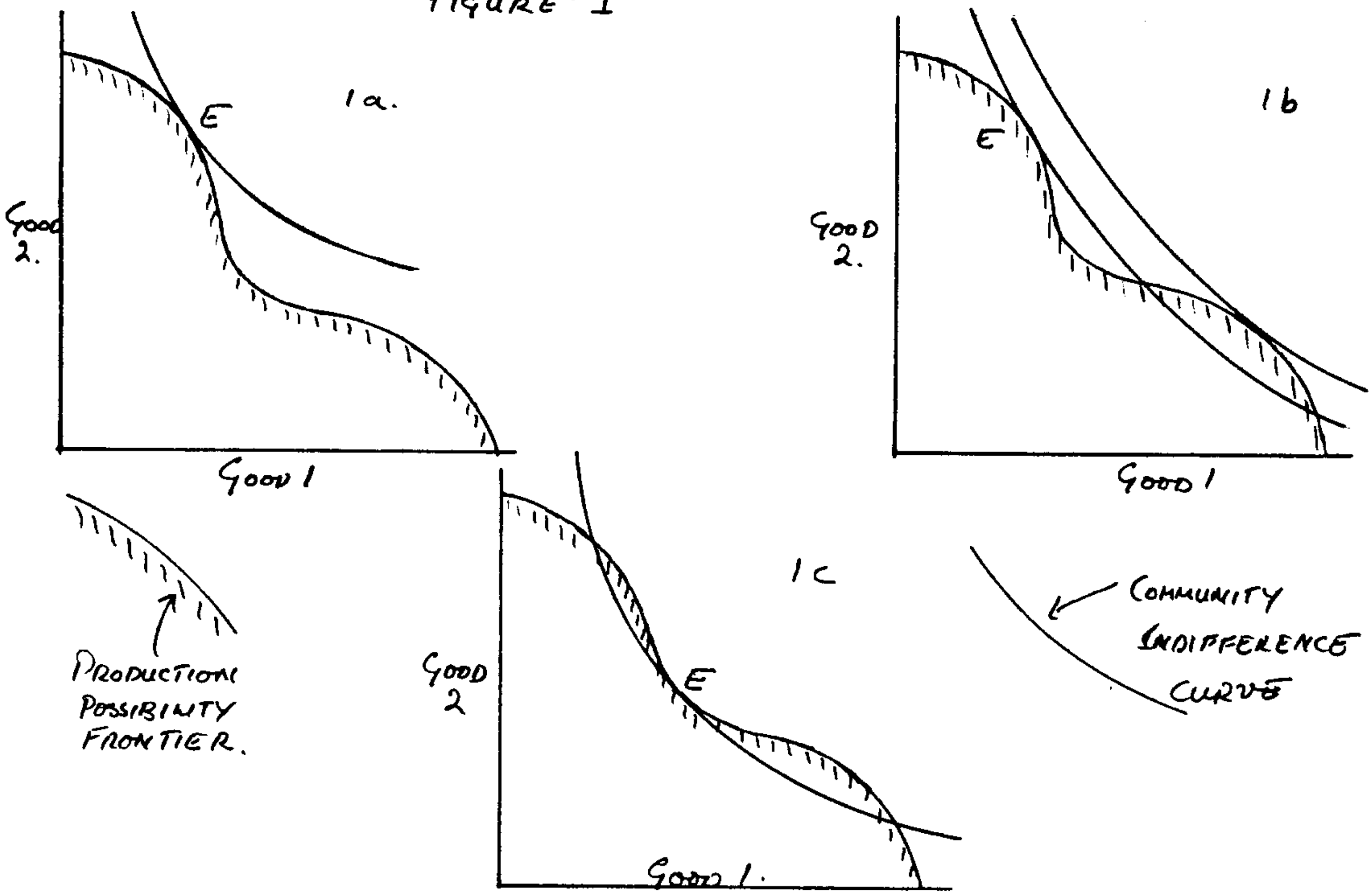
Revised, September 1978

This research was supported in parts by grants from the National Science Foundation and the Ford Foundation to the Universities of Yale and Stanford. We would like to acknowledge the considerable benefits of discussions with Robert Eastwood, Roger Guesnerie, Peter Hammond, Al Klevorick, Kotaro Suzumura, Rolf Mantel and Sidney Winter.

1. Introduction

We shall be concerned in this paper with certain properties of equilibria in which firms with increasing returns to scale in production sell their outputs at prices equal to marginal costs. We are in other words concerned with equilibria involving increasing returns and at which the standard first-order or marginal conditions for optimality are met. Figures (1a) to (1c) illustrate such equilibria in a two-good economy. It is not our concern to prove the existence of such equilibria: this we have done elsewhere {1} and existence is here assumed. Indeed, the paper will focus on examples sufficiently simple that no formal proof of existence is required. Another very complex issue that will not be our concern, is the possibility of supporting and implementing an equilibrium of this sort. There is of course a problem here, because it may require some firms to operate at a loss, which will then have to be covered by non-distortionary levies on other agents if the first-order conditions are to be preserved. Again, this is a matter that we have discussed elsewhere {2}, demonstrating that such an equilibrium can be sustained without the need for conventional lump-sum taxes. Because it is not at present our main concern, we deal with the problem here in a slightly cavalier way, and assume all shareholdings to involve unlimited liability. The owners of a firm therefore share both in its profits and its losses, and losses are covered from shareholders' incomes. In the examples considered below there is a single firm, owned entirely by one of two consumers, and it will become clear that at an efficient equilibrium it would not be in the owner's interest to close it down, even though it makes a loss. This is just a particular example of a more general proposition discussed in

FIGURE 1


1a.
E
GOOD 2.
GOOD 1
1b
E
GOOD 2.
GOOD 1
1c
E
GOOD 2
GOOD 1.
PRODUCTION POSSIBILITY FRONTIER.
COMMUNITY INDIFFERENCE CURVE

(3): the point is that the income saving from closure would be outweighed by the loss of consumer surplus otherwise provided by having the good available at marginal cost.

The focus of this paper, then, is not on the existence or decentralisation of an equilibrium satisfying the first-order conditions for optimality, but rather on its optimality. It is clear from figure 1 above, that such an equilibrium need not be optimal, for the obvious reason that with non-convex production possibility sets, the first-order conditions are not sufficient to ensure optimality. We shall not in fact be concerned with an analysis of the sufficiency of these conditions: our concern is rather with the question: is it the case that there is always at least one equilibrium which satisfies the first-order conditions and is Pareto optimal?

The earlier diagrams, and one's intuition, suggest an affirmative answer. In fact Guesnerie (4) has already shown the answer to be negative: there may be economies with many equilibria, all of which satisfy the first-order conditions and yet are inefficient. Our purpose here is to analyse an example which makes it possible to illustrate this point more fully and simply than has been done hitherto, and also to set out some of the very significant implications that this finding has for the welfare economics of economies with increasing returns - which include, of course, most economies for which welfare economics is ever likely to be practiced. Finally, we shall set out certain conditions sufficient to ensure that this problem does not arise. These conditions will imply the existence of at least one efficient equilibrium, and thus in a certain sense guarantee that the system is well-behaved.

In particular, we shall show that the answer to the question "Is there always an efficient equilibrium?" depends upon the distribution of initial endowments of both goods and services and of claims upon the profits (or liabilities to the losses) of firms. For some such distributions, the answer is affirmative, and for others, negative. That is, for a given set of firms and consumers, and a given total of initial physical endowments, some distributions of that total and of claims upon firms will ensure the existence of at least one Pareto optimal equilibrium, whereas other distributions will ensure that no equilibruum is optimal. We thus have to recognise that some distributions are efficient (in the sense of permitting the attainment of optimality), and others are not. This is in very sharp distinction to the standard competitive case, where one judges between alternative distributions purely on the grounds of equity, confident that the efficiency or otherwise of an outcome resulting from a given endowment distribution is purely a function of the allocative system used to achieve it. It is thus a standard proposition in the Arrow-Debreu model that, given any distribution of endowments satisfying certain technical conditions, there exists an associated competitive equilibrium which is efficient. Equity and efficiency are therefore independent dimensions, and much of our accepted welfare economics and cost-benefit analysis rests, explicitly or implicitly, on this fact. The examples we present below demonstrate that, once one admits increasing returns, the situation is fundamentally different. Because some are efficient and others inefficient, one can no longer judge between alternative distributions purely in terms of equity. It is necessary to consider both the equity and the efficiency dimensions simultaneously.

At the risk of seeming repetitive, we put this point in another way. In a world where firms display non-increasing returns in production, a suggested pattern of endowments of goods and services and of claims on firms, provided it permits the existence of an equilibrium, can be rejected only on the grounds of a judgement that it is inequitable. With increasing returns, however, one has also the option of rejecting some proposed endowment patterns on the grounds that they are inefficient. We shall go on to demonstrate an even more surprising point, namely, that with increasing returns in production, it may be possible to remove some endowment from one person, give it to another, and make both better off. In other words, an unrequited transfer of property from one agent to another - an expropriation - may make both better off. Though paradoxical at first sight, this point is in fact easily understood once one realises that there may be efficient and inefficient distributions of endowments. The kind of paradox just referred to arises when the initial distribution is inefficient, but the distribution after the unrequited transfer is efficient. In such a case, the transfer will move the economy from an equilibrium inside the utility possibility frontier, to one on the frontier, and it is possible that in the process all may gain.

A natural reaction to these paradoxical results, is to enquire whether they are general properties of all economies with increasing returns. hile we are far from being in a position to give a complete answer on this point, in the later sections of the paper we do present a condition weaker than convexity which is sufficient to ensure that there is always an efficient equilibrium. This exploits the intuitively obvious fact that in a one-person economy the problem cannot occur, to show that if individuals' preferences are in a certain sense similar (i.e. can be aggregated), then there is always an efficient outcome.

These observations about the interrelationship between equity and efficiency, and the possibility of unrequited Pareto-improving transfers, are somewhat surprising, and not readily linked to the earlier literature in this field. It may however be worth noting that there is a line of argument that has been advanced in development economics and mentioned to us by people in that area, which is reminiscent of our conclusions. This is an argument to the effect that in an under-developed country with a highly unequal distribution of income, a redistribution away from the very rich may in the long-run make all better off, because the acquisition of purchasing power by the middle and lower income groups may lead to the development of a mass market and a substantial increase in industrial profits. It seems that such an argument has not been formalised, but that increasing returns in production would be an essential ingredient of any attempt to do so.

The remainder of the paper is divided into six sections. In section 2 we outline the type of model and of argument to be used. In section 3 we give a geometric, and in section 4 a mathematical, treatment of the model. Both are included because the geometric analysis probably makes the intuitive basis of the work much clearer, but a formal mathematical treatment seems needed to confirm the results suggested by the diagrams. In section 5 we develop the alternative conditions sufficient to ensure the existence of at least one efficient equilibrium, and in section 6 we survey the implications of the work, and the major unanswered questions.

2. An Outline

We turn now at some length to the first issue raised above, and discuss how to demonstrate that there are economies for which there is no Pareto efficient (P.E.) equilibrium. We shall also show that this inefficiency of the equilibrium is very sensitive to the distribution of endowments. The basic intuition may be given as follows. Figure 2 shows an economy for which the feasible set Y is non-convex.

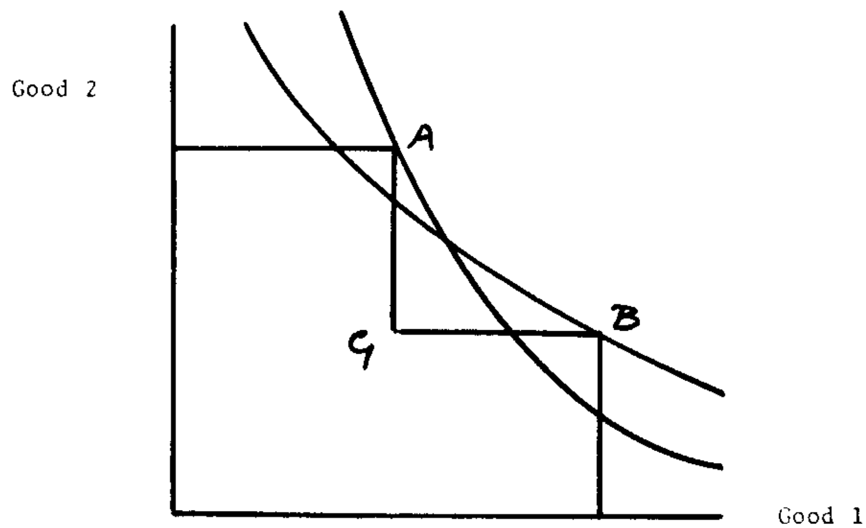


FIGURE 2

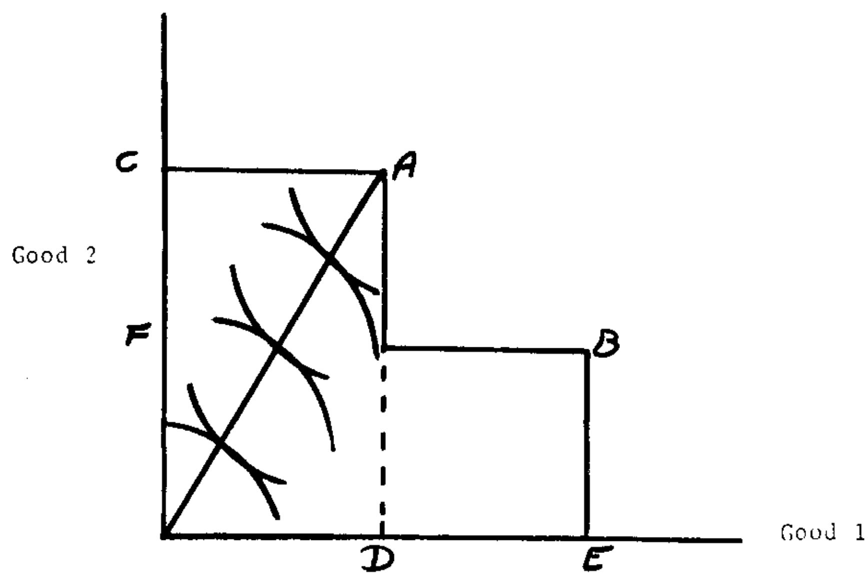


FIGURE 3

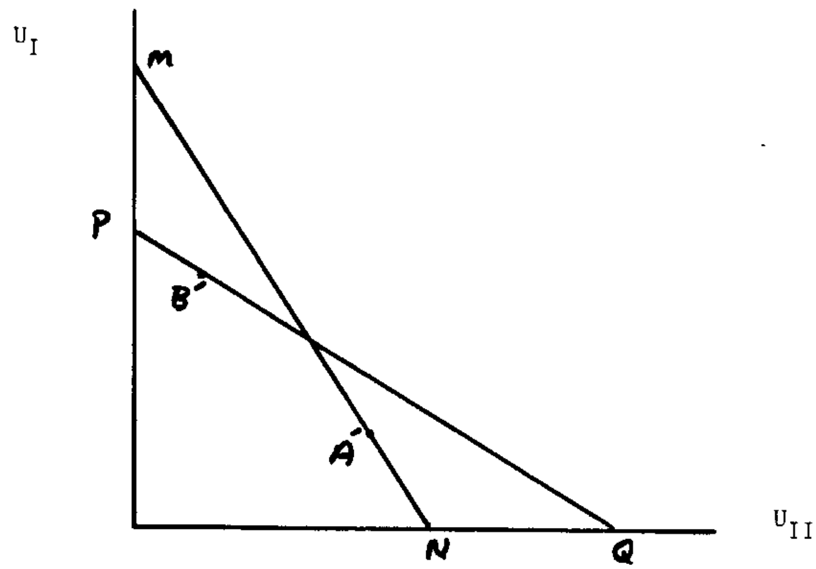


FIGURE 4

Clearly the equilibria here must be at A or B. (There may be an equilibrium at G, but as this is clearly inefficient we ignore it.) We suppose there to be two consumers (I and II) and draw the Scitovsky community indifference curve through A. This passes inside B, which is therefore Pareto-superior to A. It is therefore natural to suppose B to be Pareto-efficient. But in moving to B, relative prices change, as therefore do the distributions of income and wealth. The community indifference curves corresponding to this new distribution will in general intersect those corresponding to the previous distribution, and indeed the curve drawn through B has this property. It furthermore passes inside A, which is now Pareto-superior to B. Hence neither point is Pareto-efficient, and there is therefore no efficient equilibrium.

We shall show below how to construct an economy for which the above phenomenon occurs. Our approach will be primarily geometrical, and it will be useful if we outline in advance the way the argument will be developed. We consider a two-person, two-good model with a production set Y as represented in figure 2. The first step will be to construct the utility possibility frontier for this sytem. To do this, consider an equilibrium at point A. The possible efficient distributions of the net output represented by A amongst the two consumers, are shown by the contract curve of an Edgeworth box OCAD in figure 3. Given a numerical representation of the two preferences, this contract curve can be transposed to a curve in utility space. The curve MN in figure 4 thus corresponds to the contract curve OA in figure 3. It is clear that the equivalent curve corresponding to production at any other point in the rectangle OCAD will not lie anywhere above MN, as all other points in the rectangle correspond to smaller commodity bundles. We can now repeat this exercise for production at point B, using OFBE as an Edgeworth box. The line PQ in figure 4 corresponds to this new contract curve, and once again it is clear that this dominates the corresponding line for production at any other point in the rectangle OFBE. Hence the utility possibility frontier is the outer envelope of MN and PQ.

All that now remains to be shown is that the distribution of utilities that occurs in an equilibrium at A, corresponds to an inefficient point such as A' on MN, and likewise that an equilibrium at B leads to B' on PQ. When we have done this, we shall also be able to show that different patterns of initial endowments give rise to equilibria on the utility possibility frontier, some of which are Pareto-superior to A' or B'. This will of course imply:-

(a) that certain endowment patterns are, in the context of the present equilibrium concept, less efficient than others; and

(b) that an unrequited transfer of endowments from one consumer to another may in certain circumstances make it possible to make both better off.

3. A Geometric Analysis

We shall consider an economy with two goods, x and y, y an input produced from x according to

$$y = \begin{cases} 0, & \text{if } x < 7 \\ 7, & \text{if } x \geq 7 \end{cases} \tag{1}$$

The two individuals, I and II, have utility functions given by

$$U_I = \begin{cases} (y + \frac{4}{3}x)\frac{3\sqrt{2}}{7} & \text{if } y \geq x \\ (y + 0.04x).\frac{2\cdot 24.3\sqrt{2}}{7} & \text{if } y \leq x \end{cases} \tag{2}$$

$$U_{II} = \begin{cases} (3y + 5x)\frac{\sqrt{10}}{18} & \text{if } y \geq \frac{1}{3}x \\ \frac{x\sqrt{10}}{3} & \text{if } y = \frac{1}{3}x \\ y\sqrt{10} & \text{if } y \leq \frac{1}{3}x \end{cases} \tag{3}$$

In spite of their apparent complexity, these are really very simple functions. They give piece-wise linear indifference curves with slopes of, in the case of I, $-\frac{4}{3}$ and $-\frac{4}{100}$, and in the case of II, $-\frac{5}{3}$ and 0. These join along the lines $y = x$ and $y = \frac{1}{3}x$ respectively, and the particular numerical representations chosen ensure that the utility level of any indifference curve is given by the distance from the origin to the point where it crosses either $y = x$ or $y = \frac{1}{3}x$ respectively.

Initial endowments are

$$W_I = (0, 5) \qquad\qquad W_{II} = (15, 0)$$

so that the total endowment is (15, 5). It remains to specify ownership of the firm: this is owned entirely by I, and all shareholdings carry unlimited liability. I therefore receives all profits and meets all losses.

Figure 5 shows the economy's production possibility set, and figure 6 shows $Y = (Y' + W) \cap R^n_{\geq 0}$. The two possible equilibria are evidently at B and D in figure 6, and it is clear that B is a point at which no production occurs. There is therefore one equilibrium where the only activity is trade in the initial endowments, and a second equilibrium with production. The indifference curves implied by (2) and (3) are shown in figure 7: recall that the particular numerical representations chosen imply that I's utility can be measured by distance along the 45° line, and II's can be measured by that along the line $y = \frac{x}{3}$.

Figure 8 shows the Edgeworth boxes corresponding to production at points D and B, respectively: the piece-wise linear contract curves are indicated. In figure 9 we see the utility possibility curves implied by these contract curves: the utility possibility frontier is, by our earlier arguments, the outer envelope of these.

In figure 10, we see the two equilibrium distributions corresponding to production at D or at B: these are labelled E_D and E_B, respectively. The endowments of the two individuals are indicated by W_I and W_{II}, and as B involves no production and hence neither profit nor loss, I's budget line must pass through W_I. The contract curve is OB, and the equilibrium price ratio must equal the slope of I's indifference curves in the region $x \geq y$. Given

FIGURE 5.

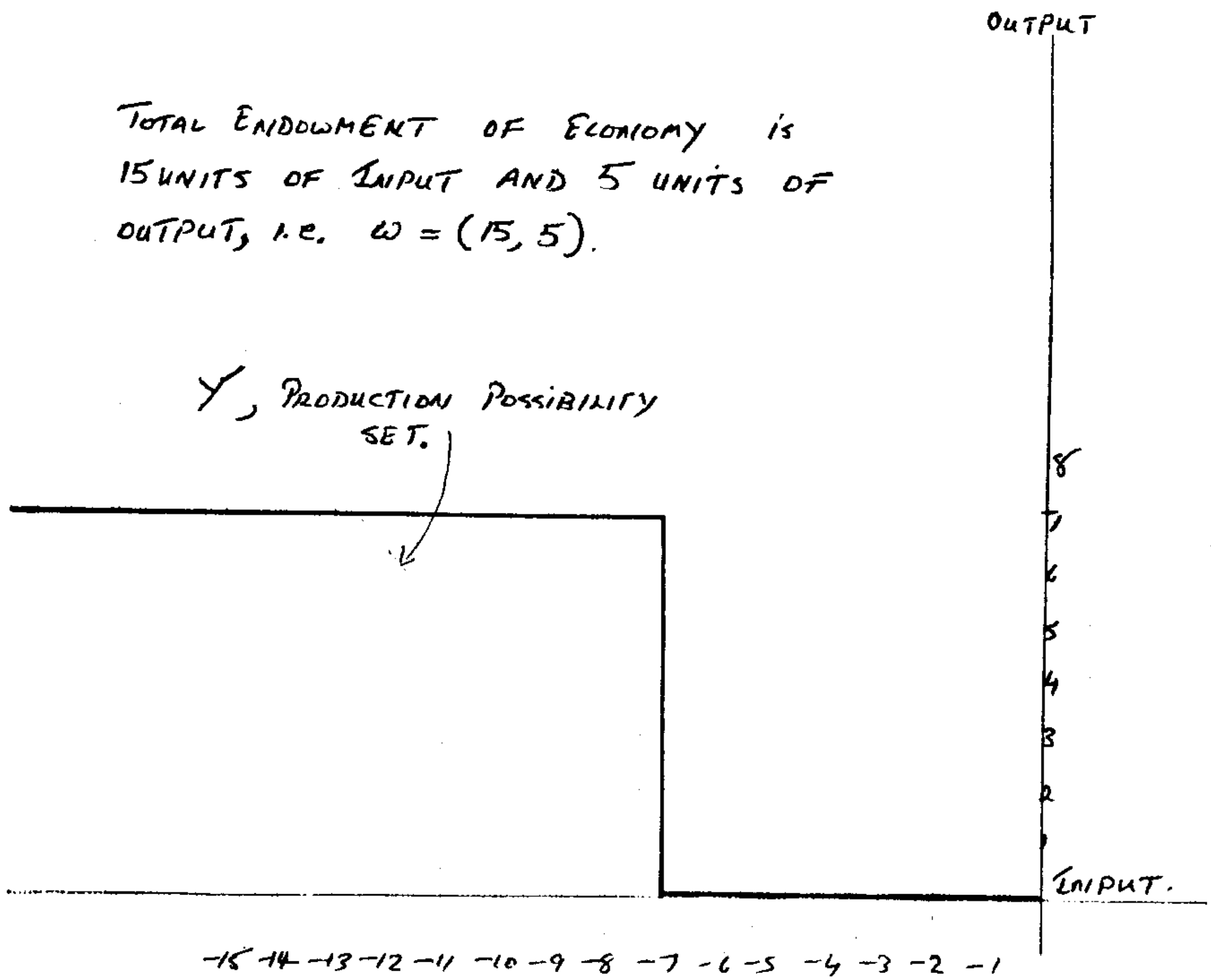

OUTPUT
TOTAL ENDOWMENT OF ECONOMY IS
15 UNITS OF INPUT AND 5 UNITS OF
OUTPUT, I.E. ω = (15, 5).
Y, PRODUCTION POSSIBILITY
SET.
8
7
6
5
4
3
2
1
INPUT.
-15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1

FIGURE 6

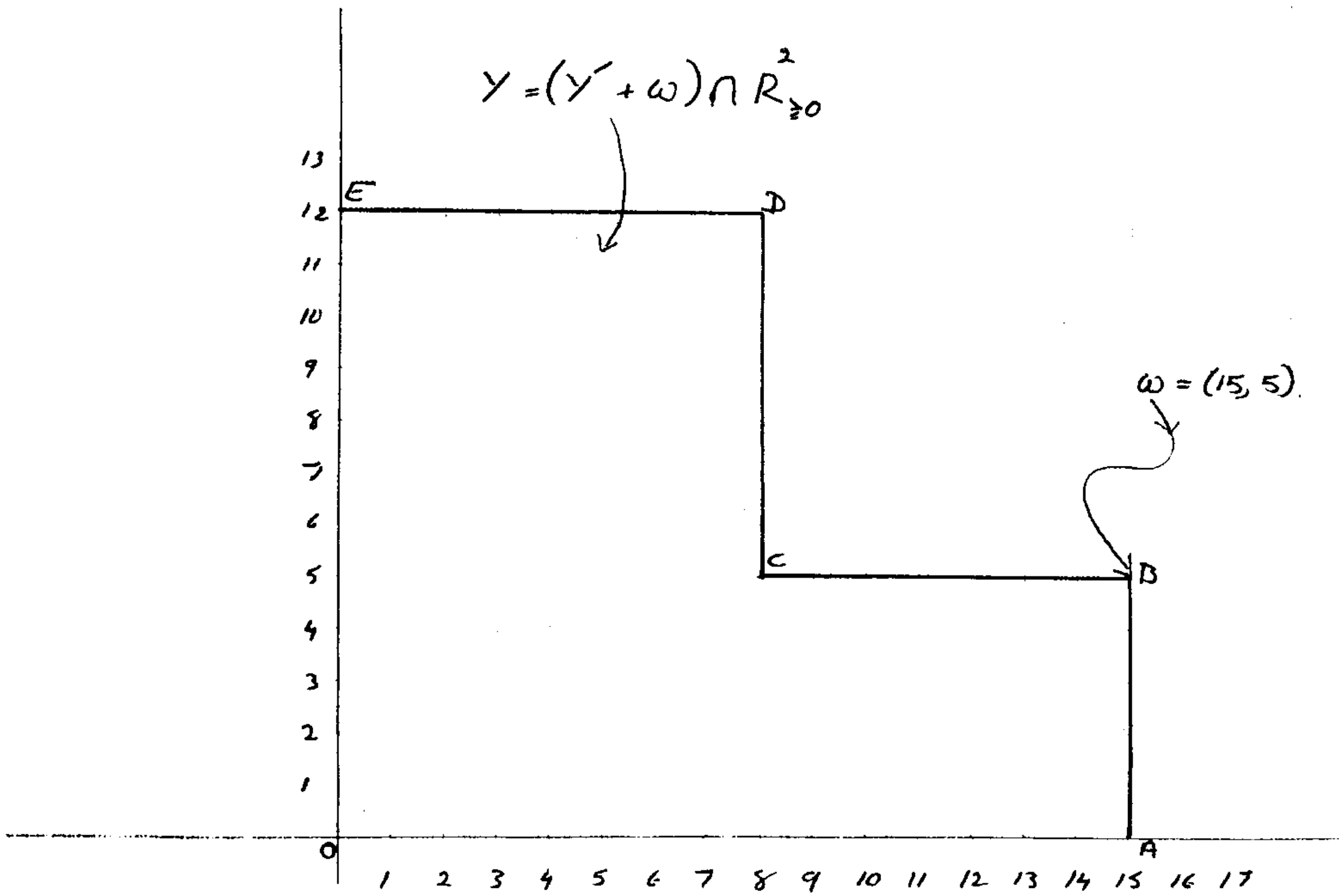

$Y = (Y' + \omega) \cap R^2_{\geq 0}$
$\omega = (15, 5)$
E
D
C
B
A
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
1 2 3 4 5 6 7 8 9 10 11 12 13

FIGURE 7
PREFERENCES.

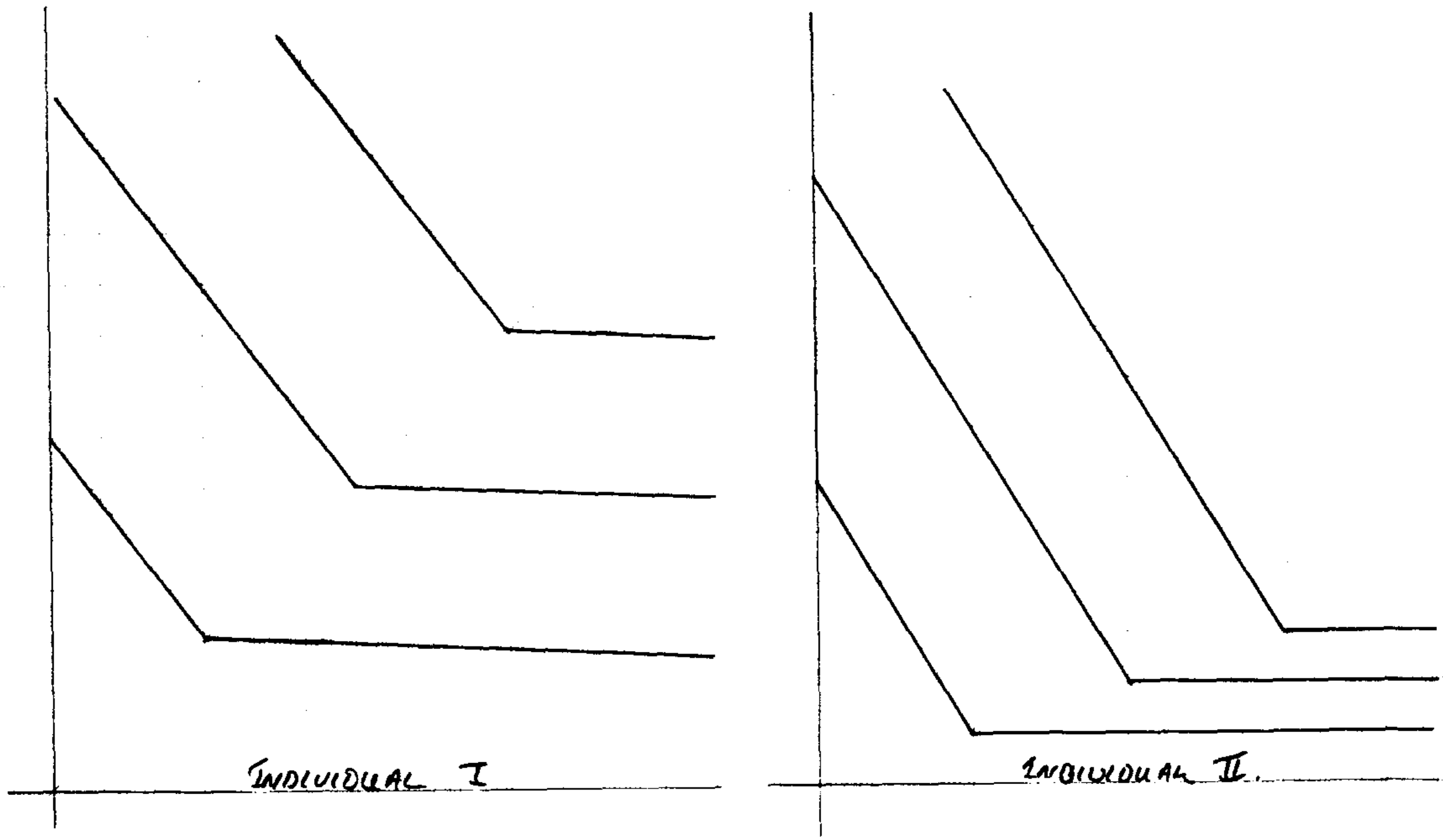


FIGURE 8

EDGEWORTH BOXES FOR PRODUCTION AT D AND AT B.

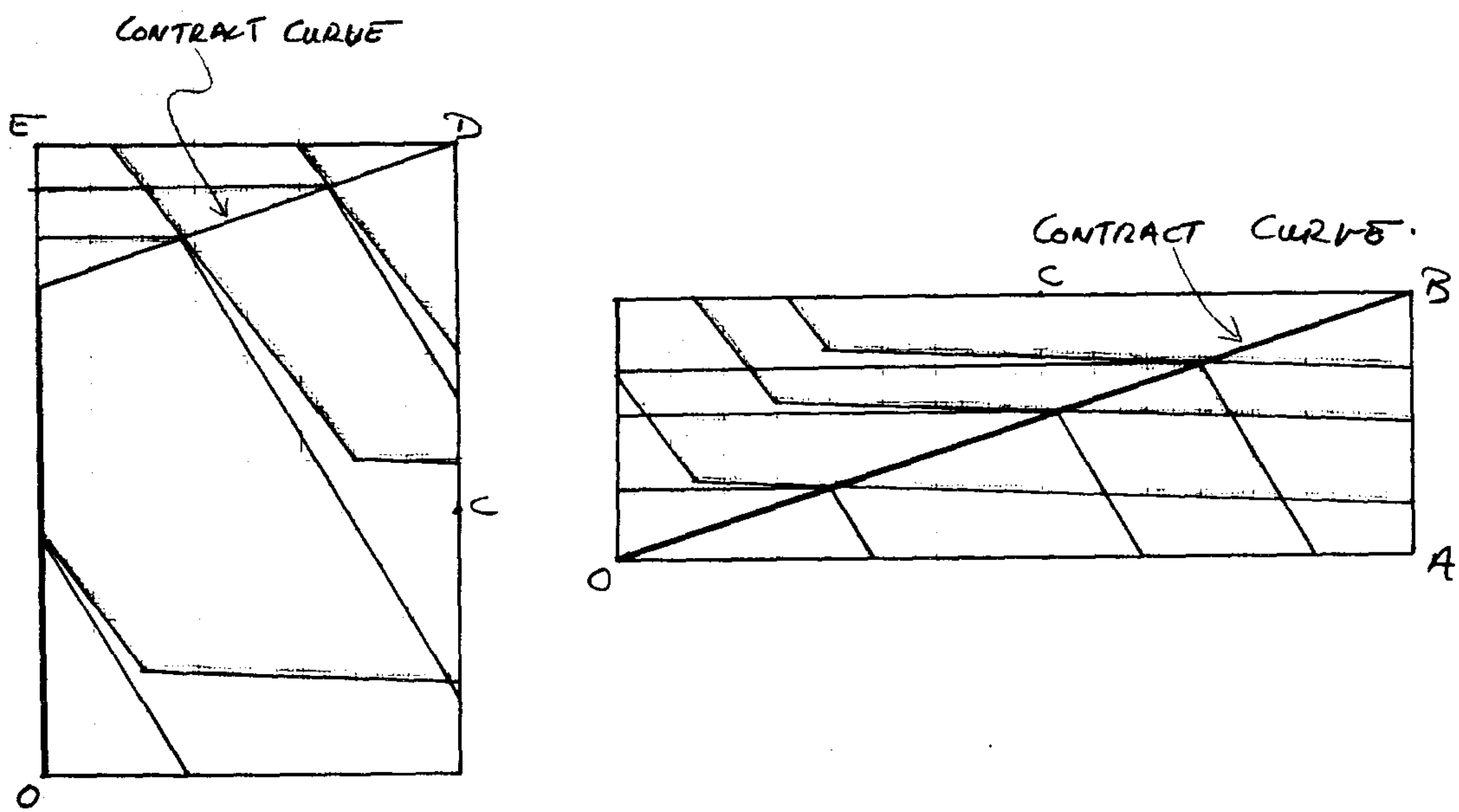


FIGURE 9

UTILITY POSSIBILITY FRONTIER.

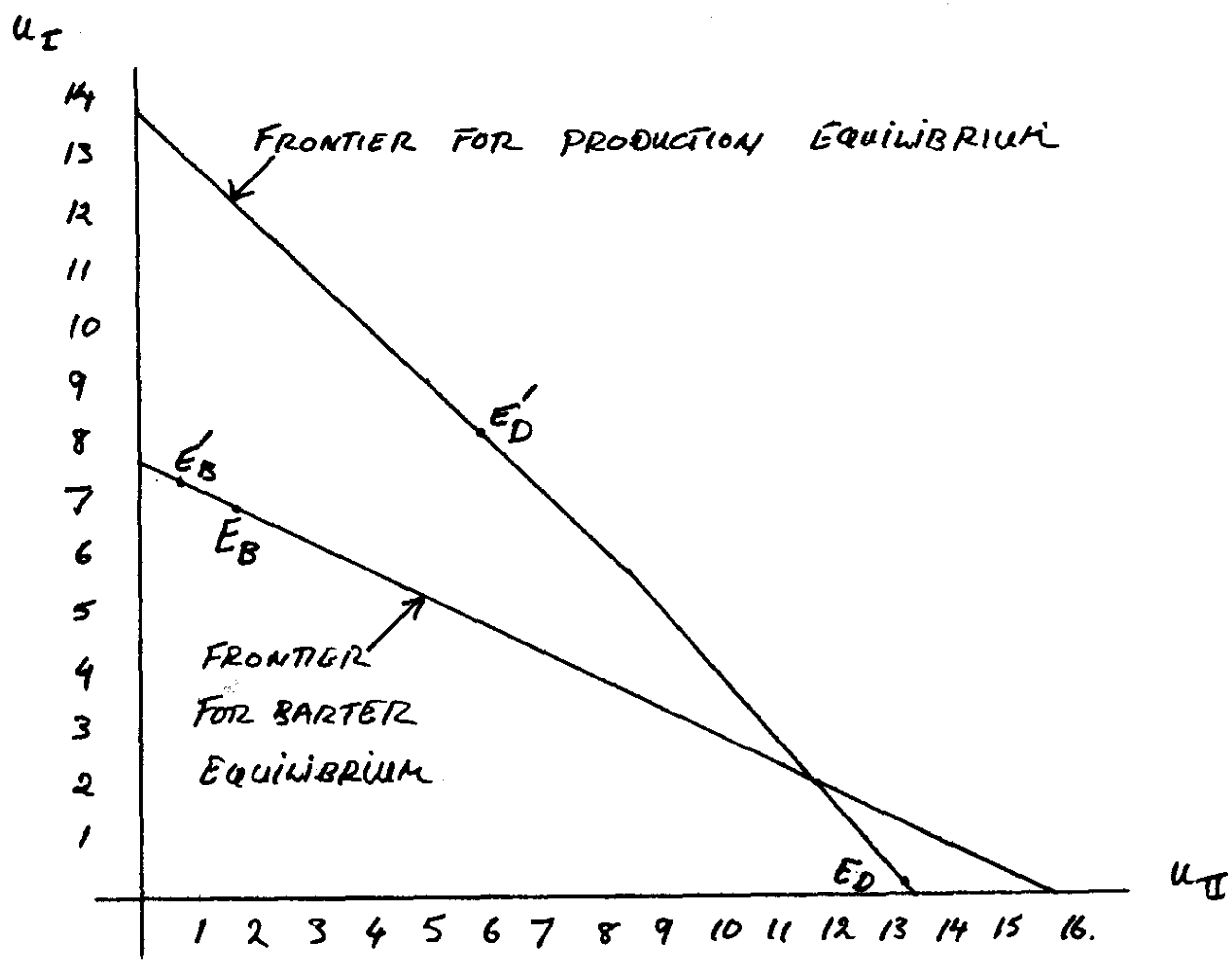
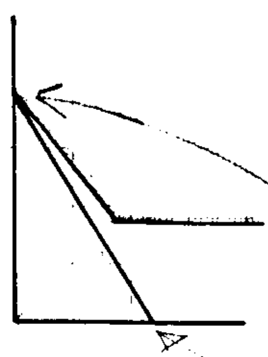


FIGURE 10

EQUILIBRIA WHEN ENDOWMENTS ARE
$\omega_I = (0,5)$
$\omega_{II} = (15,0)$.
I PAYS THE FIRM'S LOSSES.



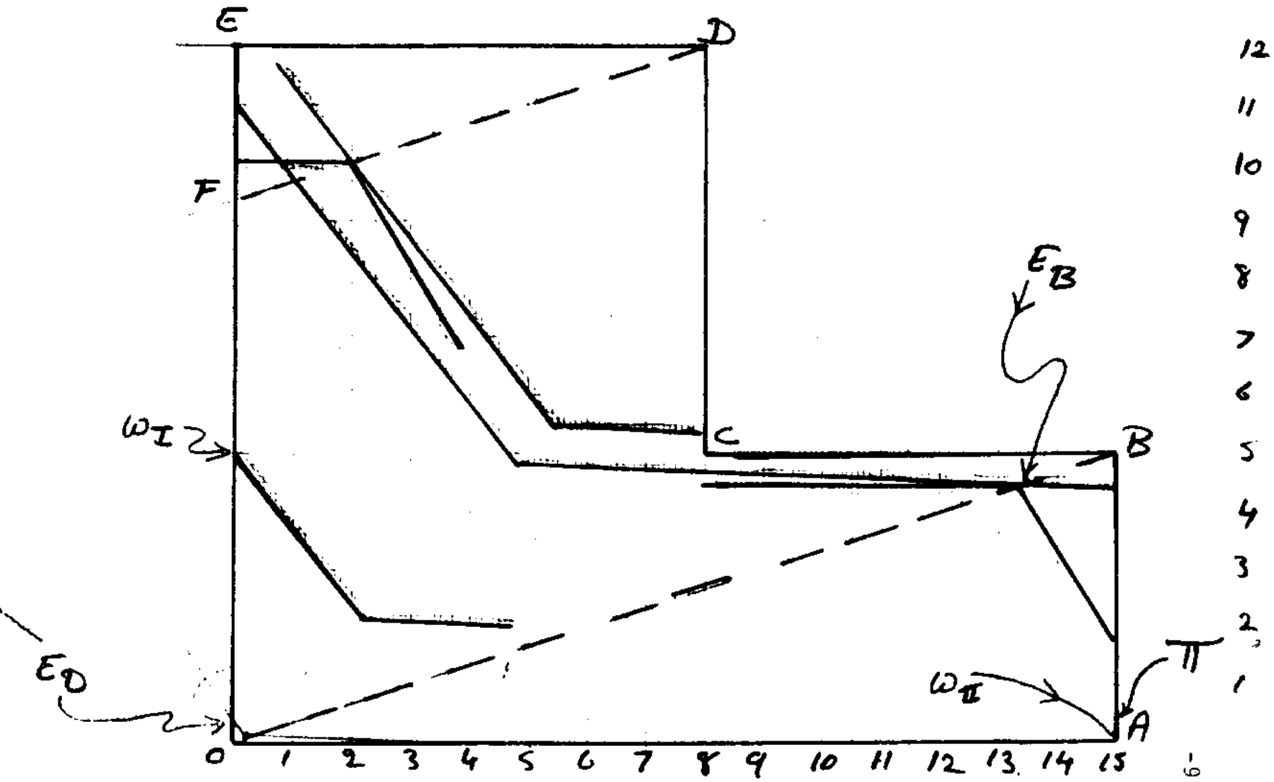
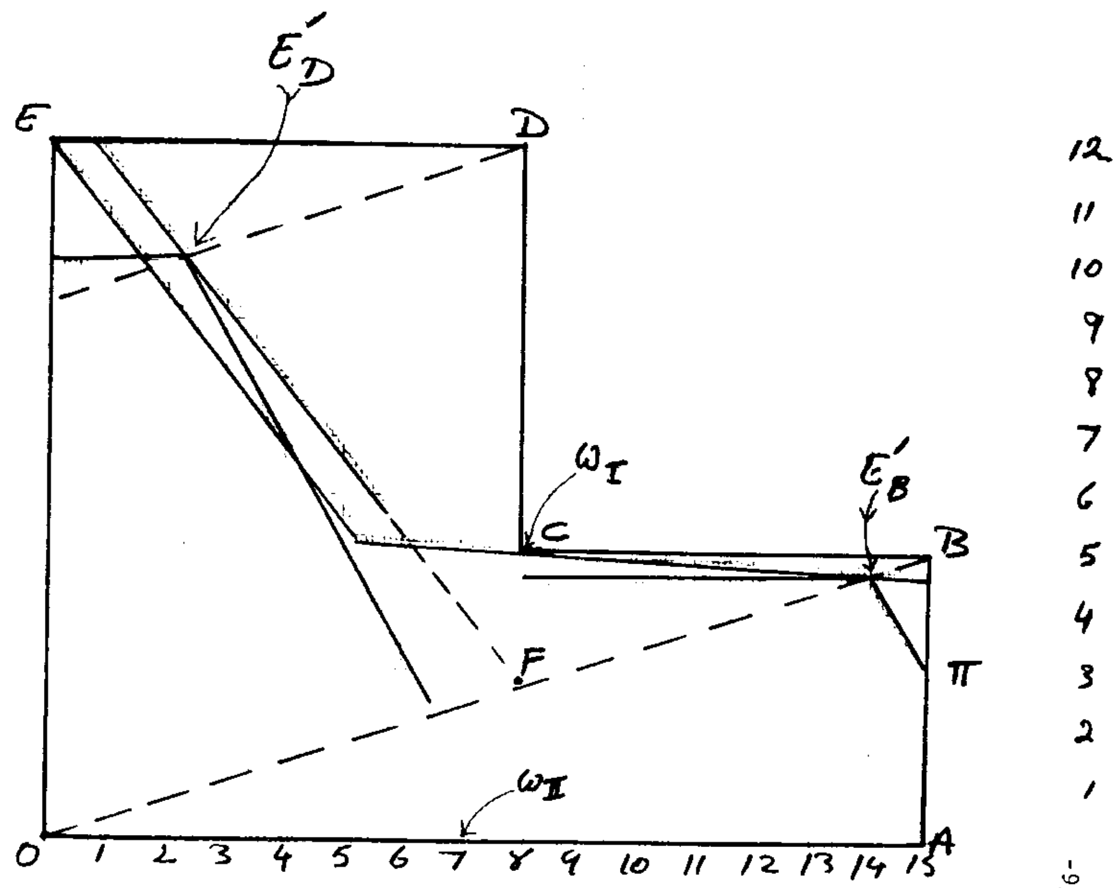


FIGURE 11.

EQUILIBRIA WHEN ENDOWMENTS ARE
$\omega_I = (8,5)$
$\omega_{II} = (7,0)$
I PAYS THE FIRM'S LOSSES.



this information and the location of one point on I's budget line, it is clear that the equilibrium is at E_B.

In the case of production at D, the contract curve consists of the segments OF and FD. Equilibrium occurs along OF, with prices given by the slope of II's indifference curves in this region. In this case, production is occurring, and the firm will make a loss given, in terms of output, by the vertical distance $B\pi$. In computing I's budget line, we have to remember his responsibility for this loss: his budget line will therefore have the slope of II's indifference curves, with a vertical intercept $B\pi$ below W_I. Such a line intersects the contract curve OFD at E_D, confirming that this is the equilibrium distribution. One can check the validity of this construction by projecting this budget line onwards to intersect the extension of DE, which it does to a point 15 units to the left of D. This is II's endowment in terms of x, so that the line is also II's budget line.

The utility levels corresponding to E_B and E_D are marked on the corresponding utility possibility curves in figure 9. Clearly both configurations are Pareto inefficient, confirming that we have indeed constructed an economy for which all equilibria are inefficient. (Recall that utility values for I and II are given by distances along the lines $y = x$ and OB, respectively.)

In figure 11 we analyse the equilibria corresponding to a different distribution of endowments. These are now

$$W_I = (8, 5) \qquad\qquad W_{II} = (7, 0)$$

so that an unrequited transfer of 8 units of x has been made from II to I. I continues to be the sole, unlimited liability, shareholder. These new endowments are shown in the figure, and the new equilibrium distributions, E_B' and E_D' are computed as before. The corresponding utility pairs are shown in figure 9, where it is clear that E_D' is Pareto efficient, and indeed is superior to both E_B' and E_B.

It is worth emphasising two conclusions that are apparant from figure 9. The first is that at equilibrium E_D', although I is responsible for meeting the losses of the firm, it would not be to his advantage to close it. This would result in a move to the equilibrium E_B', at which both parties are worse off. The second point to note is that if the economy is initially at equilibrium E_B, a transfer of resources from II to I may move it to E_D', making both I and II better off. We say may only because there is nothing in the model to indicate which of the various possible equilibria will be selected at any pattern of endowments.

Another point that we should perhaps note before moving on is that, although in this section we have demonstrated the possibility of removing the inefficiency by a change in physical endowments, there is nothing particularly special in this respect about the physical component of an individual's total endowment. We leave it as an exercise for the reader to verify that if the individuals' endowments of goods 1 and 2 were unaltered, but the ownership of the firm were transferred to individual II, then the production equilibrium would be efficient. It may therefore be possible to reach efficiency from an initially inefficient pattern of endowments by changing the ownership of financial assets - in effect, by changing the liability for financing the firm's deficit.

4. Mathematical Derivation

We shall now verify the results suggested diagramatically by computing the demand and supply functions for the economy considered there, and substituting into these the equilibrium prices suggested by the diagrams. This will enable us to verify that these are indeed equilibrium prices, and to compute precisely the associated consumption and utility levels. We shall also calculate the exact position of the utility possibility frontier.

Demand Functions

For individual I we have:-

$$U_I = \begin{cases} \frac{3\sqrt{2}}{7}(y + \frac{4}{3}x) \text{ if } y \geqq x \\ 2.24\ \frac{3\sqrt{2}}{7}(y + 0.04x) \text{ if } y \leqq x \end{cases} \qquad (4)$$

and the demand functions are

$$\frac{-4}{3} < \frac{-Px}{Py} < -\frac{4}{100}, \quad x = y = \frac{Y_I}{Px + Py} \qquad (5a)$$

$$\frac{Px}{Py} = \frac{4}{3}, \quad y \geqq x \text{ and } 4x + 3y = Y_I \qquad (5b)$$

$$\frac{Px}{Py} = \frac{4}{100}, \quad y \leqq x \text{ and } 4x + 100y = Y_I \qquad (5c)$$

$$\frac{-Px}{Py} < \frac{-4}{3}, \quad x = 0 \text{ and } y = \frac{Y_I}{Py} \qquad (5d)$$

$$\frac{-Px}{Py} > -\frac{4}{100}, \quad y = 0 \text{ and } x = \frac{Y_I}{Px} \qquad (5e)$$

Likewise, for II

$$U_{II} = \begin{cases} \frac{\sqrt{10}}{18}(3y + 5x) & \text{if } y \geqq \frac{1}{3}x \\ \frac{\sqrt{10}x}{3} & \text{if } y = \frac{1}{3}x \\ \sqrt{10}y & \text{if } y \leqq \frac{1}{3}x \end{cases} \qquad (6)$$

and the demand functions are

$$0 > \frac{-Px}{Py} > \frac{-5}{3}, \quad y = \frac{Y_{II}}{3P_x + P_y} \qquad (7a)$$

$$x = 3y$$

$$\frac{Px}{Py} = \frac{5}{3}, \quad y \geqq \frac{1}{3}x \text{ and } 5x + 3y = Y_{II} \qquad (7b)$$

$$\frac{-Px}{Py} < \frac{-5}{3}, \quad x = 0 \text{ and } y = \frac{Y_{II}}{Py} \tag{7c}$$

Equilibria with $W_I = (0, 5)$ and $W_{II} = (15, 0)$

We consider first the barter equilibrium E_B of figure 10. The figure suggests as equilibrium prices Px = 4 and Py = 100. In this case incomes are given by $Y_I = 500$ and $Y_{II} = 60$. I's demand is indeterminate {by (5c)} and II's is given by (7a). Hence II's demands are, using obvious notation

$$D_{II}(x) = 1.607, \qquad D_{II}(y) = 0.536$$

Given the total endowments of 5 and 15 respectively, I's consumptions must be 13.393 and 4.464 for x and y. It is easily verified that these satisfy (5c), so that the prices are indeed market-clearing, given the supplies available at point B. Utility levels can then be calculated, giving as the overall configuration at E_B:

$$D_I(x) = 13.393, \quad D_I(y) = 4.464, \quad U_I = 6.787$$

$$D_{II}(x) = 1.607, \quad D_{II}(y) = 0.536, \quad U_{II} = 1.694$$

These utility levels correspond to those shown in figure 9. We can now turn to the production equilibrium E_D of figure 10, for which the diagram suggests Px = 5 and Py = 3. In this case $Y_{II} = 75$. The value of I's endowments is 15, but he also has to meet the operating losses associated with production by the firm at point D. These losses are 14, so I's net income is $Y_I = 1$.

I's demand is given by (5d), so that

$$D_I(x) = 0, \quad D_I(y) = \frac{1}{3}$$

II's demands are indeterminate by (7b). Total supplies of x and y are 8 and 12 respectively, giving

$$D_{II}(x) = 8, \quad D_{II}(y) = 11\tfrac{2}{3}.$$

These satisfy (7b), so that we again have an equilibrium. The total configuration is:

$$D_I(x) = 0, \quad D_I(y) = \frac{1}{3}, \quad U_I = 0.202$$

$$D_{II}(x) = 8, \quad D_{II}(y) = 11\tfrac{2}{3}, \quad U_{II} = 13.176$$

and these utilities correspond to those shown at E_D in figure 9.

<u>Equilibria with $W_I = (8, 5)$ and $W_{II} = (7, 0)$</u>

Consider first the barter equilibrium E_B of figure 11. The prices appear to be Px = 4, Py = 100, giving $Y_I = 532$ and $Y_{II} = 28$. I's demand is indeterminate {see (5c)}, and II's is given by (7a). Hence

$$D_{II}(x) = 0.750, \quad D_{II}(y) = 0.250,$$

and

$$D_I(x) = 14.25, \quad D_I(y) = 4.75$$

These demands satisfy (5c). The overall configuration is

$D_I(x) = 14.250, \quad D_I(y) = 4.750, \quad U_I = 7.227$

$D_{II}(x) = 0.750, \quad D_{II}(y) = 0.250, U_{II} = 0.791.$

In the case of production, the suggested prices are Px $\dot{=}$ 4 and Py = 3. In this case $Y_I = 40$ and $Y_{II} = 28$, and demands are given by (5b) and (7a). Hence

$D_I(x) = 2.40, \quad D_I(y) = 10.133, \quad U_I = 8.081$

$D_{II}(x) = 5.60, \quad D_{II}(y) = 1.867, U_{II} = 5.903.$

<u>Utility Possibility Frontier</u>

With no production, total supplies are 15 and 5 of x and y. It is routine to check that

$U_I(15, 5) = 7.603, \quad U_{II}(15, 5) = 15.810,$

verifying the position of the frontier for barter equilibrium in figure 8.

With production supplies of x and y are 8 and 12. Hence

$U_I(8, 12) = 13.738 \qquad U_{II}(8, 12) = 13.352$

One can also verify that when I's consumption is $(0, \frac{28}{3})$ and II's is $(8, \frac{8}{3})$ we have

$U_I(0, \frac{28}{3}) = 5.66, \qquad U_{II}(8, \frac{8}{3}) = 8.43$

so that the utility frontier in this case passes through

(8.43, 5.66) as well as (0, 13.738) and (13.352, 0), as shown in figure 8.

5. Similar Preferences

In this section we work with a more general model similar to that used in our proof of existence results (1). The economy is represented by:

(i) a set of production possibility sets $Y_i \subset R^n$, $i = 1, 2, \ldots, f$.

(ii) a set of individual endowment vectors $w_j \in R^n_{\geqq 0}$, the non-negative orthant of R^n, where $j = 1, 2, \ldots, M$.

(iii) a set of consumption possibility sets $X_j \subset R^n$, $j = 1, 2, \ldots, M$.

(iv) a set of utility functions $U^j : X_j \to R$, $j = 1, 2, \ldots, M$.

(v) a set of profit shares s_{ij}, $i = 1, \ldots, f$, $j = 1, \ldots, M$, with $s_{ij} \geqq 0$ and $\Sigma^m_{j=1} s_{ij} = 1$ for each i.

As in section 3, we define Y by

$$Y = (\Sigma^f_{i=1} Y_i + \Sigma^n_{j=1} w_j) \cap R^n_{\geqq 0}$$

The assumptions made about producers and consumers are:

A1. Y_i is closed, non-empty and contains the negative orthant.

A2. Y is bounded.

A3. Y is supported by a pointwise bounded and equicontinuous family of prices closed in the C^1 compact topology.

A4. The X_j are closed, convex and non-empty.

A5. The U_j are continuous, monotonic and quasiconcave.

We remind the reader that a non-linear price is a function $V: R^n \to R$ which is linear homogeneous and of class C^1, and thus has the convenient property that

$$V(x) = \nabla V(x).\ x$$

We shall refer to $\nabla V(x)$ as the linear part of $V(\ )$ at x. In addition to assumptions A1 to A5, in order to ensure the existence of an equilibrium we need an assumption which ensures that each consumer is above the lower boundary of his consumption possibility set: possible forms of this assumption are discussed in (1) and (3). It was shown in (1) that these assumptions are sufficient to establish the existence of the following equilibrium:

Definition. An equilibrium is a set of consumption vectors $x_j^* \in X_j$, $j = 1, \ldots, M$, and a non-linear price V such that

(i) $x^* = \Sigma_{j=1}^{m} x_j^*$ maximises V over Y

(ii) for all j, x_j^* maximises $U_j\ (x_j)$ subject to

$$\nabla V(x^*).x_j \leq \nabla V(x^*).\ (W_j + \pi_j)$$

where π_j denotes the profits received by individual j.

For an extensive discussion of this equilibrium concept, the reader is referred to (1) and (3), and in particular to (3), which contains a detailed discussion of the possibility of decentralisation on the production side and of the relationship between this equilibrium concept and standard models of marginal-cost-pricing equilibria. For present purposes it suffices to note

(i) that at an equilibrium, the first order conditions for optimality are clearly satisfied. $\nabla V(x^*)$ is a local support to both Y and the individual preferred - or - indifferent sets so that, in traditional language, marginal rates of substitution and transformation are equal. Regarding $\nabla V\ (x^*)$ as a conventional price vector, we note that both are equal to the price ratios, so that if production sets are smooth we can speak of prices equalling marginal costs.

(ii) that the production plans of individual firms at the equilibrium (not mentioned in the definition) may well have negative value at prices $\nabla V(x^*)$, so that the value of a shareholding may be negative. As in the earlier sections we simply accept the concept of unlimited liability on the part of shareholders here. However, alternative interpretations are presented and discussed in (3).

It should now be clear that the equilibrium concept just presented is indeed a generalisation of that used in the earlier examples. The principal apparent difference is that in those, no reference was made to non-linear prices, but it clearly would be easy to establish the existence of appropriate functions supporting Y at the relevant points.

Having presented the more general model, we can now turn to the main purpose of this section, which is to establish that if individuals' preferences are sufficiently similar to allow their behaviour in aggregate to be viewed as that of a representative consumer, then there is always at least one efficient equilibrium. We approach this by showing that if there is only one consumer, then this is certainly true. Intuitively, the point is that in this case indifference curves do not intersect, so that the problem depicted in figure 2 cannot occur. The main point is then a straightforward corollary of this, for we are in effect positing conditions under which the economy behaves as if there were only one consumer.

<u>Proposition 1.</u> If there is only one consumer, then there exists an efficient equilibrium.

<u>Proof.</u> Y is compact and U continuous. Hence there exists $y^* \in Y$ such that

$$U(y^*) \geq U(y) \text{ for all } y \in Y.$$

We shall demonstrate that y^* is an equilibrium. Construct a non-linear price $V:R^n_{\geq o} \to R$ as follows:

$$V(y) = U(y^*) \text{ for } y \in (y/U(y) = U(y^*)).$$

For any other y, pick a scalar such that

$$\lambda y \in (y/U(y) = U(y^*))$$

and set $V(y) = \frac{1}{\lambda} U(y)$.

Note that up to a normalisation

$$\nabla V(y^*) = \nabla U(y^*)$$

and that by quasi-concavity y^* maximises $U(y)$ subject to

$$\nabla U(y^*) . y \leq \nabla U(y^*) . y^*$$

where the RHS can be written

$$\nabla U(y^*) . W + \nabla U(y^*) . Z^*$$

where the first term is the value of the single individual's endowment and the second is the value of the aggregate of firms' production plans at the equilibrium, evaluated at equilibrium prices.

Q.E.D.

The next proposition draws on results established by Gorman (4) and Muellbauer (6) to show that under certain circumstances the demand side of the economy behaves as if there were a single consumer, in which case one can in effect draw on the proposition 1 to establish the existence of an efficient equilibrium.

Proposition 2. If each consumer has an expenditure function of the form

$$M_j \ (U_j, p) = G_j \ (U_j, H(p)) \ B \ (p) + g_j \ (p)$$

where p is a price vector, $\Sigma^m_{j=1} g_j \ (p) = 0$, B is homogeneous of degree zero, H is homogeneous of degree zero, and the expenditure function is concave in p and monotone increasing and differentiable in p and in U_n at given p, then there is at least one efficient equilibrium.

Proof. The model meets the conditions for the existence theorem of (1), so that an equilibrium exists.

Let w_{kj} be the share of commodity k, k = 1, .., n, in individual j's budget. Then

$$w_{kj} = \partial \log m_j (U_j, p) / \partial \log p_k, \; k = 1, .., n$$

and j = 1, ..., m.

By Theorem 3 B of Muellbauer (6), there exists a function M (U_o, p) such that for some scalar U_{oj} (denoting expenditure levels by y_j)

$$\bar{w}_k = \partial \log M(u_{oj}\; p) \;/\; \partial \log p_k$$

where $\bar{w}_k = p_k \Sigma^m_{j=1} q_{kj} / \Sigma^m_{j=1} y_j \equiv \Sigma^m_{j=1} y_j w_{kj} \;/\; \Sigma^m_{j=1} y_j$

and M (U_o, p) is an expenditure function (concave and linear homogeneous in p, monotone increasing and differentiable in p and U). U_o, a function of p and y_j, is the utility level of the representative consumer whose income y_o = M (U_o, p).

We are thus in a position where the consumption side of the economy can be represented by a single utility-maximising consumer whose expenditure function is M(U_o, p). Hence we can use proposition 1, as required.

Q.E.D.

We have established conditions on preferences that are sufficient to ensure that the economy is well-behaved in the sense of having at least one efficient equilibrium. These conditions are admittedly restrictive, though they are well-known and Gorman (6) felt able to claim that they do not do undue violence to such empirical evidence as was then available. It is perhaps interesting that there should be conditions on preferences alone sufficient to ensure the desired result: there are alternative conditions that we have investigated that involve in an essential way the relationship between the curvature of preferences and that of the boundary of Y, but as these draw upon some complex considerations of differential geometry, we leave them for a further paper.

6. Conclusions

We have, as promised, produced a simple and non-paradoxical economy to illustrate a phenomenon first noted by Guesnerie, namely that with increasing returns it may be the case that none of the equilibria satisfying the first order conditions is in fact efficient. It has also been shown that a unilateral transfer of endowments may result in there being at least one efficient equilibrium, and may in consequence make all parties better off.

These facts have, as was observed in the introduction, a number of implications. One is that some endowment patterns are efficient and others inefficient. Hence one cannot regard judgements on equity and efficiency as separable: one could not for example regard a reallocation of endowments as beneficial because it reduces inequality, on the assumption, generally implicit, that efficiency is not significantly affected. The efficiency of the system may be affected substantially. Of course, the possibility of a trade-off between equity and efficiency is widely recognised, but is usually attributed to the disincentive effects of taxation. Our examples suggest that such trade-off may exist even in a first-best world.

As has already been remarked, these facts have implications both for the theory of welfare economics and for its applications in the field of cost-benefit analysis. Of course, they also have implications for pricing policies in regulated increasing-returns industries. A traditional argument is that, provided the losses can be covered in a non-distortionary fashion, these should price at marginal cost, ensuring, in a first-best world, an efficient allocation of resources. It should now be clear that this argument is incorrect, and not merely because it neglects second-order conditions. Endowments may be such that there is no way of achieving global Pareto optimality by marginal-cost pricing.

Not all non-convex economies are afflicted by these problems, and it is naturally of importance to be able to describe the boundary between those which are and those which are not. In some measure it has been possible to do this, by isolating cases which are well-behaved. If there is in effect a representative consumer, one can show that there is always at least one efficient equilibrium, though not all equilibria need be efficient. Obviously this is a rather special case and there is clearly scope for further work on this issue. And our analysis has also left open the issue of whether, in the case of an economy with some efficient and some inefficient endowments, it is possible to characterise in any way the efficient (or inefficient) endowments.

There is just one final point to note. Although in a non-convex economy it is not the case that an efficient outcome can be reached from any set of endowments, it is nevertheless still true that, given a point on the utility possibility frontier, there exists some pattern of endowments with an equilibrium yielding this utility vector. This is an immediate corollary of Theorem 3 of (3).

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