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COUNTABLY ADDITIVE MEASURES IN CORES OF GAMES

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1. Introduction. Let Σ be a field of subsets of a set X, and let v be a non-negative set function defined on Σ. A necessary and sufficient condition for the existence of a finitely additive measure μ, defined on Σ, such that

$$\mu(S) \geq v(S) \quad \text{for all} \quad S \in \Sigma \tag{1}$$

and

$$\mu(X) = v(X) \tag{2}$$

is well known (see [9]) and may be obtained directly from a general condition by Ky Fan [6] for the existence of solutions of general systems of linear inequalities in normed spaces (see section 2). The present paper deals with the problem of the existence of a countably additive measure μ satisfying (1) and (2).

This problem arises naturally in game theory. We recall that an n-person game in characteristic function form with transferable utility is just a function v from the subsets of a finite set N to the reals, such that $v(\phi) = 0$. The elements of this set are interpreted as "players"; the subsets are "coalitions"; and $v(S)$ gives, for each coalition S, the maximum expected payoff it can achieve by its own efforts. A possible outcome of such a game is an additive function x defined on the subsets of N, such that $x(N) = v(N)$, i.e., a division of the "spoils" $v(N)$ among the players. If $v(S) > x(S)$ for some $S \subset N$, then the coalition S may "block profitably" the outcome x. The core is the set of outcomes that cannot be blocked by any coalition of players, i.e., the set of x such that $x(S) \geq v(S)$ for all $S \subset N$.

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It is obvious that the core can be non-empty only if $v(N)$ is not too small. Shapley [10] and Bondareva [4] proved that a necessary and sufficient condition for the non-emptiness of the core is that the game is <u>balanced</u>. This means that if the subsets $S_1, \ldots, S_k$ of N satisfy

(3) $$\Sigma_{i=1}^{k} \lambda_i \chi_{S_i} \equiv 1$$

with positive coefficients $\lambda_1, \ldots, \lambda_k$, then

(4) $$\Sigma_{i=1}^{k} \lambda_i v(S_i) \leq v(N) \quad .$$

Here χ_S denotes the characteristic function of S.

A game (with transferable utility, in characteristic function form) with a general set of players X is a non-negative set function v defined on the field Σ of coalition (= subsets of X). An outcome of the game is a non-negative additive set function μ defined on Σ satisfying (2). If the inequality (1) is also satisfied, then no coalition can profitably block μ. The set of outcomes which satisfy both (1) and (2) is again called the core.

Schmeidler [9] proved that such a game has a non-empty core if and only it is balanced. Here the game is called balanced if

(5) $$\sup \Sigma_{i=1}^{k} \lambda_i v(S_i) \leq v(X)$$

where the sup is taken over all finite sequences λ_i and S_i, where the λ_i are non-negative numbers, the S_i are elements of Σ and

(6) $$\Sigma \lambda_i \chi_{S_i} \leq \chi_X \quad .$$

The interest in countably additive measures in the core is quite natural. In section 2 we shall show how the general theory of linear inequalities in normed linear spaces, as given by Ky Fan [6] yields, besides Schmeidler's theorem, also a necessary and sufficient condition for the existence of countably additive measures in the core. The resulting theorem, however, does not seem to be very useful in practice. In section 3 we shall give several examples of games with a countable set of players which will disprove certain conjectures that might arise. (In particular, a question left open by Schmeidler in [9] will be answered.) These examples will also motivate necessary and sufficient conditions, proved in section 4, for the existence of a countably additive measure in the core of a game defined on the subsets of a countable space or on the Borel subsets of the unit interval. More generally, we shall prove in section 4 a necessary and sufficient condition for the existence in the core of a regular countably additive measure concentrated on a countable union of compact sets, if the game is defined on the Borel subsets of a completely regular Hausdorff space.

Games with infinitely many players have not been introduced just for the sake of mathematical curiosity. Rather, they have been introduced to explain mass phenomena which could not be treated easily in finite models. This is especially true for economic applications of game theory (compare [2] and [3]). The natural context for these latter applications is the non-transferable utility theory. Unfortunately, the results for games with non-transferable utility are much less complete then those for the transferable utility case. We shall discuss core theory for non-transferable utility games with a countable set of players in section 5.

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2. <u>An application of a theorem by Ky Fan.</u> The following is a fundamental result by Ky Fan (Theorem 13 in [6]) concerning systems of linear inequalities in normed linear spaces.

<u>Theorem</u>. Let $\{x_\nu\}_{\nu \in I}$ be a family of elements, not all 0, in a real normed linear space B, and let $\{a_\nu\}_{\nu \in I}$ be a corresponding family of real numbers. Let

$$\sigma = \sup \Sigma_{i=1}^{n} \lambda_i a_{\nu_i} \tag{7}$$

when $n = 1, 2, 3, \ldots, \nu_i \in I$ and λ_i vary under the conditions

(8) $$\lambda_i > 0 \ (1 \leq i \leq n); \ \| \Sigma_{i=1}^n \lambda_i x_{\nu_i} \| = 1 \ .$$

Then:

(i) The system

(9) $$f(x_\nu) \geq a_\nu \qquad (\nu \in I)$$

of linear inequalities has a solution $f \in B^*$, if and only if σ is finite.

(ii) If the system (9) has solutions $f \in B^*$, and if the zero functional is not a solution of (9), then σ is equal to the minimum of the norms of all solutions f of (9).

(Here B^* denotes the conjugate space of the normed linear space B.)

Consider the Banach space $B(X, \Sigma)$ which consists of all uniform limits of finite linear combinations of characteristic functions of sets in Σ. It is well known [5, p. 258] that the conjugate space $B^*(X, \Sigma)$ is (isometrically isomorphic to) the space of all bounded additive set functions defined on Σ, normed by the total variation. Hence, one may bring the system of linear inequalities (1) to the form (9) by letting Σ be the indexing family, $\{\chi_S\}_{S \in \Sigma}$ the family of elements, $\{v(S)\}_{S\in\Sigma}$ the corresponding family of real (actually non-negative) numbers. Thus we are looking for an element f of $B^*(X, \Sigma)$, which is a solution of the system

(10) $$f(\chi_S) \geq v(S) \qquad (S \in \Sigma)$$

of linear inequalities. We consider, therefore, finite collections of sets $S_i \in \Sigma$, $1 \leq i \leq n$, and positive numbers λ_i, $1 \leq i \leq n$, which satisfy the condition (8). This condition is nothing else than (6), and the number σ, defined by (7), is the same as the left side of (5). Since v is non-negative, μ has also to be non-negative. Hence the norm of a solution f of (10) is equal to $f(\chi_X) = \mu(X)$, so that (by (ii)) (2) can be satisfied if and only if (5) holds. Thus Schmeidler's theorem may be derived from the general Ky Fan theorem.

The set of all countably additive set functions on Σ is, generally speaking, a proper subset of $B^*(X, \Sigma)$ which is not closed in the w^*-topology on $B^*(X, \Sigma)$. Hence one cannot apply directly Ky Fan's method to the main problem of the present paper (see remark 3 in [6]). It is possible, however, to proceed somewhat indirectly and to prove the following

Theorem 1. A game has a countably additive set function in its core if and only if there exists a non-negative set function $w(S)$ defined on Σ such that $w(\phi) = 0$ and such that for each decreasing sequence $\{S_i\}$ of elements of Σ with empty intersection $w(S_i) \underset{i\to\infty}{\longrightarrow} 0$, and

$$\sup(\Sigma_{i=1}^{n}\lambda_i v(S_i) - \Sigma_{j=1}^{m}\mu_j w(T_j)) \leq v(X) \tag{11}$$

where the sup is taken over all finite sequences $\{\lambda_i\}_{i=1}^{n}$, $\{\mu_j\}_{j=1}^{m}$ of positive numbers and $\{S_i\}_{i=1}^{n}$, $\{T_j\}_{j=1}^{m}$ of elements of Σ, such that

$$|\Sigma\lambda_i\chi_{S_i}(x) - \Sigma\mu_j\chi_{T_j}(x)| \leq 1 \tag{12}$$

for all $x \in X$.

Proof. If a countably additive set function μ satisfying both (1) and (2) does exist, set $w(S) = \mu(S)$ for $S \in \Sigma$. Let f be that element of $B^*(X, \Sigma)$ which corresponds to μ. Then the systems of linear inequalities

$$f(\chi_S) \geq v(S) \qquad S \in \Sigma$$

and

$$f(-\chi_T) = -\mu(T) \geq -w(T) \qquad T \in \Sigma$$

hold both, so that

$$\begin{aligned}&\Sigma_{i=1}^{n}\lambda_i v(S_i) - \Sigma_{j=1}^{m}\mu_j w(T_j) \leq \\ &\leq \Sigma_{i=1}^{n}\lambda_i f(\chi_{S_i}) + \Sigma_{j=1}^{m}\mu_j f(-\chi_{T_j}) \leq \\ &\leq \|f\|_{B^*(X,\Sigma)} \|\Sigma_{i=1}^{n}\lambda_i\chi_{S_i} - \Sigma_{j=1}^{m}\mu_j\chi_{T_j}\|_{B(X,\Sigma)}\end{aligned}$$

for all positive λ_i and μ_j, $1 \leq i \leq n$, $1 \leq j \leq m$, and the only if part of theorem 1 follows.

Suppose that (11) holds. Then the system of linear inequalities

$$\begin{aligned} f(\chi_S) &\geq v(S) \qquad && S \in \Sigma \\ f(-\chi_T) &\geq -w(T) \qquad && T \in \Sigma \end{aligned} \tag{13}$$

has, by the Ky Fan fundamental theorem, a solution $f \in B^*(X, \Sigma)$ and $\|f\| \leq v(X)$. Let μ' be the measure defined by $\mu'(S) = f(\chi_S)$ for $S \in \Sigma$. It follows from the first family of inequalities in (13) that μ' is non-negative and cannot be blocked by any $S \in \Sigma$. From the second family in (13) it follows that $\mu'(S) \leq w(S)$ for all $S \in \Sigma$. Hence $\mu'(S_i) \xrightarrow[i \to \infty]{} 0$ for every decreasing sequence $\{S_i\}$ of elements of Σ with empty intersection, which proves that μ' is countably additive. Noting that $\mu'(X) = \|f\| \leq v(X)$ we can easily find a countably additive measure μ which satisfies both (1) and (2).

Remark. While it is probably true that the usefulness of theorem 1 is limited, the method of its proof may be applied to the solution of problems like: Suppose that a game has a non-empty core. Do there exist points μ in the core such that $\mu(S) \leq a$ for a given S and a? (In particular, the set of players may be finite and S may consist of one player.)

3. Several Examples. In this section we shall present several simple examples of games with countably many players in order to exhibit some of the phenomena which take place in the infinite case. Note that all these examples may be translated to the context of the Borel subsets of the unit interval.

We denote by (k, ∞), for k positive integer, the set of integers greater than k. In the following examples X will be the set N of positive integers.

Our first example is the game defined by

$$v((k, \infty)) = 1 \qquad k = 1, 2, \ldots$$
$$v(N) = 1$$
$$v(S) = 0 \text{ for all other } S \subset N .$$

This game is obviously balanced. The existence of many finitely additive measure on N which vanish on the finite sets and which assume the value 1 on the sets which have finite complements is well known (see e. g. [7] and [11]). In fact, each ultra-filter which refines the filter of all sets with finite complements yields such a measure. On the other hand, there is no countably additive measure in the core, since a countably additive measure is (induced) by an element $x = (x_1, x_2, \ldots)$ of ℓ_1, so that for some k, $\Sigma_{i=k}^{\infty} x_i$ has to be smaller than 1 and so the coalotion (k, ∞) can block.

For any game v in a general space X define $b_v(S)$, for any $S \subset X$, by

$$b_v(S) = \sup \Sigma_{i=1}^{n} \lambda_i v(S_i) \tag{14}$$

where the sup is taken over all finite sequences of λ_i and S_i, for λ_i positive numbers and $S_i \in \Sigma$, $S_i \neq X$ and $\Sigma_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S$. (We exclude $v(X)$ from the sums in (14) because of the special role played by $v(X)$ in (2).) Note that according to our definition a game is balanced if and only if $v(X) \geq b_v(X)$. We shall delete the subscript v if no confusion will be caused by this.

A simple generalization of the preceding argument yields the following

Theorem 2. A necessary condition for the existence of a countably additive measure in the core is that $\lim b(S_i) = 0$ for every decreasing sequence $\{S_i\}$ of elements of Σ with empty intersection.

Proof. If the measure μ is countably additive, then $\mu(S_i) \underset{i \to \infty}{\longrightarrow} 0$ for every decreasing sequence $\{S_i\}$ of elements of Σ with empty intersection. It is easily seen that, for any measure μ in the core, $\mu(T) \geq b'(T)$ for all $T \in \Sigma$. In particular, $\mu(S_i) \geq b(S_i)$.

Note that if the condition of theorem 2 is violated then, no matter how large we set $v(X)$, there will exist no countably additive measure satisfying (1) and (2).

The following simple example demonstrates that the condition of theorem 2 is far from being a sufficient one. Define the game v by

$$v(\{i \cup (k, \infty)\}) = \frac{1}{i} \quad \text{for} \quad k \geq i > 0.$$
$$v(S) = 0 \qquad \text{otherwise.}$$

It is easy to verify that in this case, $b((i, \infty)) = \frac{1}{i+1}$. Since $b(S)$ is a monotone set function, it follows that $b(S_i) \underset{i\to\infty}{\longrightarrow} 0$ for all decreasing sequences $\{S_i\}$ of subsets of N with empty intersection. Suppose, however, that the vector $x = (x_1, x_2, \ldots) \in \ell_1$ satisfies the inequalities $\sum_{i\in S} x_i \geq v(S)$ for all $S \subset N$. If $x_i < \frac{1}{i}$ for some i then $\sum_{j\geq k} x_j$ has to be greater than $\frac{1}{i} - x_i > 0$ for all k, so that x cannot be in ℓ_1. If $x_i \geq \frac{1}{i}$ for every $i \in N$ then Σx_i diverges. Hence there is no countably additive measure in the core.

Schmeidler [9] proved that if $b(T_i) \longrightarrow b(X)$ for any monotone increasing sequence $\{T_i\}$ of elements of Σ, the union of which is X, then every measure in the core is countably additive. He gave an example showing that this condition is not necessary for the existence of a countably additive element of the core, and raised the question whether this condition is necessary for the stronger conclusion that every element should be countably additive. The following example answers this question negatively. Let

$$v(S) = \begin{cases} 1 & \text{if } 1 \in S \text{ and } S \text{ is infinite} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that the only outcome in the core is the one which gives 1 to the player 1, and this is a countably additive outcome. However, $b(\{1, 2, \ldots, n\}) = 0$ for all n.

Finally, consider the game defined by

$$v(\{1\} \cup (k, \infty)) = 1 \qquad k \geq 2$$
$$v(\{2\} \cup (k, \infty)) = 1 \qquad k \geq 2$$
$$v(S) = 0 \qquad \text{for all } S \subset N,\ S \neq N.$$

Here $b(N) = 1$, so that if we set $v(N) = 1$, the core will be non-empty. However, a vector $x \in \ell_1$ satisfies $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$, if and only if $x_1 = 1$ and $x_2 = 1$. Hence the minimal value which has to be set for $v(N)$ in order that there should exist countably additive measures in the core is 2.

4. The main theorems for a completely regular set of players. In this section we shall give a characterization of games which have countably additive measures (which satisfy also some technical assumptions) in their cores, if X is a completely regular Hausdorff space. This characterization differs from that of theorem 1 and may be motivated partly in the following way.

If one considers closely the last example of section 3, one cannot help feeling that this game is related to the game defined by $v(\{1\}) = v(\{2\}) = v(N) = 1$, $v(S) = 0$ otherwise. A precise concept of "relatedness" of games will now be formulated.

Definition. Let v_1 and v_2 be balanced games defined on a field Σ of subsets of X. The game v_2 is called an extension of v_1 if $v_2(S) \geq v_1(S)$ for all $S \in \Sigma$ and $v_2(X) = v_1(X)$.

It is easily verified that the pairs of games mentioned above share a common extension, while none of them extends the other. We need one more concept in order to be able to formulate our main result.

Definition. Let $\mathcal{F}$ be a subfamily of Σ, $X \notin \mathcal{F}$. The game v is said to be generated by $\mathcal{F}$ if

(i) $v(S) = 0$ for $S \notin \mathcal{F} \cup \{X\}$

(ii) $v(X) = b(X)$.

The extension w of v is said to be restricted if $w(S) \leq b_v(S)$ for all $S \in \Sigma$.

Theorem 3. Let X be a completely regular Hausdorff space and let Σ be the Borel field of X. The balanced game v (defined on Σ) has in its core a regular countably additive measure concentrated on a countable union of compact sets if and only if there exists a game which is both an extension of v and a restricted extension of a game generated by the compact subsets of X.

(We recall that the Borel field of a topological space X is the σ-field generated by the closed subsets of X. A measure μ is said to be concentrated on a set Y if $S \cap Y = \emptyset$ implies $\mu(S) = 0$.)

Proof. Assume that the regular countably additive measure μ is in the core of the game v and that there exists an increasing sequence $\{C_i\}$ of compact subsets of X such that μ is concentrated on $\bigcup_{i=1}^{\infty} C_i$. Define a game w by

$$w(S) = \mu(S) \text{ if } S \text{ is compact,}$$
$$w(S) = 0 \text{ if } S \neq X \text{ and } S \text{ is not compact,}$$
$$w(X) = \mu(X)$$

Then w is clearly generated by the compact subsets of X. Let S be any Borel subset of X. For any $\varepsilon > 0$ there exists (by regularity) a closed set $F \subset S$ with $\mu(S) < \mu(F) + \varepsilon$. But $\mu(F) = \mu(\bigcup_{i=1}^{\infty}(F \cap C_i)) = \lim_{i \to \infty} \mu(F \cap C_i)$. Hence there exists an integer n such that $\mu(F \cap C_n) + 2\varepsilon > \mu(S)$. Since $F \cap C_n$ is compact, it follows that $\mu(S) \leq b_w(S)$ for all Borel S. Thus μ is both an extension of v and a restricted extension of w.

On the other hand, assume that u is a game generated by the compact subsets of X and that w is a restricted extension of u and an extension of v. Let βX be the Stone-Čech compactification of X. Consider the system of linear inequalities in $C(\beta X)$ (the space of continuous functions on βX):

$$L(f) \geq u(S) \text{ if } f \geq \chi_S,$$

for all Borel subsets S of X, where L is a continuous linear functional on $C(\beta X)$, and f and χ_S are defined on βX with f continuous. Let $\lambda_1, \ldots, \lambda_n$ be positive numbers and let $f_1, \ldots, f_n$ be continuous functions in $C(\beta X)$, $S_1, \ldots, S_n$ Borel subsets of X, such that $\|\Sigma \lambda_i f_i\| \leq 1$ and $f_i(x) \geq \chi_{S_i}(x)$ for $x \in \beta X$. Then $\Sigma \lambda_i \chi_{S_i}(x) \leq \Sigma \lambda_i f_i(x) \leq 1$ for $x \in X$. Since u is balanced it follows that $\Sigma \lambda_i u(S_i) \leq u(X)$. Since u is generated by the compact subsets, it follows by Ky Fan's theorem (which is recorded in section 2) that there exists a linear functional L on $C(\beta X)$ with $\|L\| = u(X) = v(X)$ and $L(f) \geq u(S)$ for $f \in C(\beta X)$ such that $f(x) \geq \chi_S(x)$ $(x \in \beta X)$. By the Riesz representation theorem, there exists a regular countably additive measure μ' defined on the Borel subsets of βX such that

$$L(f) = \int_{\beta X} f d\mu'$$

for all $f \in C(\beta X)$. We note that a compact subset S of X coincides with its closure in βX and therefore is also closed in the topology of βX. Hence the σ-Ring R of subsets of X, generated by the compact subsets of X, is contained in the Borel σ-field of βX. We note next that $\mu'(S) \geq u(S)$ for all compact subsets S of X. Assume, on the contrary, that $\mu'(S) < u(S)$ for some compact $S \subset X$. It follows from the regularity of μ' that there exists an open subset U of βX with $S \subset U$ and $\mu'(U) < u(S)$. Since βX is normal, we know, by Urysohn's lemma, that there exists a function $f \in C(\beta X)$ such that $0 \leq f \leq 1$, $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in \beta X - U$. In particular, $f(x) \geq \chi_S(x)$, so that according to the construction $L(f) \geq u(S)$, but $\int f d\mu' < \mu'(U)$, a contradiction.

We proceed now to prove that for every positive integer i there exists a compact set $C_i \subset X$ with $\mu'(C_i) \geq v(X) - \frac{1}{i}$. Since $v(X) = b_u(X)$, there exist compact sets $S_1, \ldots, S_n$ and positive numbers $\lambda_1, \ldots, \lambda_n$ such that $\Sigma \lambda_j \chi_{S_j} \leq 1$ and $\Sigma \lambda_j u(S_j) \geq v(X) - \frac{1}{i}$. Set $C_i = S_1 \cup S_2 \cup \ldots \cup S_n$. Then

$$\mu'(C_i) = L(\chi_{C_i}) \geq L(\Sigma \lambda_j \chi_{S_j}) = \Sigma \lambda_j \mu'(S_j) \geq \Sigma \lambda_j u(S_j) \geq v(X) - \frac{1}{i}$$

(we use the fact that L is a non-negative functional). Using the non-negativity of L once again we find that the measure μ' is concentrated on $\bigcup_{i=1}^{\infty} C_i = D$. It is easily seen that for every Borel subset F of X, $D \cap F$ is an element of R. Hence we can define a measure μ on the Borel subsets of X by $\mu(F) = \mu'(D \cap F)$. Clearly, μ is a countably additive regular measure concentrated on a countable union of compact sets. Moreover, $\mu(X) = v(X)$ and $\mu(S) \geq u(S)$ for all compact subsets S of X. It follows that for any Borel subset T of X, $\mu(T) \geq b_u(T)$. But w is a restricted extension of u. Hence $\mu(T) \geq w(T)$. Since w is also an extension of v we conclude that $\mu(T) \geq v(T)$, i.e., μ is an element of the core, and the theorem is proved.

In several cases theorem 3 may be sharpened. Consider first the case where X is the set of positive integers N. Under the discrete topology N becomes a completely regular Hausdorff space. However, a simpler argument yields

Theorem 4. A balanced game v defined on the subsets of the integers has a countably additive measure in its core if and only if there exists a common extension of both v and a game which is generated by the finite subsets of N.

Proof. If $x = (x_1, \ldots, x_n, \ldots) \in \ell_1$ is in the core of v then the game w defined by

$$w(S) = \sum_{i \in S} x_i$$

is an extension both of v and of the game u defined by $u(\{i\}) = x_i$, $u(N) = v(N)$, $u(S) = 0$ otherwise. The game u is clearly generated by the finite subsets.

Assume, now, that u is a game which is generated by the finite subsets. Then

$$\lim_{n \to \infty} b_u(\{1, \ldots, n\}) = b_u(N)$$

since every S with $S \neq N$, $u(S) > 0$ is contained in one of the segments $\{1, \ldots, n\}$. Since w is an extension of u, $\lim_{n \to \infty} b_w(\{1, \ldots, n\}) = w(N)$.

Since w is balanced, its core is non-empty. Let μ be a measure in the core of w. Then $\lim_{n \to \infty} \mu((n, \infty)) = 0$ and therefore μ is countably additive. But w is also an extension of v. Hence μ is also in the core of v.

Consider next the case where X is the unit interval I. Every countably additive measure on the Borel subsets of I is regular since I is metric [5, p. 170, exercise 22]. Since I is compact, every measure on I is concentrated on a countable union of compact sets. Hence the condition of theorem 3 is necessary and sufficient for the existence of just a countably additive measure in the core of a balanced game which is defined on the Borel subsets of I --or, indeed, on the Borel subsets of any σ-compact metric space.

Finally, let us observe that a theorem analogous to theorem 3 may be obtained for normal "spaces" in the sense of Aleksandrov [1] (i.e., spaces in which uncountable unions of open sets are not necessarily open).

5. Remarks on games with non-transferable utility. Core theory for games with non-transferable utility is less developed than the corresponding theory for the transferable utility games even in the case of finitely many players. Consequently, one cannot hope (at the present) for a detailed theory in the infinite case, and the results are necessarily fragmentary.

We recall the definition of an m-person game with non-transferable utility [8]. The set of m players will be denoted by M. For each coalition $S \subset M$, E^S will mean the Euclidean space of dimension equal to the number of players in S and whose coordinates have as subscripts the players in S. If u is a vector in E^M then u^S will be its projection onto E^S. With each coalition S a set $V_S \subset E^S$ is associated. (Intuitively, V_S represents the set of possible utility levels that can be obtained by that coalition.) We require that

1. For each S, V_S is a closed non-empty set.
2. If $u \in V_S$ and $y \in E^S$ with $y \leq u$, then $y \in V_S$.
3. The set of vectors in V_M in which each player receives no less than he could obtain by himself is a non empty bounded set.

Let us note that in the transferable utility case V_S consists of the vectors $u \in E^S$ such that $\sum_{i \in S} u_i \leq v(S)$.

Let u be a point in V_M and u^S its projection onto E^S. The vector u is blocked by the set S if we can find a point $y \in V_S$ with $y > u^S$, or in other words if the coalition S can obtain a higher utility level for each of its members than that given by the vector u. A point $u \in V_M$ will be in the core if it cannot be blocked by any set S.

It is obvious that unless V_M is large enough the core will be empty. A game is balanced in the sense of Scarf if for every collection $S_1, \ldots, S_k$ of coalition such that there exist positive numbers $\lambda_1, \ldots, \lambda_k$ with

$$\Sigma_{i=1}^{k} \lambda_i \chi_{S_i} \equiv 1$$

the vector u must be in V_M if $u^{S_i} \in V_{S_i}$ for all $1 \leq i \leq k$. Scarf succeeded in generalizing the sufficiency part of the Shapley-Bondareva theorem and proved [8] that a balanced m-person game has a non-empty core. (The proof goes considerably deeper than just linear inequalities theory.)

We shall assume from now on that each V_S contains the origin of E^S. Then by 1 and 2 and by the definition of the blocking concept, it suffices to consider the intersection of V_S with the non-negative orthant of E^S, and we shall denote this intersection by V_S. Then V_S is compact.

In trying to generalize this theory, even for the case of countably many players, the problem of choosing a topology on the outcome space is of utmost importance. On one hand, the blocking relation is not continuous in the usual (Tychonoff) product topology if S is infinite. On the other hand, you do not get enough compact sets using other topologies.

Let us define a topology P on the linear space E^N of real sequences by requiring that the set $\{x \in E^N; a_i < x_i < b_i\}$ will be open for all vectors a, b with $a < b$. We replace 1 by assuming that for each

S, V_S is a closed subset of the non-negative orthant of E^S, where closed means "closed in P". The assumption 2 is left unchanged except for replacing E^S by the non-negative orthant of E^S.

For any game V we define the balanced cover of V with respect to S, $B_V(S)$, to be the closure (in the topology P of E^S) of the set of vectors $u \in E^S$ which satisfy $u^{S_i} \in V_{S_i}$, $1 \leq i \leq n$, for all collections $S_1, \ldots, S_n$ of subsets of S (with $S_i \neq N$) for which there exist positive constants $\lambda_1, \ldots, \lambda_n$ such that

$$\Sigma_{i=1}^{n} \lambda_i \chi_{S_i} = \chi_S .$$

A balanced game W is said to be an extension of the balanced game V if for all $S \subset N$, $V_S \subset W_S$ and $V_N = W_N$. It is called a restricted extension of V if in addition the inclusion $W_S \subset B_V(S)$ holds for all $S \underset{\neq}{\subset} N$. A game V is generated by the finite subsets of N if $V_S = \{0\}$ for all infinite subsets S of N other than N and V_N is the closure (in the product topology) of $B_V(N)$.

We can now state our (rather weak) only main positive result (in non-transferable utility theory).

Theorem 5. A sufficient condition for the non-emptiness of the core of a balanced game V is that

(i) There exists a restricted extension W' of a game W generated by the finite subsets of N, such that W' is also an extension of V.

(ii) The set V_N is compact in the Tychonoff topology.

Remark. An outcome in V_N is obviously analogous to a countably additive measure. There is no very natural analogue of finitely additive outcomes.

Proof. We shall prove first the existence of an element of V_N (= W_N) which cannot be blocked (in the W game) by any finite coalition. Define $F_S \subset W_N$ to be the set of all vectors $u \in W_N$ such that for no $y \in W_T$ one has $y > u^T$, $T \subset S$. Since $W_N = B_W(N) \supset B_W(S)$ it follows from Scarf's theorem that the sets F_S are not empty. Moreover, $F_{S_1} \cap F_{S_2} \cap \ldots \cap F_{S_n} \supset F_{S_1 \cup \ldots \cup S_n} \neq \emptyset$. The sets F_S are closed in the product topology and therefore compact by (ii). Hence the intersection $\bigcap_{|S| < \infty} F_S$ ($|S|$ denotes the cardinality of S) is non empty. Thus there exists a vector $y \in W_N$ which cannot be blocked by any finite coalition. We prove that y cannot be blocked by any coalition $T \neq N$ whether $|T|$ is finite or not, in the W' game. Otherwise, there would exist a vector $x \in W'_T$ with $x > y^T$. But $W'_T \subset B_W(T)$. Hence there exists a vector $u \in E^T$ with $u > y^T$ and a system $S_1, \ldots, S_n$ of sets such that positive constants $\lambda_1, \ldots, \lambda_n$ exist with

$$\Sigma_{i=1}^{n} \lambda_i \chi_{S_i} = \chi_T$$

and $u^{S_i} \in W_{S_i}$. Thus $u^{S_i} > y^{S_i}$, but this is impossible (by the construction of y) if $|S_i|$ is finite. If $|S_i| = \infty$ then $W_{S_i} = \{0\}$ and the zero vector cannot block y^{S_i} (since y is contained in the non-negative orthant). Since W'_S contains V_S for all $S \subset N$ we thus proved that y cannot be blocked (in the V game) by any proper subset of N.

If there exists no vector x in V_N with $x > y$, we are through. Otherwise, consider the set $D = \{x \in V_N : x \geq y\}$. The set D is closed in the product topology, and is therefore compact. The series

$$\Sigma_{i=1}^{\infty} \frac{x_i}{2^i}$$

converges uniformly on D to a continuous function $f(x)$ (in the product topology). Let z be an element of D which maximizes f. Then z cannot be blocked via N, and since $z \geq y$, it follows that z is in the core of V.

To illustrate theorem 5 consider the game V defined by

$V_{\{1,\ldots,n\}} = \{x \in E^{\{1,\ldots,n\}};\ 0 \le x_i \le 1 - \frac{1}{n},\ 1 \le i \le n\}$, $V_S = \{0\}$ for all other $S \underset{\neq}{\subset} N$. If we define V_N by $V_N = \bigcup_{S \underset{\neq}{\subset} N} V_S$ we get a balanced game, where, however, no point y exists in V_N which cannot be blocked by a finite subset. Defining V_N by $V_N = \{x \in E^N;\ 0 \le x_i < 1$ for all $i\}$, we may find lots of vectors y in V_N which cannot be blocked by any coalition $S \neq N$, but for each $y \in V_N$ there exists an $x \in V_N$ with $x > y$. Only by letting V_N be equal to the set $\{x \in E^N;\ 0 \le x_i \le 1$ for all $i\}$ we get a non-empty core.

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