

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER No. 143

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

On the Existence of a Subinvariant Measure

Tjalling C. Koopmans and Richard E. Williamson

July 31, 1962

ON THE EXISTENCE OF A SUBINVARIANT MEASURE

by *

Tjalling C. Koopmans and Richard E. Williamson **

Haar measure is invariant under the homeomorphisms induced by the group operation in the measure space. Consider instead the problem of finding a subinvariant measure for a locally compact space with respect to a set \mathcal{G} of homeomorphisms. That is, we look for a measure λ such that $\lambda(GB) \leq \lambda(B)$ for all $G \in \mathcal{G}$ and Borel sets B . Clearly the existence of such a measure when \mathcal{G} is a group implies that λ is already invariant, so it is natural to consider semigroups \mathcal{S} of homeomorphisms instead. Furthermore, for a monotone set function λ the relation $GB \supset B$ implies $\lambda(GB) \geq \lambda(B)$, and it is therefore natural to require $GB \not\supset B$ for all $G \in \mathcal{S}$.

In this paper we take the underlying space to be the open unit interval I . The construction of the set function λ given below follows the construction of Haar measure for compact sets as described in [1, Ch. XI].

* Research on this paper was started in 1960-61 while both authors were visiting at Harvard University, and was continued in the summer of 1962 at the Cowles Foundation under Task NR 047-006 with the Office of Naval Research.

** Department of Mathematics, Dartmouth College.

The problem of a subinvariant measure on an interval has arisen from an economic problem in the axiomatics of utility [2,3]. The latter problem concerns choice between consumption programs consisting of an infinite sequence of future consumption vectors. The points of I on which \mathcal{S} operates are utility levels of these programs. The elements G of \mathcal{S} represent the effect on utility levels of postponement of programs by a stated number of time units. Each G is labeled by that number and by the "momentary" utility levels associated with the consumption vectors inserted in the gaps created by postponement. The existence of a measure on I subinvariant for \mathcal{S} signifies a certain lack of patience with regard to the time of availability of desirable goods.

Theorem 1.

Let \mathcal{S} be a semi-group of homeomorphisms from I , the open unit interval, to I , having the property that $GU \supset U$ never holds for an interval U of I and a $G \in \mathcal{S}$. Assume that for any given open interval U of I an arbitrary point of I can be covered by GU for some $G \in \mathcal{S}$. Then there exists a real function λ defined on closed intervals D of I , finitely additive on intervals with disjoint interiors, positive on non-degenerate intervals, monotone, and such that $\lambda(D) \geq \lambda(GD)$ for all $G \in \mathcal{S}$ and all $D \subset I$.

Proof:

Fix a point p in I and let U be an open interval containing p . If D is a closed subinterval of I let

$$(D:U) = \min \left\{ n \mid D \subset \bigcup_{i=1}^n U_i, U_i = G_i U, G_i \in \mathcal{G} \right\}.$$

It follows from the compactness of D that $(D:U)$ is finite. Define, for A fixed closed and non-degenerate in I ,

$$\lambda_U(D) = (D:U)/(A:U).$$

Then, if D is non-degenerate, since a cover of D by images of U can be constructed from a cover of D by images of A and a cover of A by images of U ,

$$(1) \quad 0 < 1/(A:D) \leq (D:U)/(A:U) = \lambda_U(D) \leq (D:A).$$

Let Φ be the set of functions f defined for closed intervals D in I and such that $0 \leq f(D) \leq (D:A)$. Provide Φ with the topology of convergence on finite sets $\{D_1, D_2, \dots, D_n\}$. [4, p. 92]. Then Φ is compact,

by Tychonoff's theorem on the compactness of a product of compact spaces [4, p. 143].

Let $\mathcal{A}(U) = \{ \lambda_V \mid U \supset V, p \in V \}$. It is straightforward

to verify [1, p. 255] that the family of all sets $\mathcal{A}(U)$ has the property that any finite subfamily $\{ \mathcal{A}(U_1), \dots, \mathcal{A}(U_n) \}$ has a non-empty

intersection. Since Φ is compact there is therefore a function λ in

$$\bigcap_{p \in U} \mathcal{A}(U). \quad \text{For } D \text{ non-degenerate, (1) implies } \lambda(D) > 0.$$

To show that λ is finitely additive we use two lemmas.

Lemma 1 . If $\{U_n\}_{n=1, 2, \dots}$ is a nested sequence of neighborhoods of p converging to p , then $\lim_{n \rightarrow \infty} (A:U_n) = \infty$.

Proof of Lemma 1 . Clearly $(A:U_n)$ is a nondecreasing function of n .

Suppose $(A:U_n) \leq N$ for all n for some fixed integer N . Choose $2N$

disjoint open intervals $A_i, i = 1, \dots, 2N$, in A . Find open intervals

U'_i about p such that $G_i A_i = U'_i$ for some $G_i \in \mathcal{S}$. Let $U_0 = \bigcap_{i=1}^{2N} U'_i$.

Then no A_0 , with $G U_0 = A_0$ for some G , can contain an A_i . Hence N

images of U_0 under \mathcal{S} could not cover A , which contradicts the premise.

Lemma 2 .
$$- \frac{1}{(A:U)} + \lambda_U(D) + \lambda_U(E)$$

$$\leq \lambda_U(D \cup E) \leq \lambda_U(D) + \lambda_U(E) ,$$

where U is a neighborhood of p and D and E are closed intervals with disjoint interiors and a common end point.

Proof of Lemma 2 . The first inequality holds because minimal coverings of D and E can be turned into a minimal covering of $D \cup E$, perhaps by removing an interval that covers the common endpoint of D and E .

The second inequality holds because the union of minimal coverings of D and E is a covering of $D \cup E$, perhaps not minimal.

To prove additivity of λ , notice that $\lambda \in \overline{\mathcal{A}(U)}$, for all neighborhoods U of p , implies that there is a nested sequence U_n converging to p , such that $\lambda_n \equiv \lambda_{U_n} \in \mathcal{A}(U_n)$ and λ_n converges to λ in the topology of Φ . For let U'_n be a sequence of neighborhoods of p converging to p . Since $\lambda \in \bigcap_{p \in U} \overline{\mathcal{A}(U)}$, we have $\lambda \in \bigcap_{n=1}^{\infty} \overline{\mathcal{A}(U'_n)}$.

Then, for any given finite set $\{D_1, \dots, D_N\}$, there is for all n a $U''_n \subset U'_n$ such that $|\lambda_{U''_n}(D_k) - \lambda(D_k)| < \frac{1}{n}$, $k = 1, \dots, N$. Since this sequence converges to p , it contains a nested subsequence U_n converging to p such that λ_n converges to λ .

Now given $\epsilon > 0$ there is an n_ϵ such that $n > n_\epsilon$ implies

$$|\lambda_n(D \cup E) - \lambda(D \cup E)| < \epsilon/3$$

$$|\lambda_n(D) - \lambda(D)| < \epsilon/3$$

$$|\lambda_n(E) - \lambda(E)| < \epsilon/3 .$$

From these inequalities it follows that

$$\begin{aligned} & -\epsilon + \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) \\ & \leq \lambda(D) + \lambda(E) - \lambda(D \cup E) \\ & \leq \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) + \epsilon . \end{aligned}$$

But by Lemma 2 ,

$$\frac{1}{(A:U_n)} \geq \lambda_n(D) + \lambda_n(E) - \lambda_n(D \cup E) \geq 0 .$$

Therefore, for all n ,

$$-\epsilon \leq \lambda(D) + \lambda(E) - \lambda(D \cup E) \leq \frac{1}{(A:U_n)} + \epsilon$$

By Lemma 1, $\frac{1}{(A:U_n)}$ tends to zero. Since ϵ is arbitrary, λ is additive.

To check that $\lambda(GD) \leq \lambda(D)$ it is enough to check the same condition for arbitrary λ_U . Now $(GD:U) \leq (D:U)$ because a minimal covering of D by sets $G_i U$ gives rise to a covering, not necessarily minimal, of GD by sets $G G_i U$. The desired result follows on division by $(A:U)$.

Monotonicity follows similarly.

Corollary. λ is zero on one-point sets.

Proof: If D is a one-point set, $(D:U) = 1$ for all U . The corollary follows by Lemma 1.

Theorem 2 . The interval function λ of Theorem 1 is continuous in the sense that, if D_n is a sequence of intervals, containing a fixed point q and converging to it, then $\lim_{n \rightarrow \infty} \lambda(D_n) = 0$.

Proof.* There are in I intervals having λ - measure at most ϵ for any $\epsilon > 0$. To see this take an interval having finite positive measure M and partition it into at least $M \epsilon^{-1}$ non-degenerate intervals. One of these must have measure less than ϵ . Let D_ϵ be a non-degenerate interval of measure at most ϵ . Then, for some $G \in \mathcal{S}$, GD_ϵ contains q in its interior and $\lambda(GD_\epsilon) \leq \lambda(D_\epsilon) \leq \epsilon$. For n sufficiently large $D_n \subset GD_\epsilon$ so $\lambda(D_n) \leq \epsilon$.

* We are indebted to R. Strichartz for this simple proof.

REFERENCES

1. Halmos, P. R., Measure Theory, van Nostrand, 1950.
2. Koopmans, T. C., Stationary Ordinal Utility and Impatience, Econometrica, April 1960, pp. 287-309.
3. Diamond, P. A., T. C. Koopmans, and R. E. Williamson, Stationary Utility and Time Perspective, Cowles Foundation Discussion Paper No. 142, July 31, 1962.
4. Kelley, J. L., General Topology, van Nostrand, 1955.