COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 120

Economic Growth and Wicksell's Cumulative Process

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June 13, 1961

ECONOMIC GROWTH AND WICKSELL'S CUMULATIVE PROCESS

by

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Abstract

Assume a homogeneous production function, a constant rate of population growth and a monetary policy which keeps the rate of interest fixed. With initial unemployment there exists an unstable process of balanced growth. Under initial full employment there exists a natural rate of interest (possibly zero or infinite) such that 1) at interest rates below the natural rate balanced growth is attained and the price level is rising; 2) at interest rate above the natural rate both employment and the price level are falling. Explicit solutions are given for Cobb-Douglas production functions. The "optimal" rate of interest which maximizes per capita consumption at any time during balanced growth is shown to equal the rate of growth.

Introduction

In developing aggregate growth models consistent with the static Keynesian theory of income determination, the most common procedure is to assume that the price level is fixed through suitable measures of monetary and fiscal policy so that the liquidity preference equation may be dropped. Alternatively one may retain it with Tobin [1] in a generalized form (as a portfolio balance equation) into which capital stock enters as one of the determinants of the demand for real balances. However even if one prefers the first approach for simplicity one must further assume (or consider it to be an implication of a constant level of prices) that the rate of interest is fixed in order that the accelerator principle, say, may follow from a homogenous production function.

In this paper we shall explore the implications of a fixed interest rate for economic growth without necessarily assuming a constant price level. To some extent this is just a variation on the theme of Mr. Tobin's Dynamic Aggregative Model [1] if one considers that the rate of interest is held fixed by suitable adjustments of the money supply. However in the present model the emphasis will be on possible disequilibria and on the effects on accumulation and on the rate of economic growth of interest rate policies.

This is, of course, a classical case of Wicksell's cumulative process, and it may be instructive to reconsider it from the current point of view with emphasis on economic growth.

1. Notation and Assumptions

Consider a static Keynesian model in -- more or less -- standard form.

(1) $Y = F(K, N)$

(2) $I = I(i, K, Y)$

(3) $C = C(i, Y)$

(4) $Y = C + I$

(5) $M(i, Y) = pL(i, Y, K)$

(6) $w = \frac{\partial F(K, N)}{\partial N}$

(7) $pw = H(N, L)$

where

Y	national income in real terms
F	the production function
K	capital stock in real terms
N	employment
$\dot{K}$	rate of change of capital stock
I	net investment
i	rate of interest
C	consumption in real terms
M	money supply function
p	price level

L	demand for real balances
$\frac{\partial F}{\partial N}$	marginal productivity of labor
w	real wage
H	labor supply function
L	labor force

This system is made dynamic by inclusion of $\frac{\dot{p}}{p}$ among the variables in the I , C , M and L functions, by observing that

$$(8) \qquad L = L(t)$$

and by including the identity

$$I = \dot{K} .$$

The time path of the system is determined completely by the variables K(t) , N(t) , i(t) and p(t) . The equation system may be reduced to four equations in terms of these unknowns

$$(2a) \qquad \dot{K} = I\left(i , K , F(K, N) , \frac{\dot{p}}{p}\right)$$

$$(4a) \qquad \dot{K} + C\left(i , F(K , N) , \frac{\dot{p}}{p}\right) = F(K , N)$$

$$(5a) \qquad M\left(i , F(K , N), \frac{\dot{p}}{p}\right) = p\, L\left(i , F(K, N) , K , \frac{\dot{p}}{p}\right)$$

$$(7a) \qquad p \frac{\partial F(K , N)}{\partial N} = H (N, L)$$

In order that growth should occur monetary policy (5a) must determine such interest rates i and induced changes in the price level $\frac{\dot{p}}{p}$ that a positive rate of investment $\dot{K}$ is brought forth by (2a) after allowing for the forces that changing consumer demand and labor force exert on the price level through (4a) and (7a).

For more definite conclusions the various functions F , I , C , M , L , H must be specified.

1) Let F be homogeneous of degree 1 , i.e., we assume constant returns to scale.

$$
\begin{aligned}
F(K , N) &= N \cdot F\left(\frac{K}{N} , 1\right) \\
&= N \; f \; (k)
\end{aligned}
$$

where

$$k = \frac{K}{N}$$

$$f(x) = F(x , 1)$$

We disregard technological progress, or rather assume it offsets diminishing returns.

Some times f will be further specified to be a Cobb Douglas production function. (It was known already to Wicksell [2]).

$$f(k) = k^{\alpha} \qquad 0 < \alpha < 1 \tag{1b}$$

2) Let the induced rate of investment be proportional to the difference of the marginal productivity of capital and the market rate of interest.

$$I\ (i,\ K\ ,\ N)\ = \mu\ K \cdot \left(\frac{\partial F(K,\ N)}{\partial K} - i \right) \qquad \text{or}$$

$$\frac{\dot{K}}{K} = \mu \cdot \left(\frac{\partial F}{\partial K} - i \right)$$

$$(2b) \qquad = \mu \cdot (f'(k) - i)$$

For Cobb Douglas functions this is equivalent to the assumption that

"Investment is proportional to the difference between the level of capital stock which is desired at the given rate of interest and actual capital stock." This, in turn, is equivalent to assuming an exponentially distributed lag in the adjustment of capital to income.

In particular, when $\mu = \infty$

$$\frac{\partial F}{\partial K} = i$$

For the Cobb Douglas function

$$\frac{\partial F}{\partial K} = \alpha \frac{Y}{K}$$

$$\therefore \quad K = \frac{\alpha}{i} Y$$

Thus differentiation with respect to time one obtains the acceleration principle

$$I = v\dot{Y}$$

with an accelerator

$$v = \frac{\alpha}{i}$$

3) Consumption expenditure is assumed to be a fixed proportion of money income and a lag intervenes between production and income payments and possibly between the receipt and spending of income. Consequently during inflation physical consumption is less than the intended proportion by a term which is proportional to the rate of change of the price level, the coefficient of this term being larger, the larger the lag between production and spending.

(3b) $$C = cY = \left(c_0 - c_1 \frac{\dot{p}}{p}\right) Y$$

4) Ex post income equals ex post consumption plus ex post investment, where ex post consumption is given by (3b) and ex ante investment by (2b) and it is assumed that investment plans are realized.

Equation (4) implies in particular the absence of fiscal policy.

5) Monetary policy is assumed to consist in fixing the interest (i.e., the discount) rate and supplying all money demanded at that rate.

This is the Wicksellian assumption of the model.

(5b) $$i = \hat{i} = \text{constant.}$$

Hence (5) does not restrict Y to K directly.

6) The demand for labor is determined by equating the marginal productivity of labor to the real wage rate.

(6b) $$w = \frac{\partial F}{\partial N} = \frac{\partial}{\partial N}\left(N\, f\left(\frac{K}{N}\right)\right) = f\left(\frac{K}{N}\right) - N\, f'\left(\frac{K}{N}\right)\frac{K}{N^2}$$

$$= f(k) - k\, f'(k) > 0$$

The law of diminishing returns to substitution assures the positivity of the right hand side of (6b).

7) At less than full employment labor is offered at the going money wage rate. At full employment wage rates are flexible in an upward direction.

(7b) $$H = h = \text{constant} \qquad N < L$$

$$\frac{dH}{dt} \geqq 0 \qquad N = L$$

8) Population and hence the labor force grow exponentially at a given rate $\hat{n}$.

(8b) $$L = L_0 e^{\hat{n}t}$$

With these specifications of our functions it is convenient to introduce new variables as unknowns

$k = \frac{K}{N}$ capital labor ratio

$m = \frac{\dot{K}}{K}$ rate of growth of capital stock

$n = \frac{\dot{N}}{N}$ rate of growth of employment

In terms of these and of $\frac{\dot{p}}{p}$, c we have the relationships

(2c) $$m = \mu\,(f'(k) - i)$$

(3c) $$c = c_0 - c_1 \frac{\dot{p}}{p}$$

(4c) $$c + m \frac{k}{f(k)} = 1$$

(7c) $$p \cdot [f(k) - k\, f'(k)] = h \qquad N < L$$

(7c') $$\frac{d}{dt} \left\{ p \cdot [f(k) - k\, f'(k)] \right\} \geqq 0 \qquad N = L$$

Under full employment alternative (7c') applies. Since employment grows at the rate of the labor force, i.e., of population $\hat{n}$, the identity

(9c) $$\frac{\dot{k}}{k} = m - n \qquad \text{with } n = \hat{n} \quad .$$

is relevant. The complete set of conditions is (2c), (3c), (4c), (7c')
and (9c).

For unemployment (9c) merely serves to find n from m and k. The system is determined by (2c), (3c), (4c) and (7c').

2. Full Employment

Under full employment the rate of growth of employment n is equal to that of population $\hat{n}$ and money wages are flexible in an upward direction. The following set of conditions was derived in Section 1.

(2d) $$m = \mu\,[f'(k) - i]$$

(4d) $$c_0 - c_1\frac{\dot{p}}{p} + m\frac{k}{f(k)} = 1$$

(9d) $$\frac{\dot{k}}{k} - m = \hat{n}$$

(7d) $$\frac{d}{dt}\Big(p \cdot [f(k) - k\,f'(k)]\Big) \geqq 0$$

From (2d) and (9d)

(13) $$\frac{\dot{k}}{k} = \mu\,f'(k) - (\mu\,i + \hat{n}) \equiv R(i,k) \quad \text{say.}$$

Since $\frac{\partial R}{\partial k} = f''(k) < 0$ by the law of diminishing returns to substitution,* the right hand side of (13) is a decreasing function of k. If

$f'(0) > \frac{\mu i + \hat{n}}{\mu} > f'(\infty)$ then there exists a unique equilibrium at which

$\frac{\dot{k}}{k} = 0$. It is stable since $\frac{d}{dk}[\mu f'(k) - \mu i - \hat{n}] = \mu f'' < 0$. The equilibrium level $k = \bar{k}$ is determined by

$$(14) \qquad f'(\bar{k}) = i + \frac{\hat{n}}{\mu}$$

It defines a process of balanced growth of capital and employment (3), (4).

Since at equilibrium $\frac{\partial R}{\partial k} < 0$ $\frac{\partial R}{\partial i} < 0$ we have

$$(15) \qquad \frac{d\bar{k}}{di} = - \frac{\frac{\partial R}{\partial i}}{\frac{\partial R}{\partial k}} < 0$$

The equilibrium ratio of capital to labor is a decreasing function of the interest rate. The rate of balanced growth $m = \hat{n}$ is, of course, independent of i.

From (4d)

$$(16) \qquad \frac{\dot{p}}{p} = \frac{c_0 - 1}{c_1} + \frac{\hat{n}k}{c_1 f} = \frac{1}{c_1 f(k)} [k\hat{n} - (1 - c_0)f(\bar{k})] = G(\bar{k}) \text{ , say .}$$

$$= \frac{1}{c_1 f} \text{ [investment demand - planned saving]}$$

$$> 0 \text{ if ex ante investment exceeds ex ante saving.}$$

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*or else the equilibrium ration $\bar{k}$ is infinite: the capital labor ration increases indefinitely, if below.

Consider

$$\frac{d}{di}\left(\frac{\dot{p}}{p}\right) = \frac{d}{d\bar{k}} G(\bar{k}) \cdot \frac{d\bar{k}}{di}$$

$$= \frac{d}{d\bar{k}}\left[\frac{\hat{n}}{c_1} \cdot \frac{\bar{k}}{f(\bar{k})} - \frac{1 - c_0}{c_1}\right] \cdot \frac{d\bar{k}}{di}$$

$$= \frac{\hat{n}}{c_1} \cdot \frac{f(\bar{k}) - \bar{k} f'(\bar{k})}{f^2(\bar{k})} \cdot \frac{d\bar{k}}{di}$$

> 0 by (6b) $\qquad < 0$ by (15)

(17) $$\frac{d}{di} \frac{\dot{p}}{p} < 0$$

The rate of inflation increases as the interest rate is decreased. Suppose that

(18) $$f'(0) > \frac{\hat{n}}{1 - c_0}$$

This is time, for instance, for the Cobb Douglas production function since then $f'(0) = \infty$.

By the law of diminishing returns

(19) $$\lim_{k \to \infty} \frac{k}{f(k)} = \infty$$

From (18) and (19) it is seen that there exists a $k = \hat{k}$ for which

$$\frac{\hat{k}}{f(\hat{k})} = \frac{1 - c_0}{\hat{n}}$$

Substitution in (16) shows that at $k = \hat{k}$ $\frac{\dot{p}}{p} = 0$. Write

$$\hat{k} = \bar{k}(\hat{i})$$

(20) $$\frac{\bar{k}(\hat{i})}{f(k(\hat{i}))} = \frac{1 - c_0}{\hat{n}}$$

The associated $i = \hat{i}$ is called the normal or natural rate of interest according to Wicksell (cf. below). It is defined as that rate of interest at which the price level is constant. Alternatively it may be defined as that rate of interest at which investment demand equals planned saving. From (17) it follows that

(21) $$\frac{\dot{p}}{p} \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \qquad \text{for} \qquad i \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \hat{i} \quad .$$

Observe now that $\frac{\dot{p}}{p} < 0$ and $k = \bar{k} = \text{constant}$ are incompatible with downward rigidity of money wages (7d). Therefore at $i > \hat{i}$ the present model cannot apply: there can be no full employment.

If $f'(0) < \frac{\hat{n}}{1 - c_0}$ contrary to assumption (18) then a positive natural rate of interest does not exist. According to (16) there is then inflation at all capital labor ratios $\bar{k}$, i.e., at all positive rates of interest.

If, on the other hand, $\lim_{r \to \infty} \frac{k}{f(k)} < \frac{1 - c_0}{\hat{n}}$ contrary to assumption (19) then deflation and hence unemployment result at all levels of interest.

In the former case the natural rate of interest may be said to be infinite, in the latter to be zero.

3. Unemployment

We restate the set of equations that determine m , c , k and p .

(2c) $m = \mu \; [f'(k) - i]$

(3c) $c = c_0 - c_1 \frac{\dot{p}}{p}$

(4c) $c + m \frac{k}{f} = 1$

(7c) $p \cdot [f(k) - k f'(k)] = h$

From (7c)

(7d) $\frac{\dot{p}}{p} = \frac{f''k \dot{k}}{f-kf'}$

Substituting (7d) in (3c) , and (2c) and (3c) in (4c)

$$c_0 - c_1 \frac{f''k\,\dot{k}}{f-kf'} + \frac{k\mu}{f}(f' - i) = 1$$

$$(22) \qquad \dot{k} = \frac{f-kf'}{c_1 f'' k}\left[\mu \frac{k}{f}\cdot(f' - i) + c_0 - 1\right]$$

$$= \frac{f-kf'}{c_1 f'' k f}\left[\mu\, k\,(f' - i) - (1 - c_0)\, f\right] = G(i,h), \text{ say .}$$

where $\mu k(f'-i)$ is the investment demand and $(1 - c_0)f$ the supply of saving per unit of labor employed. If

$$(23) \qquad \lim_{k \to 0} \frac{k[f'(k) - i]}{f(k)} > \frac{1 - c_0}{\mu} > \lim_{\mu \to \infty} \frac{k\cdot[f'(k) - i]}{f(k)}$$

there exists at least one positive equilibrium capital labor ratio $k = k^*$, i.e., one for which $\dot{k}$ by (22) is zero. (When assumption (23) is not satisfied, then the "equilibrium" capital labor ratio is zero or infinite.)

An equilibrium capital labor ratio is determined by

$$(24) \qquad \frac{k}{f(k)}[f'(k) - i] = \frac{1 - c_0}{\mu}$$

With each equilibrium ratio of capital to labor is associated a path of balanced growth. On it, according to (7c), the price level is constant. The equilibrium growth rate $\overset{*}{m}$ is therefore given by (4c) with $c = c_0$ according to (3c).

$$(25) \qquad m^* = \frac{(1 - c_0)}{k^*} f(k^*) = \frac{1 - c_0}{\frac{f(k^*)}{k^*}}$$

$$= \frac{\text{savings ratio}}{\text{capital output ratio}}$$

Theorem: The largest root of (24) is unstable.

Proof: By (23) and continuity for sufficiently large k

$$\frac{1 - c_0}{\mu} > \frac{k(f' - i)}{f}$$

The bracket on the right hand side of (22) is then negative. Since $f'' < 0$ the right hand side of (22) is positive. Therefore if $k > k^*$ $\dot{k} > 0$ and if $k < k^*$ then $\dot{k} < 0$. k tends to move from its equilibrium value k^*; k^* is unstable.

In particular when (22) has but a single root -- which is the case e.g., for the Cobb-Douglas production function -- it is unstable.

In (22)

$$\frac{\partial G}{\partial i} > 0$$

and as just shown

$$\frac{\partial G}{\partial k^*} < 0 .$$

therefore

(26) $$\frac{dk^*}{di} = - \frac{\frac{\partial G}{\partial i}}{\frac{\partial G}{\partial k^*}} > 0$$

The (unstable) equilibrium ratio of capital to labor is larger the higher the rate of interest.

Consider next

$$\frac{dm^*}{di} = (1 - c_0) \frac{d}{di} \left[\frac{f(k^*)}{k^*} \right]$$

$$= (1 - c_0) \frac{d}{dk^*} \left[\frac{f(k^*)}{k^*} \right] \quad \frac{dk^*}{di}$$

$$= (1 - c_0) \frac{f'k^* - f}{k^{*2}} \cdot \frac{dk^*}{di}$$

$$< 0 \text{ by } (6b) \qquad > 0 \text{ by } (26)$$

(27) $$\frac{dm^*}{di} < 0 \quad .$$

Consider now the (unstable) equilibrium ratio of capital to labor that is obtained at the natural rate of interest

$$k^* (\hat{i}) \; .$$

From (16)

$$\bar{k}(\hat{i}) \cdot \hat{n} = (1 - c_0) \; f(\bar{k}(\hat{i}))$$

and (14)

$$f'(\bar{k}(\hat{i})) = \hat{i} + \frac{\hat{n}}{\mu}$$

one has

$$\bar{k} \ \mu \ [f'(\bar{k}) - i] = (1 - c_0) \ f(\bar{k}) \qquad \text{or}$$

$$\frac{\bar{k}\ [f'(\bar{k}) - i]}{f(\bar{k})} = \frac{1 - c_0}{\mu}$$

Therefore $\bar{k}(\hat{i})$ satisfies the condition (24) which determines $k^*(\hat{i})$. Hence at $i = \hat{i}$ (one of) the equilibrium capital labor ratio(s) is given by

$$\bar{k}(\hat{i}) = k^*(\hat{i})$$

Moreover

$$m^*(\hat{i}) = \frac{(1 - c_0) \ f(k^*(\hat{i}))}{k^*(\hat{i})} = (1 - c_0) \ \frac{f(\bar{k}(\hat{i}))}{\bar{k}(\hat{i}))} = \hat{n}$$

The natural rate of interest determines, therefore, the borderline between full employment and inflation, and unemployment and deflation.

4. Graphical Analysis

Figures 1, 2, 3 are "phase diagrams" showing $\dot{k}$ as a function of k for both full employment and unemployment. From these the histories of k (and by inference of the other variable) may be traced. The downward sloping curve is the full employment relation (13), the upward sloping curve the unemployment relation (22). Points below the unemployment curve are unattainable because the implied rate of deflation is incompatible with non-falling money wages. Initially all points on the unemployment curve are attainable.

Figure 1 shows the case $i < \hat{i}$. Consider a situation of initial full employment. Full employment may be maintained only if $k(0) < \tilde{k}$. If $k(0)$ is to the left of $\bar{k}$ then $\dot{k} > 0$ and one moves along the full employment curve to the right until $\bar{k}$ is reached. If $k(0)$ lies between $\bar{k}$ and $\tilde{k}$ then $\dot{k} < 0$ and one moves to the left along the full employment curve to $\bar{k}$. Initial values of $k > \tilde{k}$ and full employment are inconsistent: The system falls at once into a state of unemployment at the same or a higher capital labor ratio. In the former case some capital must be unemployed, too. The present model does not distinguish between bringing back to productive use an existing capital stock on the one hand and new investment on the other. Therefore, this situation is handled in the same way as one of initial unemployment of labor alone.

With initial unemployment and $k < k^*$ k falls along the unemployment curve. As seen from (2c) the rate of growth of capital increases. Because of the fall in the capital labor ratio, the rate of growth of employment increases *a fortiori* . If assumption (23) is fulfilled, the growth rate of employment eventually exceeds the growth rate of population, and finally full employment is reached. This means a vertical jump from the unemployment curve to the full employment curve above. Next the capital labor ratio increases as one moves to the right along the full employment curve until the stable equilibrium $\bar{k}$ is reached.

When initially $k(0) > k^*$ one moves off to the right along the unemployment curve. No equilibrium is approached and the capital labor ratio grows indefinitely.

In figure 2 $i > \hat{i}$ implies that the full employment curve lies too high for $\bar{k}$ to be attainable. From every initial situation the state of unemployment at rising capital labor ratios is reached.

Figure 3 shows that at the natural rate of interest

$$k^*(\hat{i}) = \bar{k}(\hat{i})$$

is stable from the left and unstable to the right.

Figure 1 $i<\hat{i}$

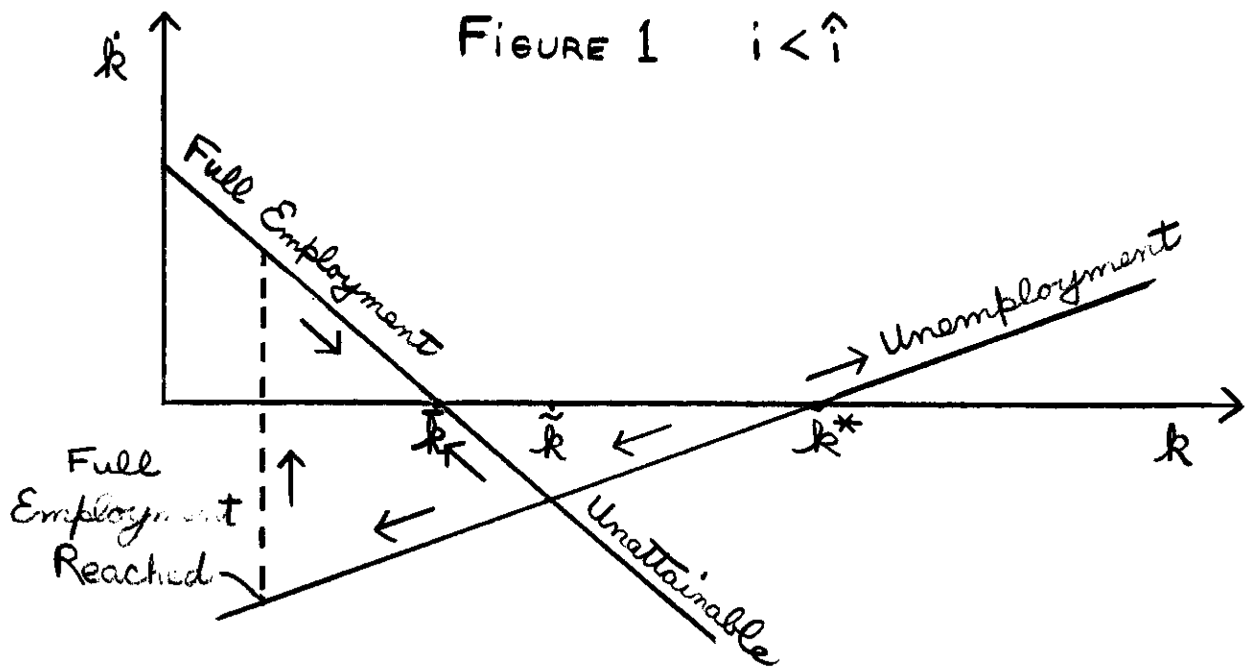


Figure 2 $i>i$

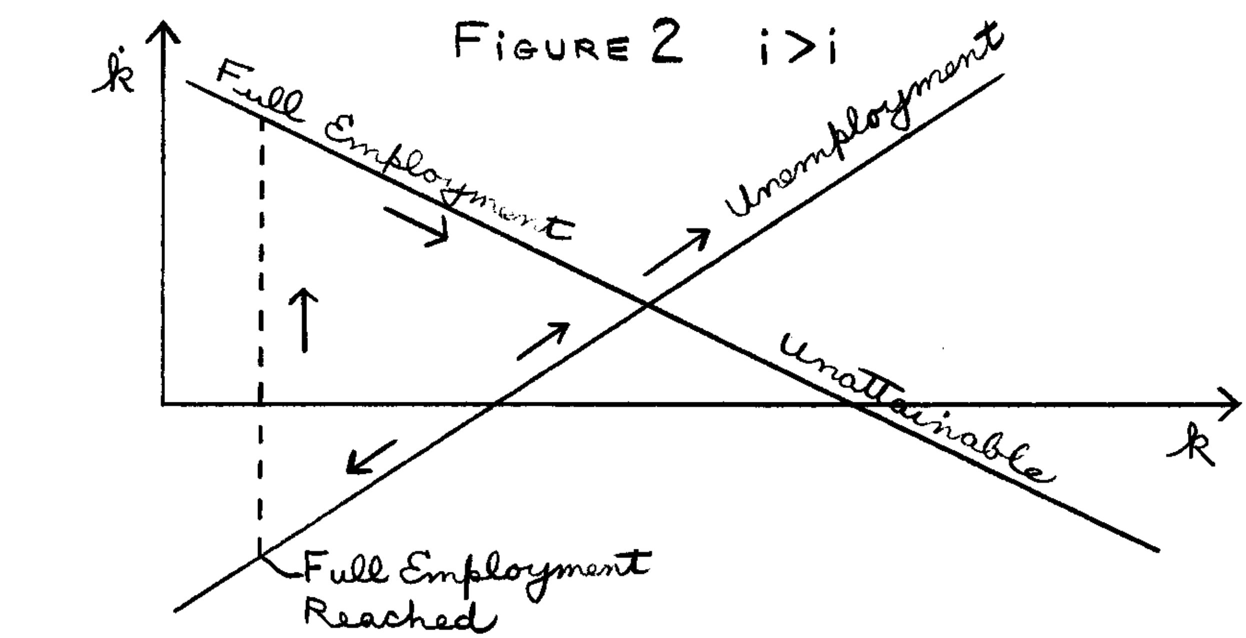
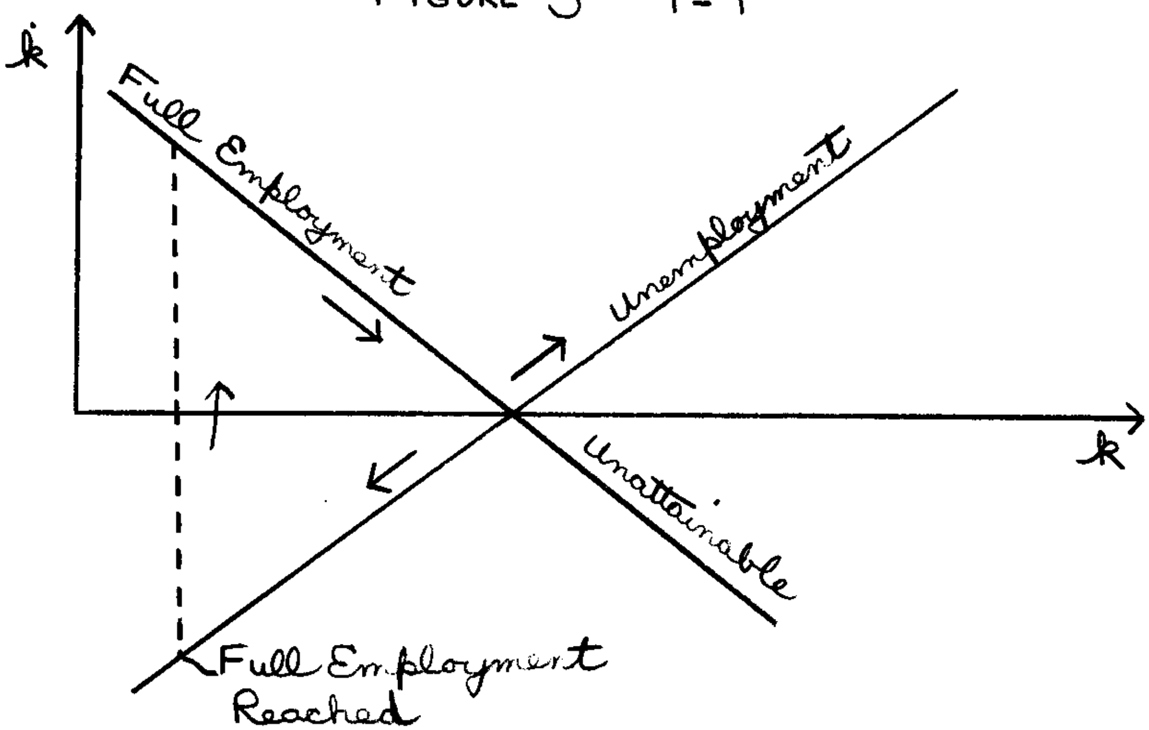


FIGURE 3 i = î


k̇
k
Full Employment
Unemployment
Unattainable
Full Employment Reached

5. Cobb Douglas Production Function

Equations (13), (16), (22) etc. can be explicitly solved when f is a Cobb Douglas production function. This section summarizes the previous discussion in terms of these convenient production functions. One has

$$f(k) = k^{\alpha}$$

$$f'(k) = \alpha k^{\alpha-1}$$

$$f - kf' = (1-\alpha) k^{\alpha}$$

$$f'' = \alpha(\alpha-1) k^{\alpha-2} < 0$$

5.1. Full employment. Equation (13) becomes

$$\frac{\dot{k}}{k} = \mu \alpha k^{\alpha-1} - \mu i - \hat{n}$$

$$\dot{k} = \mu \alpha k^{\alpha} - (\mu i + \hat{n})k = b k^{\alpha} - c k \text{ , say .}$$

Rearranging terms and multiplying by e^{ct}

$$e^{ct}(\dot{k} + ck) = b k^{\alpha} e^{ct}$$

$$\text{Let} \quad k e^{ct} = u$$

$\dot{u} = b u^{\alpha} e^{(1-\alpha)ct}$. Integrating we have

$$u(t)^{1-\alpha} = u(0)^{1-\alpha} + \frac{b}{c} \left[e^{(1-\alpha)ct} - 1 \right]$$

$$k(t)^{1-2} = k(0) \cdot e^{(\alpha-1)ct} + \frac{b}{c}\left[1 - e^{(\alpha-1)ct}\right]$$

$$k(t) = \left(k(0)\, e^{(\alpha-1)ct} + \frac{b}{c}\left[1 - e^{(\alpha-1)ct}\right]\right)^{\frac{1}{1-\alpha}}$$

$$\lim_{t \to \infty} k(t) = \left(\frac{b}{c}\right)^{\frac{1}{1-\alpha}} = \left(\frac{\mu\, \alpha}{\mu i + \hat{n}}\right)^{\frac{1}{1-\alpha}}$$

Thus

(27) $$\bar{k} = \left(\frac{\alpha}{i + \frac{\hat{n}}{\mu}}\right)^{\frac{1}{1-\alpha}}$$

is reached from every possible initial state of full employment.

Clearly $\bar{k}$ is a decreasing function of i. So is the capital output ratio

$$\frac{k}{f} = k^{1-\alpha} = \frac{\alpha}{i + \frac{\hat{n}}{\mu}}$$

Again, this is also the value of the accelerator. The rate of inflation as given by (17) becomes

$$\frac{\dot{p}}{p} = \frac{c_0 - 1}{c_1} + \frac{\hat{n}}{c_1} \cdot k^{1-\alpha}$$

$$= \frac{c_0 - 1}{c_1} + \frac{\hat{n}}{c_1} \cdot \frac{\alpha}{i + \frac{\hat{n}}{\mu}}$$

The natural rate of interest $i = \hat{i}$ is then determined by

$$0 = \frac{c_0 - 1}{c_1} + \frac{\hat{n}}{c_1} \cdot \frac{\alpha}{\hat{i} + \frac{\hat{n}}{\mu}}$$

(28) $$\hat{i} = \hat{n} \left(\frac{\alpha}{1-c_0} - \frac{1}{\mu} \right)$$

It is positive for sufficiently large μ. If μ is small

$$\mu < \frac{1-c_0}{\alpha}$$

then $\frac{\dot{p}}{p}$ is positive for all nonnegative interest rates i. In this case inflation is unavoidable and there exists no finite natural rate of interest.

The natural rate of interest -- when it exists -- is proportional to the rate of populatic growth, it is an increasing function of labor productivity α and of the speed of capital adjustment μ; it is inversely related to the savings coefficient $1 - c_0$.

For

$$\mu = \infty$$

$$\alpha = .75$$

$$c_0 = .75$$

the natural interest rate turns out to be that of population growth. (Cf. also the concluding section.)

5.2. <u>Unemployment</u>. Equation (22) becomes

$$\dot{k} = \frac{\mu i}{\alpha c_1} \quad k^{2-\alpha} + \frac{1 - c_0 - \mu \alpha}{\alpha c_1} \quad k$$

(29) $$= b \, k^{2-\alpha} - c \, k \qquad \text{where}$$

$$b = \frac{\mu i}{\alpha c_1} \qquad c = \frac{\mu \alpha + c_0 - 1}{\alpha \, c_1} > 0 \quad \text{if} \quad \bar{k} > 0 \, .$$

Define $u = e^{ct} k$ and integrate as before to obtain

(30) $$k(t) = \left(\left[k(0)^{\alpha - 1} - \frac{b}{c} \right] e^{(1-\alpha)ct} + \frac{b}{c} \right)^{\frac{1}{\alpha - 1}}$$

Suppose first that $k(0)^{\alpha-1} > \frac{b}{c} = \frac{\mu i \cdot \alpha \, c_1}{\alpha \, c_1 (\mu \, \alpha + c_0 - 1)}$

$$= \frac{\mu i}{\mu \, \alpha + c_0 - 1}$$

i.e., $$k(0) < \left(\frac{\mu i}{\mu \, \alpha + c_0 - 1} \right)^{\frac{1}{\alpha - 1}}$$

$$= \left(\frac{\mu \, \alpha + c_0 - 1}{\mu i} \right)^{\frac{1}{1 - \alpha}} = k^*$$

Then as $e^{(1-\alpha)ct}$ increases over time the term in parenthesis in (30) increases indefinitely and the right hand side of (30), being a negative power, decreases. Thus the capital labor ratio falls and the rate of growth of employment.

$$(31) \qquad n = m - \frac{\dot{k}}{k}$$

$$= (1 - c_0)\ \frac{f}{k} - \frac{\mu i}{\alpha c_1}\ k^{1-\alpha} + \frac{1-c_0-\mu\ \alpha}{\alpha c_1}$$

$$= (1-c_0)\ k^{\alpha-1} - \frac{\mu i}{\alpha c_1}\ \ k^{1-\alpha} + \frac{1-c_0-\mu\ \alpha}{\alpha c_1}$$

increases as k falls. Eventually full employment is reached and we are back in Case 5.1.

Secondly, when $k(0) > k^* = \left(\frac{\mu\ \alpha + c_0 - 1}{\mu i}\right)^{\frac{1}{1-\alpha}}$

then in

$$k = \frac{1}{\left(\left[k(0)^{\alpha-1} - \frac{\mu\ i}{\mu\ \alpha + c_0 - 1}\right]e^{(1-\alpha)ct} + \frac{b}{c}\right)^{\frac{1}{1-\alpha}}}$$

the first term in the denominator is negative. As its absolute value increases the denominator falls toward zero. The capital output ratio continues to increase and the rate of growth of employment as determined by (31) decreases, perpetuating unemployment.

6. Wicksell Revisited

The analysis of the previous section shows that in a state of initial unemployment for every market rate of interest there exists a critical capital labor ratio -- namely, its unstable equilibrium value -- such that any capital labor ratio exceeding it leads off to a process of increasing capital labor ratios and persistent unemployment and falling prices. The situation arising from an initial state of full employment -- which is reached also from initial unemployment when the capital labor ratio is below the equilibrium ratio -- has been fully analyzed by Wicksell more than 60 years ago. His conclusions still stand:

> "At any moment and in any economic situation there is always a certain rate of interest, at which the exchange value of money and the general level of commodity prices have no tendency to change. This can be called _the normal rate of interest_; its level is determined by the current natural rate of interest, the real return on capital in production, and must rise or fall with this.
>
> "If the rate of interest on money deviates _downwards_, be it ever so little, from this normal level prices will, as long as the deviation lasts, rise continuously; if it deviates _upwards_, they will fall indefinitely in the same way." [5, p. 82-83].

Wicksell's position is not entirely clear on the question of whether capital accumulation stimulated by the inflationary Wicksellian process might not bring down the natural rate of interest to the point where it equals the pre-set market rate. In the Lectures he writes "The objection has been raised ... that a lowering of the loan rate must also depress the real rate so that the difference between them is more and more leveled out and thus the stimulus to a continued rise in prices eliminated. This possibility certainly cannot be entirely rejected. Ceteris paribus a lowering of the real rate unconditionally demands new real capital, i.e., increased saving. But, this would certainly occur, even if unvoluntarily owing to the fact that higher prices would compel a restriction of consumption on the part of those people who had fixed money incomes, such as civil servants, unless they were able to secure increases in their salaries corresponding to the rise in prices. Against this, however, would have to be set the decrease in voluntary saving which a lowering of interest rates tends to produce. But if the former influence prevails, and if production is unable to absorb unlimited quantities of new capital without a reduction in net yield, then the incipient rise in prices, though it would certainly not recede, might yet be arrested, unless the banks reduced their rate still further. ..." [6].

However, the present analysis shows that Wicksell's original and unqualified position was the correct one. If the market rate of interest is below the natural rate, then even with capital accumulating to the point where its marginal productivity equals the loan rate, the process of inflation is not thereby stopped but once set in motion continues indefinitely.

Conclusion: The "Optimal" Rate of Interest

Since this analysis has explored the implications of fixing the interest rate, it might well be concluded by asking what is the best level at which the interest rate should be fixed if it were to be fixed. There is no doubt as to Wicksell's answer: at the natural rate. So staunch was his opposition to inflation that he urged the Swedish government repeatedly on grounds of moral justice to restore the prewar purchasing power of the Swedish crown after World War I which had dropped to about half its former value. [7].

Without entering into any value judgments concerning price stability, let us ask the purely technical question as to the rate of interest which, in the course of a process of balanced growth, maintains the maximal standard of per capita consumption. (Since the rate of growth of output is that of population, there can be no increase in the standard of living over time owing to the fact that technological change was excluded.) The condition for a maximum of $\frac{C}{N}$ is that

$$\frac{d}{di}\left(\frac{c}{N}\right) = 0 \qquad \text{Now}$$

$$\frac{d}{di}\left(\frac{c}{N}\right) = \frac{d}{di}\left[\left(c_0 - c_1 \frac{\dot{p}}{p}\right) f\right]$$

$$= \frac{d}{di}\left[\left(1 - \hat{n}\ \frac{k}{f}\right) f\right] \qquad \text{by (3b), (4b)}$$

$$= \frac{d}{d\bar{k}}\left[f(\bar{k}) - \hat{n}\ \bar{k}\right] \frac{d\bar{k}}{di}$$

$$= \left[f'(\bar{k}) - \hat{n}\right] \frac{d\bar{k}}{di} = 0$$

if and only if

$$f'(\bar{k}) \;=\; \hat{n}$$

(since $\frac{d\bar{k}}{di} < 0$ by (15)) .

Recalling

(14) $$f'(\bar{k}) \;=\; i + \frac{\hat{n}}{\mu}$$

we conclude that

$$\bar{I} + \frac{\hat{n}}{\mu} = \hat{n}$$

(32) $$i \;=\; \hat{n}\,(1 - \frac{1}{\mu})$$

For infinite speeds of adjustment this reduces to

(32a) $$i \;=\; \hat{n}$$

The "optimal" rate of interest at which per capita consumption is maximized at any time in a process of balanced growth equals(approximately for finite speeds and exactly for infinite speeds of capital adjustment) the rate of balanced growth $\hat{n}$.

This may be put also as follows: Under balanced growth the rate of growth is the appropriate measure of the opportunity cost of current consumption. However, only if this optimal interest rate is equal to or less than the natural rate is there an associated process of stable balanced growth (Figure 1).

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