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Remarks on a Rational Selection of a Decision Function
(Sequel to Discussion Papers, Statistics, 326, 326a)

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Here we shall consider the general problem G_n of all mixed problems corresponding to problems (u, D, S) where S is a set of n elements, and D contains a finite number of pure strategies. We shall prove:

Theorem 1.

The only solution of G_n is given by C(Q) when C(Q) is the set of strategies d which maximize $\sum_{s \in S} u(d,s)$.

As in the two state problem we may consider the geometrical representations of a problem Q. To each strategy d there corresponds a point in n dimensional space whose coordinates are $u(d,s_1)$, $u(d,s_2)$, ... $u(d,s_n)$ where s_1, s_2, ... s_n are the elements of S. A geometrical representation of a problem Q of G_n is convex because Q is mixed.

Most of the axioms and consequences in 326a may be geometrically interpreted. For example

A: If the representation of Q_1 is transformed to that of Q_2 by one of the following transformations, the set of points representing $C(Q_1)$ are similarly transformed into the set representing $C(Q_2)$.

1. relabeling of axes $x_i' = x_{j(i)}$ j(i) takes all values from 1 to n
2. translation $x_i' = x_i + \mu_i$
3. expansion or contraction $x_i' = cx_i$

B: The set of points corresponding to C(Q) form a convex set.

C: If x corresponds to an element of C(Q) and x = pu + (1 - p)v where $0 < p \leq 1$ and u, v are points of the geometrical representation of Q then u corresponds to and element of C(Q).

D: To any convex set in n dimensional space which is generated by a finite number of points [i.e. there are m points $u_1, u_2, \ldots, u_m$ so that for all x in the convex set there is a $p_1, p_2 \cdots p_m$ $0 \leq p_i$ $\Sigma p_i = 1$ so that $x = p_1 u_1 + p_2 u_2 + \ldots + p_m u_m$] such that every component of every point of this set is an attainable utility, there is at least one problem Q of G_n for which this convex set is a geometrical representation.

E: If x and y are points of a geometrical representation of Q and x is uniformly dominated by y [i.e. $x_i \leq y_i$ for all i and $x_i < y_i$, for some i] then x does not correspond to an element of C(Q).

Before proceeding with the proof it is well to note that the proof for the two dimensional case was considerably complicated by a desire to go as far as possible without axiom 12 [convexity]. Here we shall use the convexity axiom freely but not apply axiom 7 that $C(Q_1 + Q_2) \supset C(Q_1) \cap C(Q_2)$.

The Theorem is an immediate consequence of Lemmas 1, 2, 3, and 4.

Lemma 1. If Q is the problem whose geometrical representation is given by the convex set of x for which $m \leq x_i \leq M$ and $\Sigma x_i = c$, then C(Q) consists of all strategies of Q.

Lemma 2. If Q is a problem of G_n the set of strategies d which maximize $\sum_{s \in S} u(d,s)$ are in C(Q).

Lemma 3. If Q is a problem of G_n and $\sum_{s \in S} u(d^*,s) < \max_{d \in D} \sum_{s \in S} u(d,s)$. Then d^* is not in S.

Lemma 4. The solution C(Q) of Theorem 1, satisfies axioms one to twelve.

Lemma 1.

The case where there is only one point of the convex set is trivial. Hence we assume that $m n < c < n M$ and thus no point has all components equal to M nor all components equal to m. Suppose that $x = (x_1, x_2 \ldots x_n)$ corresponds to an element of C(Q) and has r components equal to m and s equal to M where $1 \leq r + s \leq n$.

Then by a relabeling of axes one may obtain another point y corresponding to an element of C(Q) so that $\frac{x + y}{2}$ which by convexity also corresponds to an element of C(Q) has less than r + s components equal to m or M. By induction it follows that there is an element z for which $m < z_i < M$ which corresponds to an element of C(Q). Given any point u of the convex set there is a p, $0 < p < 1$ and a point v of the convex set so that $z = p\,u + (1 - p)v$ [1] and hence all strategies are elements of C(Q).

Lemma 2.

$$\text{Let } m = \min_{\substack{d \in D \\ s \in S}} u(d,s) \qquad M = \max_{\substack{d \in D \\ s \in S}} u(d,s) \qquad c = \max_{d \in D} \sum_{s \in S} u(d,s)$$

Construct a problem $Q_1 \;\textcircled{D}\; Q$ so that Q_1 corresponds to the convex set [generated by a finite number of points] of points x so that $m \leq x_i \leq M$, $\Sigma x_i \leq c$. Each point of this convex set for which $\Sigma x_i < c$ is uniformly dominated by a point of this set for which $\Sigma x_i = c$. By axiom 6 $C(Q_1) = C(Q_2)$ where Q_2 is the problem derived from Q_1 by deleting strategies d^* for which $\sum_{s \in S} u(d^*,s) < c$. By Lemma 1, $C(Q_2) = C(Q_1)$ consists of all strategies of Q_2 and hence those strategies of Q for which $\sum_{s \in S} u(d,s) = c$. By axiom 5, these strategies are in C(Q) which establishes Lemma 2.

Lemma 3.

Consider two points a, b so that $\Sigma a_i > \Sigma b_i$. There is a point c so that c is uniformly dominated by a and $\Sigma c_i = \Sigma b_i$. Let $d = \frac{a + b}{2}$ $e = \frac{c + a}{2}$. [See Figure 1]

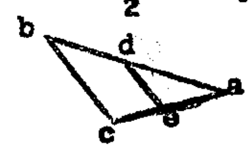

Fig. 1

Note 1. Every point in Δ a b c not on a b is uniformly dominated by a point on ab.[2]

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1. Let $v(p) = \frac{z - p\,u}{1 - p}$. For p sufficiently close to 0, v(p) is in the convex set.

2. Let $x = p_1 a + p_2 b + p_3 c$ where $p_i \geq 0$ $\Sigma p_i = 1$ $p_3 > 0$. If we set $y = (p_1 + p_3)c + p_2$ b then x is uniformly dominated by y.

Note 2. The quadrilateral b c e d is a convex set generated by b, c, d, e. $\Sigma e_1 \geq \Sigma x_1$ for x in this quadrilateral.

Consider Q_1 a problem for which the line ab is a geometrical representation.

$Q_2 \textcircled{D} Q_1$ a problem for which the triangle abc is a geometrical representation.

$Q_3 \textcircled{D} Q_2$ a problem for which the quadrilateral bced is a geometrical representation.

To establish this lemma, it suffices by axiom 5 to show that b does not correspond to an element of $C(Q_1)$. Suppose that it did. Then by note 1 and axiom 6 b would correspond to an element of $C(Q_2)$ and hence of $C(Q_3)$. By Lemma 2 and note 2, e corresponds to an element of $C(Q_2)$. By convexity $\frac{b + e}{2}$ does also. But $\frac{b + e}{2}$ is uniformly dominated by d and axiom 3 is not satisfied. Hence b does not correspond to an element of $C(Q)$ and lemma 3 is established.

Lemma 4.

This is easily established.