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NON-LINEAR SELF-EXCITED OSCILLATIONS AND BUSINESS CYCLES

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Introduction

This paper is an attempt to apply the theory of non-linear self-excited oscillations in mechanics to business cycle theory. The discussion is graphical rather than analytical. The analytical approach, though useful and indispensable at an advanced stage of research, will not be followed here.

The paper consists of three parts. Part I purports to be an exhaustive graphical study of linear systems. Parts II and III discuss non-linear systems. In Part II a simple model of self-excited economic fluctuations will be presented by piecing together two linear systems. Part III deals with Kalecki-Kaldor theory of business cycles in terms of the van der Pol-type equation and compares it with Hicks' and Goodwin's theory which, by the latter, has been cast into the Rayleigh-type equation.

I

Let us begin with linear systems. It is convenient to start with some well-known linear models of economic fluctuations. First, Samuelson's famous equations of an interaction between the acceleration principle and the multiplier, when transformed into differential equations according to Georgescu-Roegen,[1] are as follows:

(1.1) $$c = \alpha y - \alpha' \dot{y}, \qquad I = \beta \dot{c}, \qquad y = I + c + g,$$

where c = consumption, I = induced investment, y = income, g = government expenditure (a constant), and coefficients α, α', β are respectively constants, and $1 > \alpha > 0$, $\alpha' > 0$, $\beta > 0$. Deriving a single equation for y from (1.1), we have

(1.2) $$\alpha' \beta \ddot{y} + (\alpha' - \alpha\beta)\dot{y} + (1 - \alpha)y = g$$

Putting $Y = y - \frac{g}{1 - \alpha}$, this equation becomes

(1.3) $\alpha'\beta \ddot{Y} + (\alpha' - \alpha\beta)\dot{Y} + (1 - \alpha)Y = 0.$

Secondly, Goodwin's linear model consists of the two equations [2/]

(1.4) $e\dot{y} = \dot{k} - (1 - \alpha)y, \quad \xi\dot{k} = \delta y - k,$

where y = income in terms of deviation from the equilibrium value, k = the stock of capital, $\dot{k}$ = the rate of change of the capital stock, i.e., investment, α = the marginal propensity to consume, δ = the accelerator (δ y means the "ideal" stock of capital), α, e, and ξ are proportionality factors such that $1 > \alpha > 0$, $e > 0$ and $\xi > 0$. Deriving a single equation for y from (1.4) we get

(1.5) $e\xi \ddot{y} + [\xi(1 - \alpha) + e - \delta]\dot{y} + (1 - \alpha)y = 0.$

As the equations (1.3) and (1.5) are both homogeneous linear differential equations of the second order with constant coefficients, it is evident that we can express these equations in the following general form

(1.6) $m\ddot{x} + b\dot{x} + kx = 0. \quad (m, k > 0.)$

This is a fundamental equation of linear systems and corresponds to Hicks' difference equation in his "elementary case." In what follows we will investigate the nature of the solution of this equation.

The above equation can be solved graphically on the $(x, \dot{x})$ - plane; i.e., the phase plane, by means of the so-called method of isoclines. But it is more convenient for us to make use of Lienard's graphical construction which was originally developed with respect to a non-linear equation. For this purpose we transform the equation (1.6) into

(1.7) $$\frac{d^2x}{dt_1^2} + \frac{b}{\sqrt{km}}\frac{dx}{dt_1} + x = 0$$

by putting $t_1 = \sqrt{k/m}\, t$, or again

$$(1.8) \qquad v\frac{dv}{dx} + \frac{b}{\sqrt{km}}v + x = 0,$$

by putting $v = dx/dt_1$.

When we plot the following expression

$$(1.9) \qquad \Gamma : \qquad x = -\frac{b}{\sqrt{km}}v$$

on the phase plane (x, v), it is readily seen that (1.9) is a straight line passing through the origin, having a positive or negative slope according as b is negative or positive. This straight line will be called hereafter a characteristic. Suppose for a while $b < 0$, and the characteristic Γ be such as is depicted in Figure 1. Take any arbitrary (non-zero) point P on the plane, and let PN be a normal at this point to a curve which satisfies the equation (1.8) and passes through P. Then, NM: the projection of PN on the axis Ox is $-v\frac{dv}{dx} = \overline{NM}$ and $-\frac{b}{\sqrt{km}} = \overline{ON}$, $x = \overline{OM}$. Hence the equation (1.8) can be written as $\overline{NM} = \overline{OM} - \overline{ON}$. This relation gives us the procedure to construct the integral curves of the equation (1.8) the details of which we need not enter.

The shapes of these integral curves depend upon the slope of the characteristic Γ. To clarify the point, it is convenient to plot two lines $x = 2v$ and $x = -2v$, and divide the half plane above the x-axis into four domains A, B, C and D. (See Figure 2). We find that the integral curves can be classified into some categories according to the domains which the characteristic Γ passes through. The results are summarized in Table 1 and the corresponding seven diagrams of Figure 3. Only a brief explanation of them will be given here.

In all diagrams the characteristic is shown as a dotted line, and the characteristic roots in the last column of Table 1 refer to the roots of the characteristic equation for (1.7); namely, $\lambda^2 + \frac{b}{\sqrt{km}}\lambda + 1 = 0$.

I. The case $b > o$. This can be further classified into the following three subcases.

(i) $b > 2\sqrt{km}$. In this case Γ falls in the domain A (see Fig. 3.1). Apart from the straight lines: $\alpha\alpha$ and $\beta\beta$, the integral curves are of the shape of deformed parabolas. These curves are each of them tangent to $\alpha\alpha$ at the origin, and the further the curves are away from the origin, the more closely will they become parallel to $\beta\beta$. The tangents to the curves at intersections of the curves and the x-axis are parallel to v-axis, and those at intersections of the curves and the characteristic are parallel to the x-axis (this is true not only for the integral curves of the case under consideration, but for those of all other cases.). As $dt = dx/v$, the direction of the movement will clearly be such as shown by arrows in the diagram.

Since $b > 2\sqrt{km}$, the two characteristic roots: λ_1 and λ_2, are distinct negative numbers. Assuming $\lambda_1 > \lambda_2$, $\alpha\alpha$ is given by $v = \lambda_1 x$; $\beta\beta$ by $v = \lambda_2 x$. It is obvious that these two lines also satisfy (1.8). This is the case of "subsidence" or "overdamping," and the origin (the only singular point of the original differential equation) is called a stable nodal point.

(ii) $b = 2\sqrt{km}$. Here Γ coincides with a straight line determined by $x = -2v$ and falls on the boundary between the domains A and B. The integral curves are again deformed parabolas and touch $\gamma\gamma$ at the origin (See Fig. 3.2). The $\gamma\gamma$, which is determined by $v = -x$, is also an integral curve. The characteristic roots are equal negative numbers; strictly, $\lambda_1 = \lambda_2 = -1$. This is the case of "critical damping," and the origin is a stable nodal point.

(iii) $b < 2\sqrt{km}$. Γ passes through the domain B, and the integral curves are spirals around the origin (see Fig. 3.3). The characteristic roots are conjugate complex numbers with negative real parts. This is the case of damped oscillations, and the origin is called a stable spiral point.

II. The case b = 0. Here Γ coincides with the v-axis which divides the domain B and C. The integral curves represent circles, whose center is the origin. The characteristic roots are pure imaginaries. This is the case of simple harmonic motions, and the origin is a vortex point (see Fig. 3.4).

III. The case b < 0. This can be further classified into three cases: (i) $b > -2\sqrt{km}$, (ii) $b = -2\sqrt{km}$, and (iii) $b < -2\sqrt{km}$. With signs reversed, these correspond respectively to the three subcases under I. No further explanation will be called for.*

The above classification of integral curves in the phase plane is evidently the one corresponding to that of solutions made by Samuelson and Hicks with respect to their difference equations. It should be noticed, however, that the slope of Γ determines stability or instability of the origin, no matter of what shape the integral curves are.

II

Now we can proceed to non-linear systems. The simplest way to obtain a non-linear model of self-excited oscillations is to piece together two linear systems. Let us take Samuelson's equation (1.2) again. It is clear from the explanations above that the shapes of integral curves

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* Note that there is no integral curve for which the origin is a saddle point, because our differential equation cannot have characteristic roots which are real but of opposite signs.

which can be obtained from the reduced equation of (1.2) depend upon the relationships between the coefficients α, α' and β. We suppose these relationships to be such as to permit damped oscillations. This requires that $\alpha' - \alpha\beta > 0$ and $\alpha' - \alpha\beta < 2\sqrt{\alpha'\beta(1-\alpha)}$. Moreover, in the equation (1.2) government expenditure g is assumed constant whether y is increasing or decreasing. We relax this assumption here and assume g takes one positive constant m when y is increasing, whereas it takes another constant n, which is less than m, when y is decreasing. Under this assumption the behavior of y will be determined by

$$(2.1) \qquad \alpha'\beta\ddot{y} + (\alpha' - \alpha\beta)\dot{y} + (1 - \alpha)y = m,$$

when $\dot{y}$ is positive, and by

$$(2.2) \qquad \alpha'\beta\ddot{y} + (\alpha' - \alpha\beta)\dot{y} + (1 - \alpha)y = n,$$

when $\dot{y}$ is negative.

In order to solve graphically the two equations (2.1) and (2.2), it is more convenient to make a transformation

$$(2.3) \qquad Y = y - \frac{n}{1-\alpha}, \quad m - n = r$$

and obtain

$$(2.4) \qquad \alpha'\beta\ddot{Y} + (\alpha' - \alpha\beta)\dot{Y} + (1 - \alpha)Y = r$$

$$(2.5) \qquad \alpha'\beta\ddot{Y} + (\alpha' - \alpha\beta)\dot{Y} + (1 - \alpha)Y = 0.$$

Further to simplify the solution we make another transformation:

$$t_1 = \sqrt{(1-\alpha)/\alpha'\beta}\; t, \qquad v = \frac{dY}{dt_1}$$

and obtain

$$(2.6) \qquad v\frac{dv}{dY} + \frac{\alpha' - \alpha\beta}{\sqrt{\alpha'\beta(1-\alpha)}}v + Y = \frac{r}{1-\alpha}$$

$$(2.7) \qquad v\frac{dv}{dY} + \frac{\alpha' - \alpha\beta}{\sqrt{\alpha'\beta(1-\alpha)}}v + Y = 0.$$

Thus, as y behaves according to (2.1) when $\dot{y} > 0$; and according to (2.2) when $\dot{y} < 0$ in our original equations, Y in our new equations will behave according to (2.6) when $v > 0$; and according to (2.7) when $v < 0$.

Looking at our problem from the view-point of new equations, we plot on the (Y, v)-plane the characteristic of the equation (2.7); i.e.,

$$(2.8) \qquad Y = -\frac{\alpha' - \alpha\beta}{\sqrt{\alpha'\beta}\,(1-\alpha)}\, v.$$

It is evident that the characteristic mm in Figure 4 is in the domain B of Figure 2. Since, however, the equation (2.7) determines the movement of Y only when v is negative, the segment of its characteristic that may be used in the construction of integral curves is limited to the part <u>below</u> the horizontal axis. The counterpart of the characteristic <u>above</u> the horizontal axis can be obtained as follows. Draw a straight line parallelly to mm and at a distance $r/(1 - \alpha)$ from the origin ($= OO_1$), and this line nn will be the characteristic of the equation (2.6). Since (2.6) is only valid for Y when v is positive, this characteristic can be used only in the part <u>above</u> the horizontal axis.

Now we are in a position to obtain the graphical solution of our problem. Take arbitrarily a non-zero point on the horizontal axis, say, a in Figure 4. So long as Y is increasing, the trajectory of Y—i.e., the integral curve—can be drawn by using the upper part of the latter characteristic nn. But when the trajectory reaches a point on the horizontal axis, say, b in Figure 4, the equation in question changes from (2.6) to (2.7). Hence the trajectory thereafter should be drawn by using the lower part of the former characteristic mm. In this manner we finally get a limit cycle, which is independent of any initial condition.[3/] Though this limit cycle as such is obviously not to be taken as a business cycle, it may still be of some interest to economists.

III

There are two fundamental forms of non-linear differential equations for self-excited oscillations. One is

(3.1) $\ddot{x} + (A\dot{x}^2 - B)\dot{x} + x = 0, \qquad (A, B > 0)$

which was studied by Lord Rayleigh in 1883. It must be noticed here that the damping coefficient— $(A\dot{x}^2 - B)$ in the above equation—depends upon $\dot{x}$ in such a way that it is negative when the absolute value of $\dot{x}$ is small, and positive when it is large. We can generalize (3.1) into

(3.2) $\ddot{x} + f(\dot{x})\dot{x} + x = 0$

where $f(\dot{x})$ is a continuous function and takes negative values when $|\dot{x}|$ is small; positive values when it is large.

(3.2) is usually called the Rayleigh-type equation.

The second form is

(3.3) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0 \qquad (\varepsilon > 0)$

which was investigated by van der Pol in 1926. (3.3) is distinguished from the Rayleigh-type equation by the fact that its damping co-efficient depends on x instead of on $\dot{x}$. In other respects, however, the above equation is similar to the Rayleigh-type. In the van der Pol equation also the damping co-efficient is negative or positive according as the absolute value of x is small or large; more strictly, according as it is smaller or greater than unity.

Generalizing (3.3) into

(3.4) $\ddot{x} + f(x)\dot{x} + x = 0$

where $f(x)$ is a continuous function and takes negative or positive values according as $|x|$ is small or large, we shall designate equation (3.4) as the van der Pol-type equation.

Viewed from a purely mathematical-formal standpoint, (3.2) and (3.4) may be proved to be equivalent to each other in some cases. But it is preferable for us to regard them as different equations. The graphical solution of the Rayleigh-type equation can be obtained in the manner explained above, allowing for the fact that the characteristic $x = -f(\dot{x})\dot{x}$ is now a non-linear continuous curve. And it is evident from the earlier explanations that, with the characteristic of this form, the solution curve leads to a stable self-excited oscillation. The economic significance of this solution has been discussed by Goodwin.[4] As for the van der Pol-type equation, however, the graphical method of integration, as expounded above, could not be applied without some modifications; moreover, its economic significance seems to be left unexplained. It is to this type of equation that we approach here from the viewpoint of business-cycle theory.

Let us take up the Kalecki-Kaldor theory of business cycles, which is understood here as the theory originated by Kalecki and elaborated by Kaldor.[5] Hicks praised this theory in 1942 as "perhaps the only strikingly original contribution to the theory of fluctuations which has seen the light since September, 1939."[6] Even today some economists (such as S. C. Tsiang)[7] have evaluated it more highly than Hicks' own new theory of business cycles. It seems to me that Kalecki and Kaldor's is one of the most general and elegant among those theories which do not accept the acceleration principle as a valid explanation of the inducement to invest. I should like to reserve my opinion as to the merits and demerits of the acceleration principle, and I will not discuss here the relationships between the Hicks-Goodwin and the Kalecki-Kaldor types of theory. My main interest is to transform the Kalecki-

Kaldor theory into one of self-excited oscillations of the van der Pol type, just as Goodwin has, so to speak, transformed Hicks' into one of the Rayleigh type.

In the Kalecki-Kaldor theory, income, saving, and investment are measured respectively in gross and not net terms. As a first approximation, saving may be taken as a linear increasing function of income, but investment is alleged to be a non-linear increasing function of income as well as a decreasing function of the stock of capital. When we measure income (Y) along the horizontal axis and investment (I) and saving (S) along the vertical axis (Figure 5), the saving function may be represented by the straight line S. It does not matter whether or not this line passes through the origin. We assume for convenience that it does. The investment function, assuming the capital stock as given, will take a shape such as is illustrated in Figure 5. That is, its slope is flatter than the slope of the saving function for low and high levels of income, and steeper for "normal" levels. As the capital stock, K, is a parameter, the investment function will be different for different values of K. Assume for simplicity that investment (I) is an additive function of Y and K, i.e., $I = F(Y) + G(K)$. Then the investment function will shift upwards when K decreases, and downwards when K increases, in view of the fact that investment is a decreasing function of K. Thus we get a family of investment curves, as shown in Figure 5.

There are two investment curves, one touching the saving line from above at A, the other touching it from below at B. Although there are innumerable intersections of investment curves and the saving line, it must be noted that the investment curves lying between A and B all intersect the saving line three times, whereas the investment curves which lie above A or below B intersect it only once. Among all these points we can find the only one which represents the position of long-run or stationary equilibrium where investment (i.e. gross investment) is not only equal to saving, but also to the depreciation allowance of the stock of capital. That is to say, where net investment is zero. According to the Kalecki-Kaldor theory, this stationary equilibrium point is given by C, which is one of the middle meeting-points of the saving line and the investment curves that lie between A and B.

Now let us deal with our problem analytically. We suppose the depreciation allowance to be proportional to the capital stock K and denote it by δK (δ being a positive constant). Thus the condition that net investment is zero is expressed by

(3.5) $\quad I = \delta K$

and the condition that saving equals investment by

(3.6) $\quad I = sY$

where s stands for the marginal propensity to save, and is a positive constant. Though we have previously $I = F(Y) + G(K)$, we assume further that G(K) is linear. Thus our investment function becomes

(3.7) $\quad I = F(Y) - \mu K \qquad (\mu > 0).$

The solution of the last three equations gives us the value of income, investment, and the stock of capital in a position of stationary equilibrium. These values will be hereafter denoted respectively as Y_0, I_0, and K_0.

It is more convenient for us to argue in terms of deviations from the equilibrium values. If we put

(3.8) $\quad y = Y - Y_0, \qquad k = K - K_0, \qquad \overset{\circ}{k} = I - I_0$

and $f(y) = F(Y) - F(Y_0)$, our investment function may be written as

(3.9) $\quad \overset{\circ}{k} = f(y) - \mu k$.

The last expression is nothing but the equation of our investment curve with origin shifted from O to C, taking C as the stationary equilibrium point. In accordance with this shift, the equation of our saving line will be transformed from sY to sy.

Here we take into account the saving-investment theory of income determination, one of whose fundamental dynamic principles is that the rate of change of income is proportional to the difference between investment and saving.

This can be expressed as

(3.10) $\quad \varepsilon \overset{\circ}{Y} = I - sY \qquad (\varepsilon > 0)$

in our terms, or as

(3.11) $\quad \varepsilon \dot{y} = \overset{\circ}{k} - sy$

in terms of deviations from the equilibrium values. From equations (3.9) and (3.11) it follows that

(3.12) $\quad \varepsilon \overset{\circ}{y} = f(y) - \mu k - sy.$

Differentiating this with respect to t, we have

(3.13) $\quad \varepsilon \ddot{y} + [\varepsilon \mu + s - f'(y)]\, \dot{y} + \mu s y = 0.$

When we set $t_1 = \sqrt{\mu s/\varepsilon}\, t$, the above expression will be transformed into

(3.14) $$\frac{d^2y}{dt_1^2} + \frac{1}{\sqrt{\varepsilon \mu s}} [\varepsilon \mu + s - f'(y)] \frac{dy}{dt_1} + y = 0$$

It has already been assumed that $f'(y)$ is greater than s in the neighborhood of the point $y = 0$ (which corresponds to the point C in Figure 5), but smaller than s when further away from this point in either direction. Let us make an additional assumption, plausiable for the

Kalecki-Kaldor theory, that $f'(y)$ is not only greater than s, but greater than $\varepsilon\mu + s$ in the neighborhood of $y = 0$.*

Then the coefficient of dy/dt_1 (i.e., the damping coefficient) is negative or positive according as the absolute value of y is small or large. This means that equation (3.14) is precisely of the van der Pol type.

The method of graphical integration of this equation is well known, but may be summarized here. Simplifying the notation as follows

$$(3.15) \qquad \omega = \frac{1}{\sqrt{\varepsilon\mu s}}, \qquad \varphi(y) = \varepsilon\mu + s - f'(y), \qquad v = \frac{dy}{dt_1}$$

and putting further

$$(3.16) \qquad \psi(y) = \int_0^y \varphi(y)\,dy, \qquad x = v + \omega\psi(y),$$

the equation is transformed into

$$(3.17) \qquad \frac{dx}{dy} + \frac{y}{x - \omega\psi(y)} = 0$$

Plot $x = \omega\psi(y)$ on the (y, x) plane. The curve thus obtained will generally be of a shape such as that depicted in Figure 6, when we bear in mind the above relationships between $f'(y)$ and $(\varepsilon\mu + s)$ over the range of the absolute values of y, and that $\psi(y)$ means $(\varepsilon\mu + s)\,y - f(y)$. (It may be noted that ω is greater than unity for reasonable values of ε, μ, and s. Thus it may be conceived of as a magnifier of $\psi(y)$.) Let us take an arbitrary (non-zero) point, say M, on the (y,x) plane. Drop a perpendicular through M until it cuts the curve $x = \omega\psi(y)$ at P, and draw a horizontal line through P to cut the x-axis at Q, then join

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* Incidentally, this corresponds to the fact that in his equation for self-excited oscillations

$$\varepsilon\theta\ddot{y} + [\varepsilon + (1-\alpha)\theta]\,\dot{y} - \varphi(\dot{y}) + (1-\alpha)\,y = 0$$

Goodwin assumes that in the neighborhood of $\dot{y} = 0$, $\varphi'(\dot{y})$ is not only greater than ε, but greater than $\varepsilon + (1-\alpha)\,\theta$.

Table I

	domain through which the characteristic passes	integral curves	origin (singular point)	characteristic roots
$b > 2\sqrt{km}$	A	Fig. 3 (1)	nodal point (stable)	distinct negative numbers
$b = 2\sqrt{km}$	AB*	Fig. 3 (2)	nodal point (stable)	equal negative numbers
$b < 2\sqrt{km}$	B	Fig. 3 (3)	spiral point (stable)	conjugate complex numbers with negative real parts
	BC	Fig. 3 (4)	vortex point	pure imaginaries
$b > -2\sqrt{km}$	C	Fig. 3 (5)	spiral point (unstable)	conjugate complex numbers with positive real parts
$b = -2\sqrt{km}$	CD	Fig. 3 (6)	nodal point (unstable)	equal positive numbers
$b < -2\sqrt{km}$	D	Fig. 3 (7)	nodal point (unstable)	distinct positive numbers

* AB means the boundary of the domains A and B. The same for BC and CD.

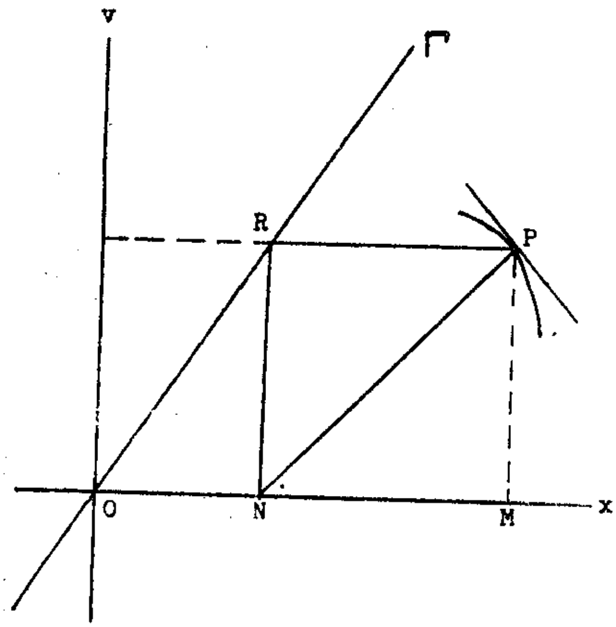

Fig. 1

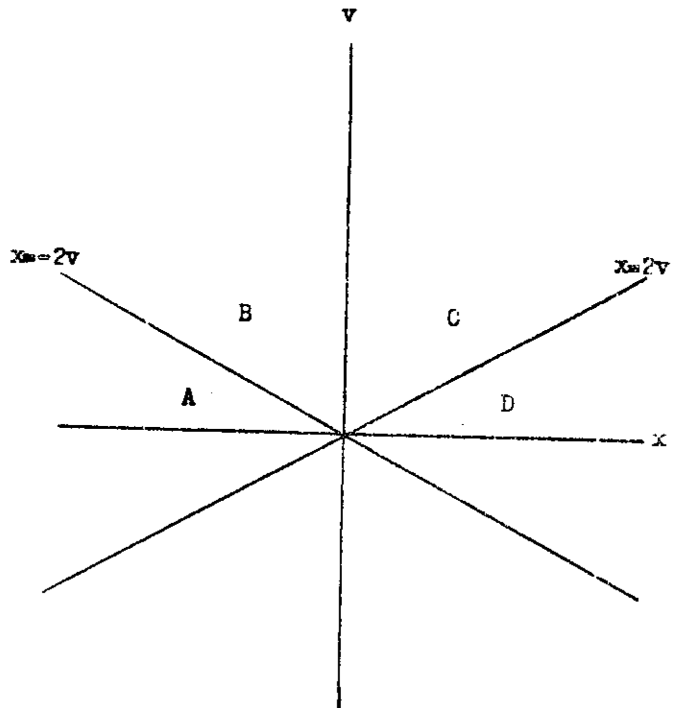

Fig. 2

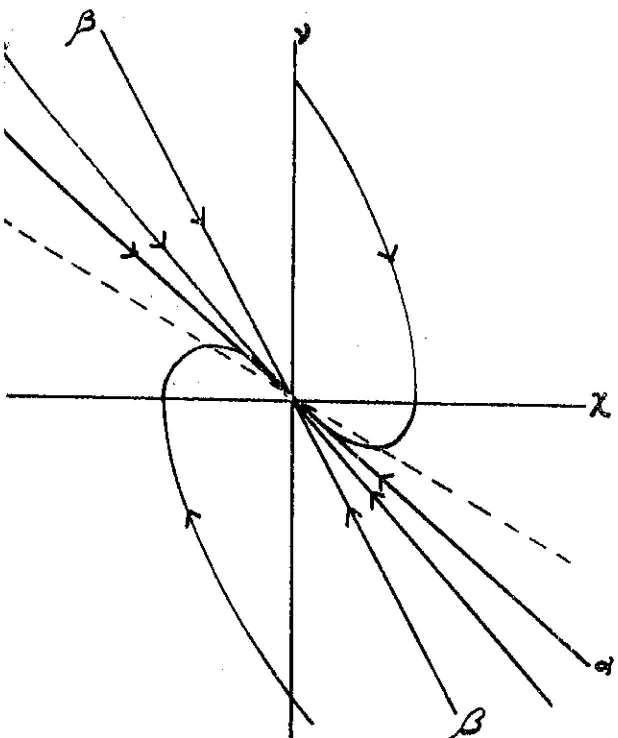

Fig. 3.1

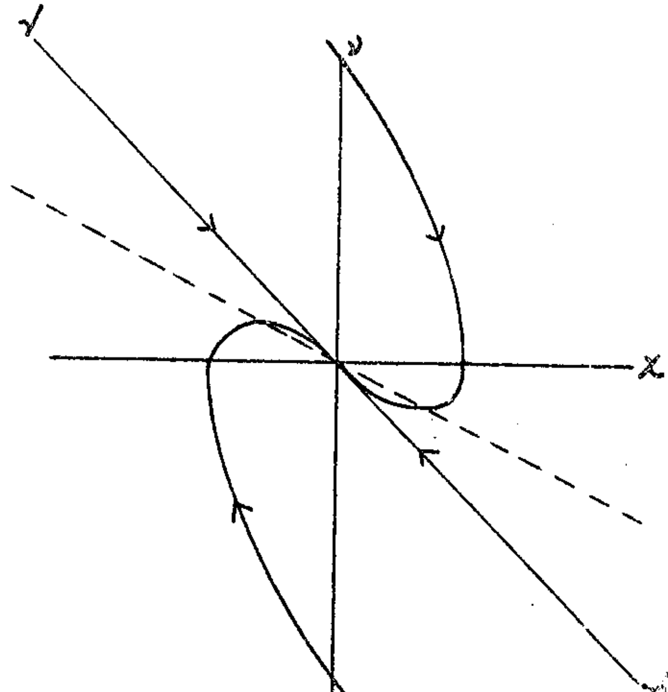

Fig. 3.2

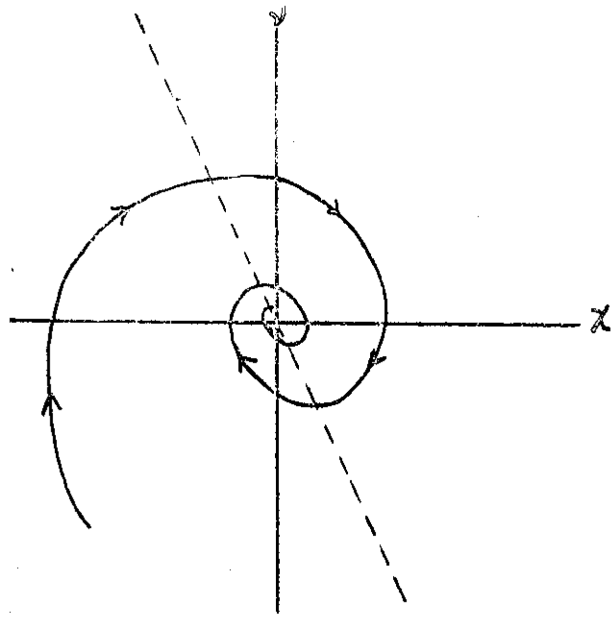

Fig. 3.3

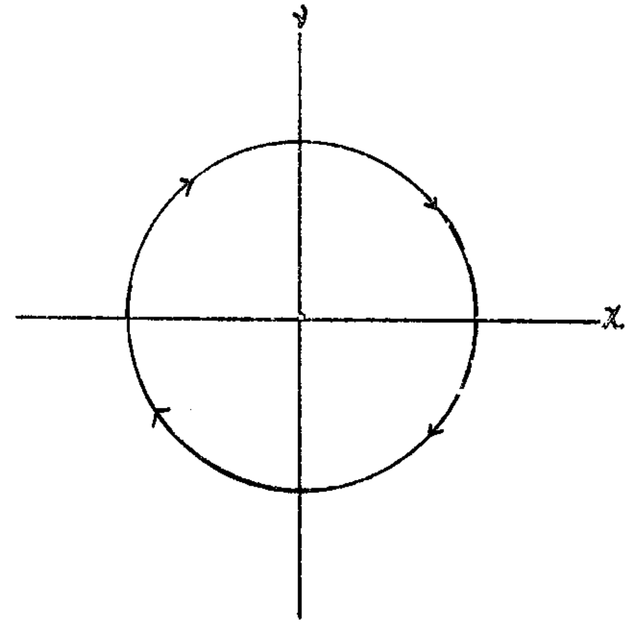

Fig. 3.4

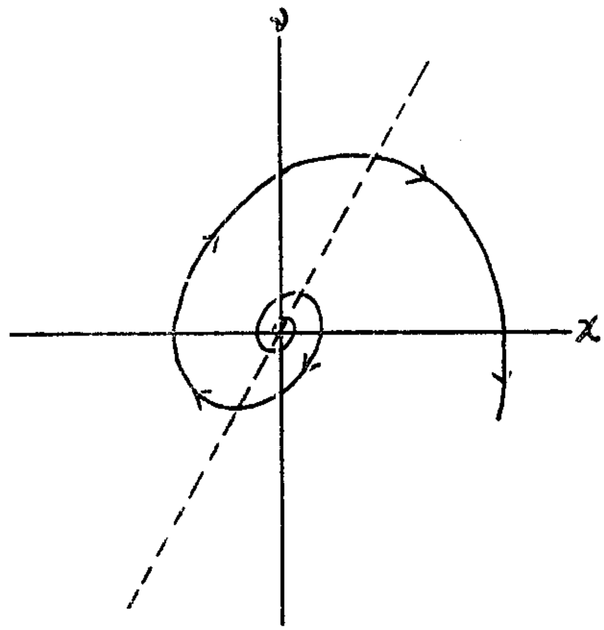

Fig. 3.5

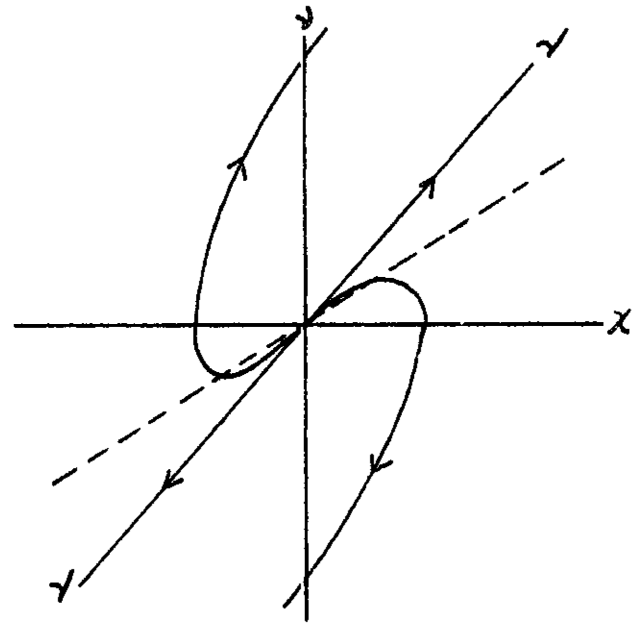

Fig. 3.6

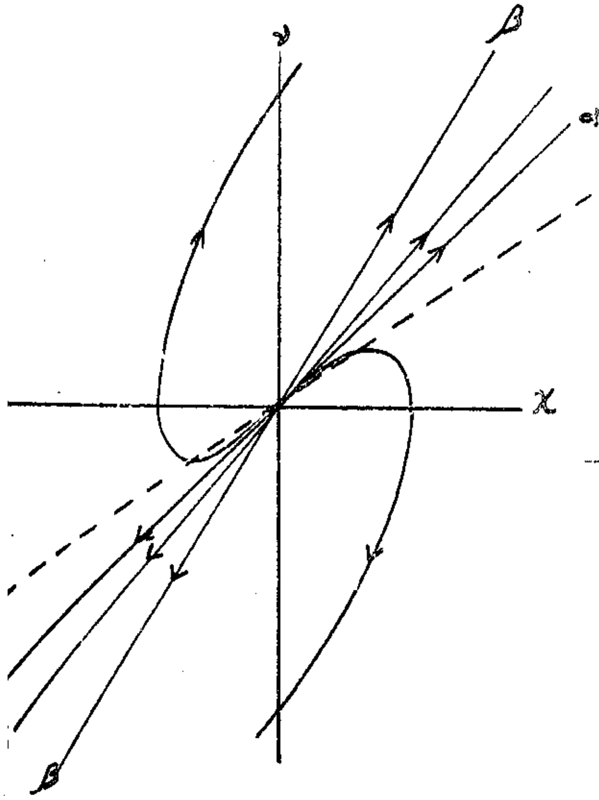

Fig. 3.7

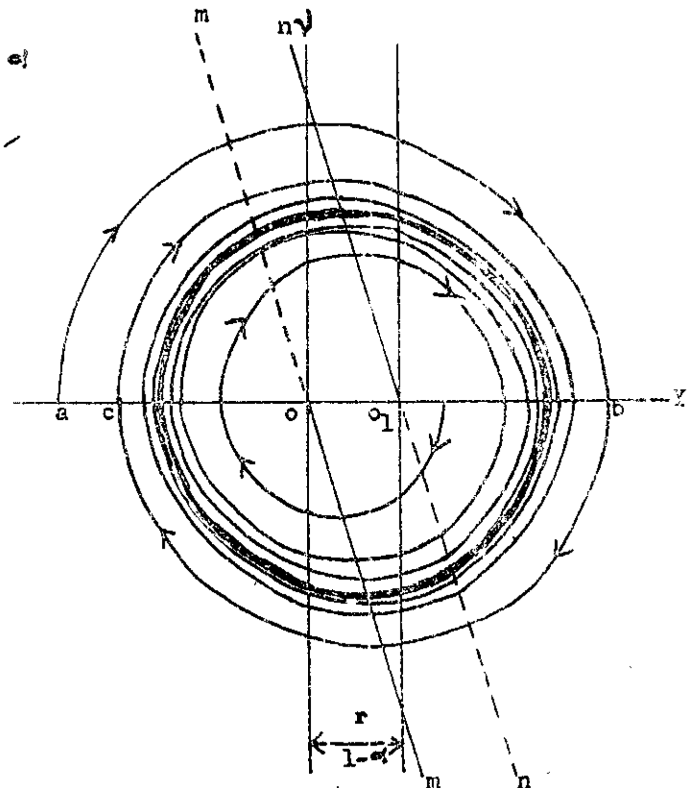

Fig. 4

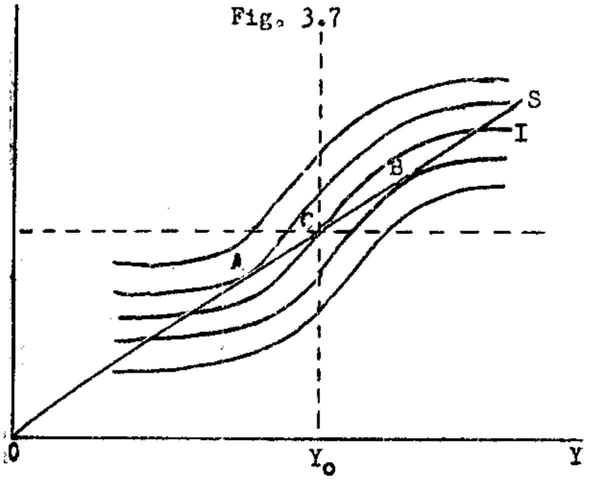

Fig. 5

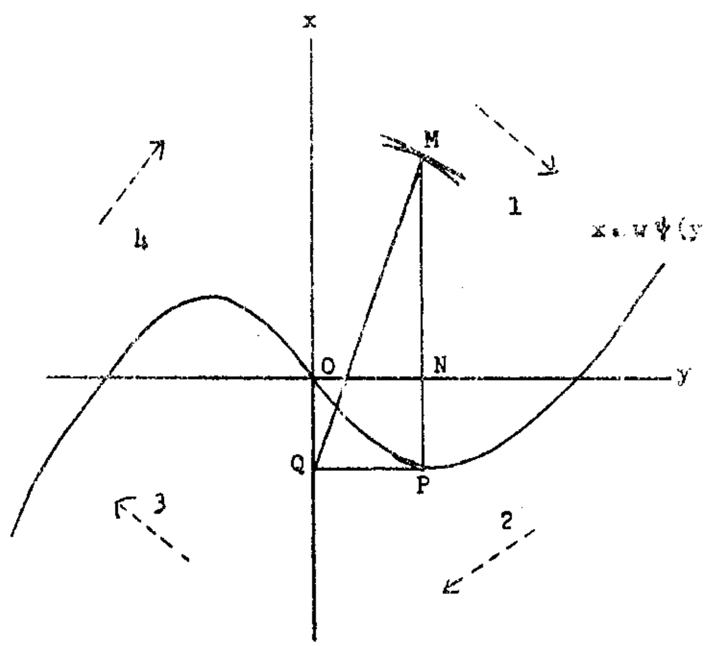

Fig. 6

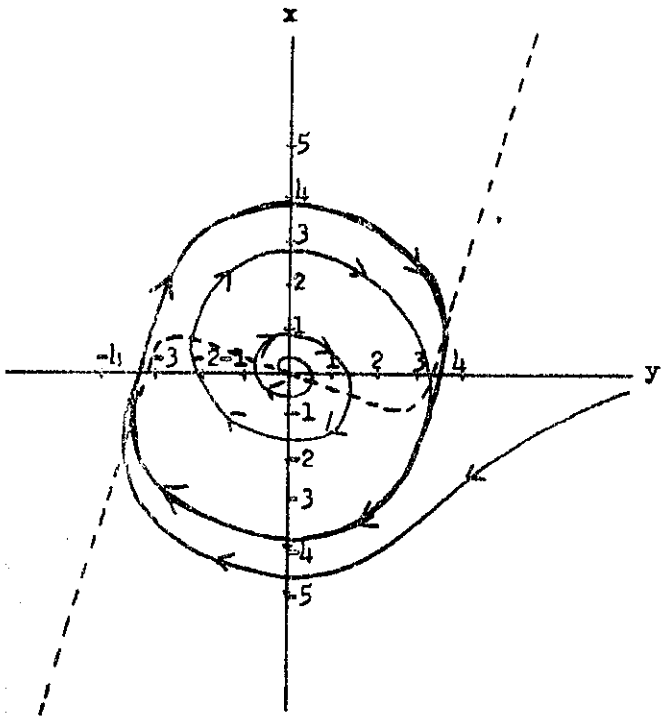

Fig. 7

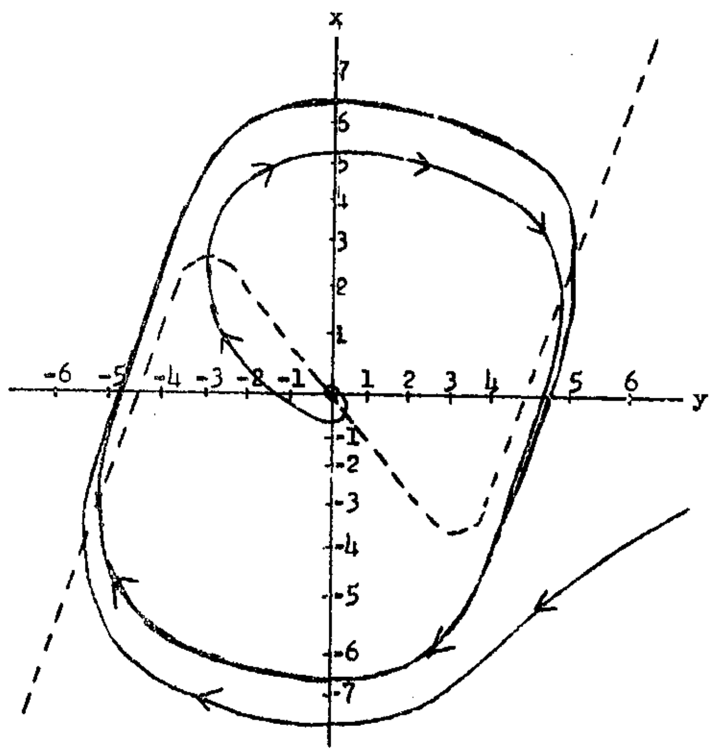

Fig. 8

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3/. For an analytical proof see A. A. Andronow and Chaikin; Theory of Oscillations, 1949. pp. 115-116, or N. Minorsky; Introduction to Nonlinear Mechanics, 1947, pp. 431-433.

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