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Incentive Functions and the Arbitrage Firm

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In the analysis, just presented here by Martin Beckmann,[1] the problem was: to find a rule that will maximize the expected profit.

He considered rules of the following form: After having observed x_i, the partner i can take one of the following three actions:

N_i : do nothing until contacted by the other partner

P_i : telephone the other partner at an arranged time

Q_i : decide to commit the firm for one unit, if no phone call is received.

The actions are symbolized by α_i. α_i can take one of the three values N_i, P_i and Q_i. The rule is a function γ_i

Rule: $\alpha_i = \gamma_i(x_i)$

which tells: For each x_i, a certain α_i should be taken. The set $\left\{ \gamma_1; \gamma_2 \right\}$ determines the behavior of the team completely.

Many teams operate under sets of rules like $\left\{ \gamma_1; \gamma_2 \right\}$. For instance, in a military organization, we would expect to find rules of this type. In the instruction-books, there are rules as to when requests for

1. See Cowles Commission Discussion Paper: Economics No. 2037.

munition should be sent in, when inventories of fuel should be reported, etc. These rules have once been designed by a central authority with the purpose of getting an optimal performance from the organization.

A team could also be organized in the following way. The central authority gives each member of the team a certain bonus function, or incentive function, and instructs him to maximize that function. In a chain store for instance, the local store manager is perhaps not given a book of rules, that state under what circumstances he should order goods from the wholesaling unit of the chain. Instead, he is paid and promoted according to how much profit he makes for his store. The way, in which this profit, is calculated, is determined by the central management. The problem to be solved by the central management is not to find an optimal set of rules, but such methods of calculating profits for each store as will induce an optimal performance for the chain-store organization as a whole. Among other things, the central management can set the rates for the charges for the services of the wholesaling unit. When setting these rates, the problem is not to find the "true" costs for the services, in order to charge these "true" costs. The problem is to find such "shadow-costs" as will induce an optimal performance.

From a mathematical point of view, an incentive function is a function that relates (actual or hypothetical) payment v_i to partner i to his actions α_i and observations x_i:

Incentive function: $v_i = v_i(\alpha_i; x_i)$.

Each partner i maximizes his incentive function v_i. When he has observed x_i he will select the value of α_i such that v_i is maximized. Call this value $\hat{\alpha}_i$. $\hat{\alpha}_i$ is a function of x_i, that we might call h_i:

$$\hat{\alpha}_i = h_i(x_i) \quad .$$

If v_i is not subject to any restrictions, it is always possible to find a v_i that gives

$$h_i \equiv \gamma_i$$

for any γ_i. In particular, this is true for an optimal $\hat{\gamma}_i$. This holds not only for our model of the arbitrage firm, but generally.

On the other hand, corresponding to a given system of incentive functions $\left\{v_1 \; v_2\right\}$, there exist rules γ_1 and γ_2 that give the same behavior of the team. In this sense, or from this point of view, the two methods are equivalent.

In most practical situations, we are not, however, free to choose the incentive function as we like. In the "shadow-cost" case, the problem was to find an optimal shadow-cost to be charged for some particular service or product in the organization, and the remainder of the incentive-function is given by accounting-rules, etc. When such restrictions are made upon the choice of incentive-functions, it is not always possible to find incentive-functions such that

$$h_i \equiv \gamma_i \quad .$$

In a practical situation, the rules cannot be too complicated. Certain restrictions have therefore to be placed upon the rules. If this is done, there might not always exist rules, that lead to the same behavior of the team as some given incentive functions.

Another reason for restrictions upon the rules and incentive functions is connected with changes in the circumstances, under which the team has to operate. If the distribution of demand in our example is changed, the optimal rules also must change, but there might exist incentive-functions that lead to optimal behavior both before and after the change. In other situations, the rules might have a higher degree of permanence than the incentive-functions.

Psychological aspects on the team situation are also important. It is, from a psychological point of view, very different things to follow a book of instructions or to maximize an incentive function. The types of motivation are quite different in these two situations. In some cases, it can be assumed that the rules will be observed strictly, in other cases not. Sometimes the members of the team have important secondary motives other than to maximize the incentive functions, and that might make it optimal for the team as a whole to control certain types of decisions by rules. The goals of the activity in the team, as the team-members perceive them, are not independent of the amount of communication. Identification with the team is a motivating force, and such identification is, up to a certain limit at least, stimulated by communication.

All these considerations restrict the possibilities to apply the kind of mathematical analysis, that we are discussing here, to practical problems. In spite of that, I believe that there are a few cases when psychological considerations, such as the ones just mentioned, are not so important, but instead the pure information and communication aspects.

We might also have the following kind of situation. A profit-function for each partner is the basis for the construction of all the incentive-functions we want to consider. The profit-function for a particular partner depends not only upon what this partner does, but also upon what the other partners do. We now have a game. The form of the incentive-functions is

$$v_i(\alpha_1 \ldots \alpha_i \ldots \alpha_n;\ x_i) \quad .$$

It is no longer possible to instruct each partner to maximize his incentive-function, because he does no longer control all variables in it.

In the von Neumann - theory of games, it is assumed that the players are able to communicate freely, without obstacles, and that they can keep and process unlimited amounts of information. These assumptions are essential for the elaborate structure of coalitions, that play a main role in von Neumann's analysis of multiperson-games. But evidently, such assumptions are not very suitable for a theory of organization, that has for purpose to direct the attention towards the influence of limited information and obstacles for communication.

Another approach to the game-problem has been developed by Nash. He studies non-cooperative games, in which communication between the players is not possible. A basic concept in his analysis is that of an equilibrium point. An equilibrium point is such a vector of strategies for all the players that each player cannot, given the strategies of all the other players, improve his position by choosing another strategy. Nash proves that each finite game has at least one equilibrium point.

In our notations, an equilibrium point would be a point $(\bar{\alpha}_1 \ldots \bar{\alpha}_i \ldots \bar{\alpha}_n)$ such that

$$\max_{\alpha_i} Ev_i(\bar{\alpha}_1 \ldots \alpha_i \ldots \bar{\alpha}_n; x_i) = Ev_i(\bar{\alpha}_1 \ldots \bar{\alpha}_i \ldots \bar{\alpha}_n; x_i)$$

for $i = 1, \ldots, n$.

For partner i, the maximum expected value of incentive payments, given the α's for the other partners, is reached for $\bar{\alpha}_i$.

If we were able to construct the incentive functions such that there existed a unique equilibrium point with this property, and also such that, in the equilibrium point, the $\bar{\alpha}_i$'s are those given by optimal rules

$$\bar{\alpha}_i = \hat{\gamma}_i(x_i)$$

then we could expect the team to behave in the same way under the incentive functions as under the optimal rules. In fact, each member could be instructed to regard present α's of his partners as constant. The efforts of each partner to maximize his incentive function would lead to a series of successive adjustments, ending in the equilibrium. Probably, we would need special arrangements to avoid cycles.

Now, let us look into the problem of incentive functions for the arbitrage firm. Evidently, any function of the following kind induces the same behavior as the Kiefer-optimum. We are, however, interested in incentive

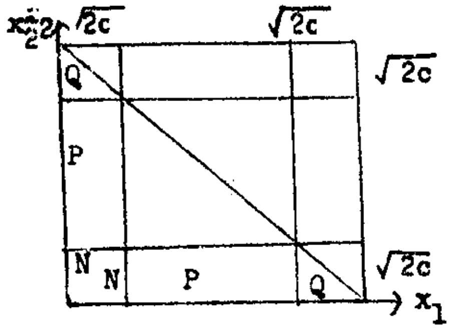
$$\frac{1}{2} + \frac{c}{8}\sqrt{c^2 + 8c}$$

functions that satisfy certain restrictions, namely: That the incentive-function is a sum of 1) a shadow-cost of communication, to be paid by a partner who decides $\alpha_i = P_i$, plus 2) the profits on such transactions as the particular partner has initiated.

For comparison, let me write down the expected profit (U) in Kiefer's solution, the best known. Each partner is assumed to be able to commit the firm for at most one unit.

$$U = \frac{1}{3} - c + \frac{c^3}{96} + \frac{c^2}{8} + \frac{c}{4} + \left(\frac{c^2}{8} - \frac{c}{3}\right)\sqrt{\frac{c}{2} - \frac{c^2}{16}} \ .$$

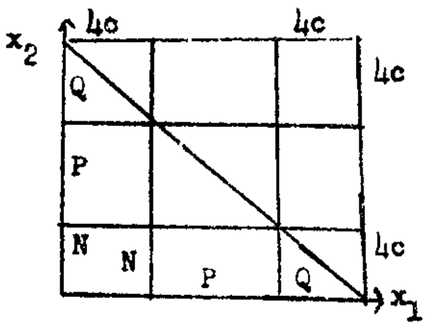
If the shadow-cost is made equal to the true cost (=c), the partners will adjust in the following way:



The lengths of the phoning and committing intervals are $\sqrt{2c}$, and the expected profit for the firm is

$$U = \frac{1}{3} - c + \frac{4}{3}c \sqrt{2c}.$$

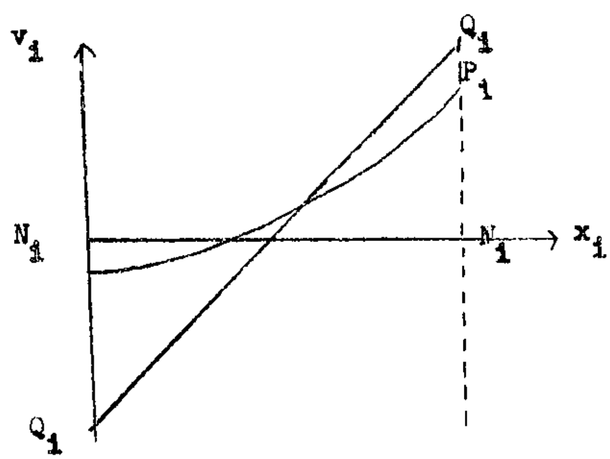
It is possible to do better than that. If the shadow-cost is made equal to $8c^2$, the lengths of the intervals become $4c$ instead of $\sqrt{2c}$. Expected profit:



$$U = \frac{1}{3} - c + \frac{64}{3} c^3$$

Better than before, but not as good as in Kiefer's case.

It does not seem possible to induce Kiefer-behavior with incentive-functions of this type. The functions for expected profit for partner i, if he makes α_i equal to N_i, Q_i and P_i, respectively, look like this:



All that we can do by changing the shadow-cost is to move the P_i-curve vertically. It is easily seen that this cannot lead to a two-interval situation like the one in Kiefer's solution.

In all the cases we have discussed, the incentive-function for each partner was independent of the other partner's actions. The rules for the splitting of profits were chosen with the very purpose of avoiding such an interdependence between the actions and the incentive-functions of the partners.

The possibility to use interdependent incentive-functions has not been analyzed. The incentive-functions would then be of the form:

$$v_1(\alpha_1; \alpha_2; x_1) \quad .$$

We could hope to be able to construct the incentive-functions in such a way that the behavior of the firm will, in a series of adjustments, approach a unique Nash-equilibrium-point, that corresponds to the optimal solution in the rule-case.