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Efficient Transportation in Networks Continued[1/]

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1. Comparative Statics 2. Applicability of the Gradient Method

3. Extensions of the Problem.

It is an empirical fact and has been assumed in the paper that the road capacity function c(u) is first increasing with speed and then decreasing. It is also true empirically, that with a reasonable value assigned to time the more economical speeds fall into the range of the declining branch of the capacity function. If we restrict ourselves to this branch then flow is a decreasing function of speed, and the inverse of this function exists and will eliminate speed from the transportation cost function and (if we assume an equipment constraint) the equipment use function. Whether this is useful depends on the convexity properties of the new functions thus obtained. For the transportation cost function the desired convexity has been derived on p. 24 of C.C.D.P. 2049A from previous plausible assumptions on the capacity functions and the cost functions. For the equipment use function in its simplest form, (the quotient of road length times flow and speed), the desired convexity also follows immediately along these lines. It must be noted that substitution of a flow variable for speed makes necessary another constraint, namely one on the maximal flow possible per unit of road width as indicated by the peak of the capacity function. From now on the problem will be considered only in terms of flow variables.

1. Comparative Statics

1.1 In order to study the effects of changes in the capacity data on the solution, the problem of efficient road utilization shall be embedded in the more general problem of determining the optimal layout of a transportation network for a static transportation program which is also of intrinsic interest.

Notations (largely those of 2049).

Indices:

m	index of commodity
i,j	indices of location

Variables:

x_{ij}^m	flow of m on road ij
x_{ij}	total flow on road ij
r_{ij}	capacity level (width) of road ij

Data:

q_i^m	net shipments of m from i (the programs)
c_{ij}	maximal flow intensity on road ij
$h_{ij}\left(\frac{x_{ij}}{r_{ij}}\right)$	transportation costs per unit flow on road ij as a function of flow intensity
f_{ij}	construction cost per unit width of road ij

The problem is that of minimizing total cost of construction and transportation with respect to flows and capacity levels subject to the requirements of a static transportation program.

Find
$$\min_{\substack{x^m_{ij} \geq 0 \\ r_{ij} \geq 0}} \sum_{ij} h_{ij} \left(\frac{x_{ij}}{r_{ij}}\right) + f_{ij} r_{ij}$$

subject to

(1) $\sum_j (x^m_{ij} - x^m_{ji}) \leq q^m_i$

(2) $x_{ij} \leq c_{ij} r_{ij}$

Here the constraints are linear, but the minimand is not jointly convex in x_{ij} and r_{ij}. Hence application of the Kuhn-Tucker theorem is not possible, but theorem I of F. John [1] gives as necessary conditions for a (local) minimum the following relation, to be called the "efficiency conditions".

(3) $\lambda^m_j - \lambda^m_i \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} h_{ij} + h'_{ij} \frac{x_{ij}}{r_{ij}} + \mu_{ij}$ if $x^m_{ij} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$

(4) $h'_{ij} \cdot \left(\frac{x_{ij}}{r_{ij}}\right)^2 + \mu_{ij} c_{ij} \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} f_{ij}$ if $r_{ij} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$

Here h'_{ij} denotes the derivative of h_{ij}; μ_{ij} is a non-negative parameter which vanishes if in the constraint (2) the $<$ sign holds; the λ^m_i are all non-negative parameters, for the "=" sign must hold in all inequalities (1) since $\sum_i q^m_i = 0$ is required for the program to be consistent.

1.2 Our aim is to show that the system (1) ... (4) has an (essentially) unique solution and thus that it is also sufficient for determination of the minimum. Now apart from the expressions after "if", x^m_{ij} and r_{ij} occur only as $\frac{x_{ij}}{r_{ij}}$ in the efficiency conditions. Writing $\frac{x_{ij}}{r_{ij}} = z_{ij}$ we show first that the z_{ij} and μ_{ij} are determined jointly and uniqely by (4) and (2) only. Suppose first that the constraints (2) are absent. z_{ij} is then determined by (4) as follows: Either $h'_{ij}(z_{ij}) \cdot (z_{ij})^2 < f_{ij}$ for all z_{ij} and then $r_{ij} = 0$ and hence z_{ij} arbitrary (say zero) or there exists one, and because of the convexity of h_{ij} only one, root z_{ij} with $h'_{ij}(z_{ij}) z_{ij}^2 = f_{ij}$. In the latter case this is the unique solution for z_{ij}. Suppose next that the constraints (2) are present. Then either $h'_{ij}(z_{ij}) \cdot (z_{ij})^2 = f_{ij}$ for some $z_{ij} \leq c_{ij}$. Then the previous considerations apply and $\mu_{ij} = 0$. Or otherwise there exists a $\mu_{ij} > 0$ such that $h'_{ij}\ (c_{ij})\ (c_{ij})^2 + \mu_{ij}\ c_{ij} = f_{ij}$ and this μ_{ij} is unique. Thus in each case μ_{ij} and z_{ij} are jointly and uniquely determined by (2) and (4).

Remark: Conversely the implications of (2) and (4) are restricted to z_{ij} and μ_{ij}. Thus nothing follows for the absolute magnitudes of either x_{ij} or r_{ij}. In particular $\mu_{ij} > 0$ does not imply that the particular road will be built at all: the cost of transportation on it may be prohibitive due to this μ_{ij}.

With z_{ij} and μ_{ij} determined, the right hand side in (3) becomes a constant (depending only on ij) and the inequalities (3) are the familiar Koopmans condition of transport efficiency with the specific transportation cost function $h_{ij} + h'_{ij}\ z_{ij} + \mu_{ij}$. It is known, that (3) in conjunction with (1) has a solution unique up to "neutral circuits" [3],

This concludes the proof that (1) ... (4) are sufficient for a solution of the road layout problem, and that its solution is essentially unique. The economic content of the preceding argument is, that the optimal flow intensity on each road (and the potentially optimal intensity on roads not built) depends only on the unit cost of construction for that road. The whole argument breaks down if also an equipment limit is introduced, for the Lagrange parameter of this constraint would depend also on the values of the x_{ij} rather than the z_{ij} alone.

In equation (3) the direct cost h_{ij} is supplemented by the term $h'_{ij}\ z_{ij} + \mu_{ij}$, to be called the social cost or efficiency toll p_{ij} (cf. p. 16 , 2049A). Multiplying (4) by r_{ij} on both sides and noting that $r_{ij}\ c_{ij} = x_{ij}$ if $\mu_{ij} > 0$, we obtain $p_{ij}\ x_{ij} \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} f_{ij}\ r_{ij}$ if $r_{ij} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$. This says that the social cost pays exactly for a given road if constructed, and does not amount to its building cost, if not constructed. This is just another way of expressing the independence of the optimal flow intensity on a given road from factors outside that road.

Finally, it is readily checked from (4) that z_{ij} is a non-decreasing function of f_{ij}, and that it is strictly increasing if h_{ij} is strictly convex and $z_{ij} < c_{ij}$. Now since the efficiency toll p_{ij} is a non-decreasing function of z_{ij} and of μ_{ij} it follows that it is non-decreasing with f_{ij}. In particular p_{ij} is a strictly increasing function of f_{ij} if h_{ij} is strictly convex. Since efficiency toll and construction cost thus move together, the question whether the efficiency toll increases with decreasing capacity, can be decided by considering whether capacity is a decreasing function of construction cost. It is for this purpose that the construction problem has been set up. Let the minimand be simply denoted by $M(x,r,f)$ where x and r stand for the optimal values of these variables given the f_{ij}. Similarly let $x + \delta x$, $r + \delta r$ be the optimal values given that construction costs are $f_{ij} + \delta f_{ij}$. Since both systems of variables satisfy the same side condition we have clearly

$$M(x,r;f) \leqq M(x + \delta x,\ r + \delta r;\ f)$$

$$M(x,\ r;\ f + \delta f) \geqq M(\ x + \delta x,\ r + \delta r;\ f + \delta f)$$

On forming the difference of the second and first line most of the (linear) terms cancel and we obtain

$$\sum_{ij} \delta f_{ij}\, r_{ij} \geqq \sum_{ij} \delta f_{ij}\, (r_{ij} + \delta r_{ij}) \qquad \text{or}$$

$$\text{(5)} \qquad \sum_{ij} \delta f_{ij}\ \delta r_{ij} \leqq 0$$

Here the strict inequality sign holds except for those $\delta f_{ij} > 0$ for which also $r_{ij} = 0$, that is unless the changes in cost affect only unconstructed roads. In particular now if all but one f_{ij} are fixed, if the corresponding

r_{ij} is > 0, and if h_{ij} is strictly convex, then $\delta f_{ij} \ \delta r_{ij} < 0$ and so $\delta p_{ij} \ \delta r_{ij} < 0$. In words: the efficiency toll is a non-decreasing function of capacity; it is a strictly decreasing function of capacity, provided that capacity is positive and the transportation cost function is strictly convex.

1.3 We conclude this section by proving the related

Theorem: The product of efficiency toll and capacity loss (gain) summed over all roads altered, is a upper bound of the economic loss[2/] (gain) to the system in terms of the transportation cost required to sustain the given program.

$$\Delta L \geq \sum_{ij} f_{ij} \, \Delta r_{ij}$$

Consider the allocation problem with given road capacities (problem 1 of 2049A), but let the latter depend on a (time) parameter $r_{ij} = r_{ij}(t)$. Write $x_{ij}(r)$ for the solution x_{ij} as a function of the paraters r_{ij}. Now the cost of transportation (the value of the minimum) associated with a solution x_{ij} is given by the Lagrangean expression

$$L(x, r, \mu, \lambda) = \Sigma\, h_{ij}\left(\frac{x_{ij}}{r_{ij}}\right) x_{ij} + \Sigma\, \mu_{ij}\,(x_{ij} - c_{ij}\, r_{ij})$$

$$+ \sum_{i,m} \lambda_i^m \left[\, \Sigma(x_{ij}^m - x_{ji}^m) - q_i^m \,\right]$$

The differential with respect to time of this expression equals

$$\sum_{k\ell, ij, m}\left[\frac{\partial L}{\partial x_{ij}}\ \frac{\partial x_{ij}}{\partial r_{k\ell}} + \frac{\partial L}{\partial \lambda_i^m}\ \frac{\partial \lambda_i^m}{\partial r_{k\ell}} + \frac{\partial L}{\partial \mu_{ij}}\ \frac{\partial \mu_{ij}}{\partial r_{k\ell}} + \frac{\partial L}{\partial r_{k\ell}}\right] dr_{k\ell}(t)$$

and here the first three terms vanish by the conditions (3), (1), (2) respectively. Thus

$$dL(t) = \sum_{k\chi} \frac{\partial L}{\partial r_{k\chi}} \frac{dr_{k\chi}}{dt} dt = \sum_{k\chi} [-h'_{k\chi} \frac{x^2_{k\chi}}{r^2_{k\chi}} + \mu_{k\chi} c_{k\chi}] dr_{k\chi}(t)$$

$$= \sum_{k\chi} f_{k\chi} \frac{dr_{k\chi}}{dt} \qquad \text{by (4)}$$

(for in the case $f_{k\chi} > \mu_{k\chi} c_{k\chi} - h'_{k\chi} \frac{x^2_{k\chi}}{r^2_{k\chi}}$

$r_{k\chi}(t)$ vanishes in an entire neighborhood of that t and hence $\frac{dr_{k\chi}}{dt} = 0$).

Thus $$L(x, r(t), \mu, \lambda) \Big|_{t_o}^{t_1} = \int_{t_o}^{t_1} \frac{dL}{dt} dt$$

$$= \int_{t_o}^{t_1} \sum_{k\chi} f_{k\chi}(t) \frac{dr_{k\chi}}{dt} dt \leq \int_{t_o}^{t_1} \sum f_{k\chi}(t_o) \frac{dr_{k\chi}}{dt} dt$$

because of relation (5),

$$= \sum_{k\chi} f_{k\chi}(t_o) [r_{k\chi}(t_1) - r_{k\chi}(t_o)] \quad \text{or in short}$$

$$\Delta L \leq \sum_{ij} f_{ij} \Delta r_{ij} \quad \text{as asserted.}$$

In a similar fashion as was done in this section for capacities, the comparative statics of different programs, in isolation or jointly with capacities, may be studied. The result is that the efficiency prices λ_i^m are non-decreasing functionsof net exports q_i^m 3/

2. Applicability of the Gradient Method

In a first approximation and for all purposes of computation the transportation cost function may be assumed linear and the capacity function piece-wise linear, with respect to speed. Note that then the minimand becomes a quadratic function of flows subject to linear contraints of capacity. The purpose of this section is to show that the straightforward gradient method will converge to a solution.

Writing $h_{ij}(x_{ij}) = \frac{1}{2} h_{ij} x_{ij}^2 + b_{ij}$ where $h_{ij} > 0$ for all ij the problem is to find

$$\min_{x_{ij}^m \geq 0} \sum_{ij} \left(\frac{1}{2} h_{ij} x_{ij}^2 + b_{ij} x_{ij}\right)$$

subject to

(1) $$\sum_j (x_{ij}^m - x_{ji}^m) = q_i^m$$

(2a) $$x_{ij} \leq c_{ij}$$

We have the efficiency conditions

(3a) $$x_{ij}^m \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \text{ if } h_{ij} x_{ij} + \lambda_i^m - \lambda_j^m + b_{ij} \begin{Bmatrix} = \\ > \end{Bmatrix} 0$$

(3b) $$\mu_{ij} \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \qquad \text{if} \quad x_{ij} \begin{Bmatrix} = \\ < \end{Bmatrix} c_{ij}.$$

The gradient method consists in considering a system of differential equations

(6) $$\begin{cases} \dfrac{dx_{ij}^m}{dt} = - h_{ij} x_{ij} + \lambda_j^m - \lambda_i^m + b_{ij} \\ \dfrac{d\lambda_i^m}{dt} = - \sum_j (x_{ij}^m - x_{ji}^m) + q_i^m \\ \dfrac{d\mu_{ij}}{dt} = x_{ij} - c_{ij} \end{cases}$$

with the provision that the derivatives be put equal to zero if the corresponding variable vanishes and the right hand expression is non-positive. This serves to maintain non-negativitiy of the x, λ , μ once these are chosen with these signs. It is convenient to write (6) in matrix form $\dot{u} = Au + a$ where A is a matrix of the form $\begin{pmatrix} D & Q \\ -Q' & 0 \end{pmatrix}$ in which D denotes a diagonal matrix whose diagonal elements are all negative, and 0 is a zero matrix. We note that after (temporary) suppression of one variable because of the non-negativity provisions), the new matrix is of this same type. For the convergence of the system it is therefore sufficient that all the characteristic roots of any matrix of that type have negative real parts. 4/

Let $i\alpha + \beta$ be a characteristic root of A and let $v = \begin{pmatrix} z \\ w \end{pmatrix}$ be the corresponding eigen-vector. Then

$$Dz + Qw = i\alpha z + \beta z$$

$$-Q'z = i\alpha w + \beta w$$

Pre-multiplying the first line by $\bar{z}'$, postmultiplying the conjugate transpose of the second by w and adding the two lines yields

$$\bar{z}' Dz = i\alpha(\bar{z}'z - \bar{w}'w) + \beta(\bar{z}'z + \bar{w}'w)$$

But the left hand expression by assumption is negative unless $z = 0$ which would entail $w = 0$, a contradiction. Hence $\beta < 0$ as asserted.

The result then is that flows and prices of the efficient solution may be found as the stationary, stable solution of the differential system (6). This, incidentally, involves $(M + 1)L + I(M - 1)$ independent variables and the same number of equations if M, L, I are respectively the number of commodities, roads and terminals.

3. Extensions of the Problem

3.1 Denote by

$h_{ij}\left(\frac{x_{ij}}{r_{ij}}, \frac{x_{ji}}{r_{ji}}\right)$	the expected cost of transportation on road ij as a function of the expected flow intensities in both directions of the two-way road,
$w_{ij}\left(\frac{x_{\chi j}}{r_{\chi j}}\right)$	the expected waiting time at exit j of road ij for one vehicle as a function of the intensities of all flows passing j,
$s_{ij}\left(\frac{x_{ij}}{r_{ij}}, \frac{x_{ji}}{r_{ji}}\right)$	the expected time requiredto travel over road ij
e	the total stock of equipment available.

Note that the relevant variables are now expected values, tentatively identified with averages over time of the respective magnitudes for the particular roads. While still in a rather crude form, this constitutes an attempt at recognizing the stochastic nature of traffic flows.

We now have a constraint on the use of equipment

$$(7) \quad \sum_{ij} \left[s_{ij}\left(\frac{x_{ij}}{r_{ij}}\right) x_{ij} + w_{ij}\left(\frac{x_{ij}}{r_{ij}}\right) x_{ij} \right] \leq e$$

in addition to the usual inequalities defining the program and the maximal flow intensities (1), (2). The following assumptions are made about the functions just introduced

$$\left.\begin{array}{l} h_{ij}(z_{ij}, z_{ji}) \\ s_{ij}(z_{ij}, z_{ji}) \end{array}\right\} = \text{jointly convex in } z_{ij}, z_{ji};$$

$$w_{ij}(\ldots z_{\chi j}) = \text{jointly convex in all } z_{\chi j}$$

These convexity properties amount to an assumption of increasing effects of substitution and scale of the various flows on the costs and (travel or waiting) times for the road in question. If cost is considered to consist largely of man-hours and equipment-hours, the first assumption is a consequence of the second (and incompatible with the third unless all variables other than z_{ij} are absent from w_{ij} (.)). The second and third assumption are capable (and in need) of verification by stochastic analysis of traffic behavior in opposing streams and at intersections of traffic.[5/] The first is the essentially plausible assertion that the rate of increase (decrease) of cost with speed is an increasing (decreasing) function of speed (cf. 2049A, reference [3]).

Problem, find

$$\min_{x_{ij}^m \geq 0} \sum_{ij} h_{ij}\left(\frac{x_{ij}}{r_{ij}}, \frac{x_{ji}}{r_{ji}} \right) \cdot x_{ij}$$

subject to

$$\sum_j (x_{ij}^m - x_{ji}^m) \leq q_i^m$$

$$x_{ij} \leq r_{ij}\, c_{ij}$$

$$\sum_{ij} \left[s_{ij}\left(\frac{x_{ij}}{r_{ij}}, \frac{x_{ji}}{r_{ji}} \right) + w_{ij}\left(\ldots \frac{x_{\ell j}}{r_{\ell j}} \right) \right] x_{ij} \leq e$$

where

$$x_{ij} = \sum_m x_{ij}^m$$

Under the above assumptions on h_{ij}, s_{ij}, w_{ij}, the hypothesis of Kuhn-Tucker's [2] theorem 3 are satisfied, and this supplies us with the following necessary <u>and sufficient</u> conditions of efficiency.

$$\lambda_j^m - \lambda_i^m \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} h_{ij} + \mu_{ij} + \nu(s_{ij} + w_{ij})$$

$$+ x_{ij} \frac{\partial}{\partial x_{ij}} [h_{ij} + \nu s_{ij}] + x_{ji} \frac{\partial}{\partial x_{ij}} [h_{ji} + \nu s_{ji}]$$

$$+ \sum_{\chi}' \nu x_{\chi j} \frac{\partial w_{\chi j}}{\partial x_{ij}}$$

$$\mu_{ij} \left\{ \begin{matrix} \geq \\ = \end{matrix} \right\} 0 \qquad \text{if } x_{ij} \left\{ \begin{matrix} = \\ < \end{matrix} \right\} r_{ij} c_{ij}$$

$$\nu \left\{ \begin{matrix} = \\ \geq \end{matrix} \right\} 0 \qquad \text{if} \qquad \sum_{ij} (s_{ij} + w_{ij}) x_{ij} \left\{ \begin{matrix} < \\ = \end{matrix} \right\} e.$$

We readily identify

λ_i^m as an efficiency price for commodity m at location i

μ_{ij} as an efficiency toll on use of road ij owing to limited capacity

ν as an efficiency rental on equipment per unit of time, and

the last three terms as the (marginal) cost to other traffic of a unit flow on road ij. Both μ_{ij} and these three terms are costs of congestion, the former that to traffic outside this road and its exit (i.e., the potential road users excluded from it) and the latter those to the direct and opposing traffic on that road and to all traffic passing through the intersection at its exit j. The cost of congestion in turn depends partly on the cost of equipment time.

In using the terms price, toll and rental in our cost imputation we have pointed to an alternative interpretation of our model, in which allocation of traffic so as to utilize optimally the available facilities is achieved by means of proper charges to individual traffic participants. These reduce to two quantities

$$p_o = \nu \quad \text{the rental for equipment time}$$

$$p_{ij} = \mu_{ij} + x_{ij} \frac{\partial}{\partial x_{ij}} [h_{ij} + \nu s_{ij}] + x_{ji} \frac{\partial}{\partial x_{ij}} [h_{ji} + \nu s_{ji}]$$

$$+ \sum_{\chi} x_{\chi j} \nu \frac{\partial w_{\chi j}}{\partial x_{ij}}$$

the toll on using road ij. If these charges are levied, individual calculation of minimum cost for each transportation activity will achieve also the overall optimal pattern of road use. Of course the determination of the proper charges is yet another problem for whose solution recourse must be had to some of the computation techniques of non-linear programming [see also section 2].

3.2 A natural next step would be to relax the conditionsof joint convexity or to include construction (and abandonment) of roads and equipment and variability of the program in the model. This shall not be undertaken here, mainly because of the difficulties involved in the fact, that now the efficiency price conditions while still necessary are no longer sufficient, and that presumably considerations in the nature of discrete choices will have to enter.

Under certain conditions, the complicated inequalities may be reduced to simpler statements. Consider the efficiency conditions with respect to the capacity levels r_{ij}, the equivalent of equation (4).

$$(4a) \quad z_{ij}^2 [\frac{\partial h_{ij}}{\partial z_{ij}} + \nu \frac{\partial s_{ij}}{\partial z_{ij}}] + z_{ij} \cdot \frac{x_{ji}}{r_{ij}} [\frac{\partial h_{ji}}{\partial z_{ij}} + \nu \frac{\partial s_{ij}}{\partial z_{ij}}]$$

$$+ \sum_{\chi} \nu z_{ij} \frac{x_{\chi j}}{r_{ij}} \frac{\partial w_{\chi j}}{\partial z_{ij}} + c_{ij} \mu_{ij} \begin{Bmatrix} = \\ \leq \end{Bmatrix} f_{ij} \quad \text{if} \quad r_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

In the special case that the cost h_{ij}, the travel time s_{ij} and the waiting time w_{ij} depend only on z_{ij} itself, this expression assumes the previous simple form

$$p_{ij}\; z_{ij} \begin{Bmatrix} = \\ \leq \end{Bmatrix} f_{ij} \qquad \text{if} \qquad r_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

or, still simpler

$$p_{ij}\; x_{ij} = f_{ij}\; r_{ij}.$$

That is, the outlay for construction must always equal the revenue from tolls on that road. If this is impossible for a finite width, the road must not be constructed ($r_{ij} = 0$). This necessary condition is not surprising in view of the assumptions made: that traffic in a given direction is not interfered with by traffic in the opposite direction or on other roads. This result remains valid also under non-static conditions; f_{ij} must then be replaced by a sum of discounted outlays $\sum_t v^t f_{ij}^t$, say, where v the discount rate.

3.3 If we now assume a situation of individual choice and relax the assumption of a fixed program, our model permits some comparisons between the states of road traffic with or without the efficiency tolls that were seen to bring about optimal utilization of roads.

Consider for this purpose the following problem. To find the optimal road capacities with respect to price dependent programs under the condition that transportation costs do not include tolls. Optimal here means that the aggregate consumers' and producers' surplusses are maximized. (However irrelevant this magnitude may seem to be, it is the maximand associated with a competitive market equilibrium under the condition of full charges to producers for social cost. In particular: a situation of maximal consumers' and producers' surplus represents an efficient point in the space of production possibilities).

Let p_i^m (q_i^m) denote the price or (marginal) value of commodity m at location i as a function of net exports q_i^m. This function is assumed increasing and, sometimes, linear. A more general situation, in which p_i^m depends on all the q_i^m for given i, can be considered along similar lines. The problem may now be formulated as finding

$$\max_{x_{ij}^m \geq 0, r_{ij} \geq 0} \sum_{i,m} - \int_0^{\sum_j (x_{ij}^m - x_{ji}^m)} p_i^m (q)\, dq - \sum_{ij} h_{ij} \left(\frac{x_{ij}}{r_{ij}}\right) x_{ij} - \sum_{ij} f_{ij} r_{ij}$$

subject to the restraints

$$(8) \qquad p_j^m (q_j^m) - p_i^m (q_i^m) \leq h_{ij}.$$

With non-negative Lagrangean parameters λ_{ij}^m we obtain, after differentiation, the following necessary conditions of efficiency

$$(9) \qquad p_j^m - p_i^m - h_{ij} - \frac{h'_{ij}}{r_{ij}} (x_{ij} - \sum_m \lambda_{ij}^m) + \sum_r \lambda_{ir}^m p_i^{m'} + \sum_s \lambda_{sj}^m p_j^{m'}$$

$$\left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} 0 \qquad \text{if} \qquad x_{ij}^m \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$$

$$(10) \quad \frac{h'_{ij}}{r_{ij}^2} (x_{ij} - \sum_m \lambda_{ij}^m) x_{ij} - f_{ij} \left\{ \begin{matrix} = \\ \leq \end{matrix} \right\} 0 \quad \text{if } r_{ij} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0$$

Here h'_{ij}, $p_i^{m'}$ denote the derivatives of h_{ij} and p_i^m. Now if $x_{ij}^m > 0$ the first condition is simplified by (8)

$$(11) \quad \frac{h'_{ij}}{r_{ij}} (x_{ij} - \lambda_{ij}) = \sum_r \lambda_{ir}^m p_i^{m'} + \sum_s \lambda_{sj} p_j^{m'}$$

where we have written λ_{ij} for $\sum_m \lambda_{ij}^m$. This equation has an obvious interpretation if the cost functions $h_{ij}(\cdot)$ as well as the price functions $p_i^m(\cdot)$ are assumed linear.[6/] In that case, the left side represents the "social cost" for a traffic flow of $x_{ij} - \lambda_{ij}$; the right side equals the decrease in geographical price difference due to increases λ_{ij}^m of flows. We see that $x_{ij} - \lambda_{ij}$ are the flows that would prevail if tolls equal to social cost were charged; in other words $x_{ij} - \lambda_{ij}$ are the optimal flows. In the absence of tolls traffic increases by (non-negative) amounts λ_{ij} until the marginal gains from interlocal commodity shipments have dropped by the amount of this toll. This conclusion holds for any road on which there is a positive flow of some commodity (some $x_{ij}^m > 0$). It is of course not surprising that dropping tolls should invite more traffic than is optimal. Our argument shows that traffic will increase or at worst remain unchaged on all roads. Sometimes it is claimed that congestion of roads tends to render urban concentration uneconomical. Our analysis shows that rather congestion is a sign of excessive transportation, i.e. of more interlocal shipment of commodities than is warranted by the social cost of transportation, and that in that sense it (or rather the free use of roads) has made possible and not curbed urban concentration to the present extent.

Comparison of (10) with the efficiency condition[6/] (4) for optimal flows $x_{ij} - \lambda_{ij}$ shows that if road use is toll free, more capacity should be constructed (at the given prices) in order to counteract somewhat the excessive use of roads. It is of course these equations rather than (3) (4) that should be considered in a realistic discussion of the road layout problem.

If h_{ij} and p_i^m are not linear functions of their arguments the interpretation of (9) is somewhat more complicated: λ_{ij} does not necessarily represent now the increase of traffic as over the optimal situation. But the conclusion remains valid, that traffic is nowhere smaller than in the case of charges for social cost.

The question whether conditions (8), (9), (10) are sufficient for a solution of the problem is still open: the maximand is not jointly concave in the r_{ij}, x^m_{ij} , even if the function $p^m_i(\cdot)$ and $h_{ij}(\cdot)$ are linear; and the uniqueness of the solution to this system is doubtful.

FOOTNOTES

1. Research undertaken under contract between the Cowles Commission for Research in Economics and the RAND Corporation.

2. measured negative

3. Finally it is possible to study the effects of changes in equipment availability (see below, section 3) provided the capacities of roads are all fixed. The outcome of joint changes in road capacities and equipment limit is complex and not capable of being studied with the above methods.

4. For the following proof I am largely indebted to K. Arrow.

5. The problems are either to show that the mean waiting times in "queues" at the exits of roads or before passing slower cars are indeed jointly convex functions of the flow densities; or to demonstrate the existence of such instability as to render mean-value concepts meaningless.

6. The constraint which gives rise to the μ- terms has been dropped in the present problem for simplicity, as has the equipment constraint.

References

[1] John, Fritz, "Extremum Problems with Inequalities as Subsidiary Conditions." Courant Anniversary Volume 1948, pp. 187 - 204.

[2] Kuhn, H. W. and A. W. Tucker, "Nonlinear Programming", Second Berkeley Symposium on Mathematical Statistics and Probability, 1951.

[3] Koopmans, Tjalling C., "A Model of Transportation", Chapter XIV, Monograph 13.

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2. The author has benefitted from comments by I. N. Herstein, H. Markowitz, and M. Slater. His main indebtedness is to T. C. Koopmans, who has suggested this problem, stimulated its treatment in various discussions, and presented an earlier version of it at the Logistic Conference, January, 1952, Washington, D. C.

3. The discussion of capacity formulas for nonuniform speed distribution is deferred to a separate paper. Cf. also B. McGuire, "Highway Capacity and Traffic Congestion: A Preliminary Study," Cowles Commission Discussion Paper, Economics No. 2048.

4. If these resources are subject to limitations these restrictions will, in general, conflict with the road capacity constraints in such a way as to render the marginal cost conditions insufficient for efficiency. The scope of the present paper does not permit our going further into this, but the study of Section 4 will convince the reader that the persistence of relative convexity of inputs with respect to flows is highly sensitive to the nature of the restrictions imposed.

5. A function is convex if (and only if) the chord spanned by any two points on its graph lies above the graph. Analytically: Let $0 \leq \lambda \leq 1$; then $f(x)$ is convex if (and only if)

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).$$

6. It may be seen directly from these conditions or, alternatively, from the general theorem of activity analysis mentioned [4; Theorem 5.6 (p. 86)] that any $\bar{x}$ satisfying (2.7)...(2.12) is also a solution of the problem.

Find $\max\limits_{x} (k_o x_o - \sum\limits_{ij,n} p^n k_{ij}^n x_{ij})$

subject to the constraints (2.1)...(2.6). Now

$$\max_{(x_{ij}^m)} - \sum_{ij,n} p^n k_{ij}^n x_{ij}$$

subject to (2.1)...(2.6),

is a continuous function of x_o whose derivative to the right is piecewise continuous in $0 \leq x_o \leq 1$. This derivative is also a piecewise continuous function of the p^m in the domain $0 \leq p^n \sum\limits_n p^n = 1$. It is therefore only necessary to choose

$$k_o > \max_{\substack{(p^n) \\ p^n \geq 0 \\ \sum_n p^n = 1}} \max_{\substack{x_o \\ 0 \leq x_o \leq 1}} \frac{d^+}{dx_o} \left(\min_{(x_{ij}^m)} \sum_{ij,n} p^n k_{ij}^n x_{ij} \quad \text{subject to } (2.1)\ldots(2.6) \right)$$

A rigorous analysis of the invoked continuity relations lie outside the scope of this study.

7. This in conjunction with (2.7) can be used to show that in (2.7) for k_o sufficiently large π_o must become positive and hence $\bar{x}_o = 1$. This demonstration fails however if the program for $x_o = 1$ cannot be sustained by the network. For then the marginal cost may increase indefinitely.

8. Comparing (3.10) with (3.4) shows that

$$_{+}\bar{p}_{ij} \leqq p^{*}_{ij} \leqq {}_{-}\bar{p}_{ij} \qquad \text{if} \qquad \frac{dc_{ij}}{du_{ij}} > 0$$

and (3.8)

$$_{-}\bar{p}_{ij} \leqq p^{*}_{ij} \leqq {}_{+}\bar{p}_{ij} \qquad \text{if} \qquad \frac{dc_{ij}}{du_{ij}} < 0$$

p^{*}_{ij} may therefore be different from either limit of $\bar{p}_{ij}$.