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The Problem of Optimal Assignment of Locations and an Equivalent 2-Person Game[1/*]

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1. Statement of the Problem.

The problem of optimal assignment of locations to (indivisible) production activities is as follows. Given are n plants and n locations, a set of real non-negative numbers $a_{\ell m}$ representing the flows from the ℓth plant to the mth plant, and a set of non-negative real numbers k_{ij} representing the cost of transportation from location i to location j. What is the assignment of plants to locations which minimizes the total cost of transportation as among plants?

This model is not intended to be realistic enough for immediate applications. Among its limitations, note first, that the total flows origination at a plant ℓ are exhausted by the coefficients $a_{\ell m}$ only when the notion of plant is taken in the broad sense that encompasses retail outlets and finally, households; second, substitutions among flows from plants of the same industry are disregarded. It seems, however, that the problems

1/ This paper was stimulated by an unpublished paper of J. von Neumann, "The Problem of Optimal Assignment and a Certain 2-Person Game," (October 26, 1951), of which extensive use is made in the proofs to follow. Errors are, of course, my own.

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involved in indivisibilities of production are serious enough to make a rather simple model desirable at the outset.

This problem is related to the personnel assignment problem [ℓ.c., footnote] which may be formulated in terms of plants and locations as follows. Given n plants and n locations and a set of numbers c_{ij} representing the profit of production for plant i at location j, what is the assignment of plants to locations which maximizes total profits? One will note that in the former problem the cost to each plant of transportation and hence its profit is a function of the locations of all other plants.

von Neumann has shown [ℓ.c.] that the personnel assignment problem is equivalent to the following game. Let there be n x n double indexed cells, say, fields in a matrix. Player I hides in one cell. Player II attempts to "find" I by guessing either of the indices of the cell in which player I has hidden. The payoff to player II is $\frac{1}{c_{ij}}$ if I is found in cell i, j and zero otherwise.

The purpose of this paper is to show the equivalence of the problem of optimal assignment of locations with the following game. Two n x n chess boards are given. On each of these player I chooses a row, player II a column. The choices are compared and player I pays to player II the amount $k_{ij} \cdot a_{\ell m}$ where i,ℓ are the rows chosen, j,m the columns chosen. This game may also be formulated abstractly in terms of strategies. For player I strategy g is defined as follows. Let $g = in + \ell$ where i,ℓ are positive integers not exceeding n. Then strategy g is to choose row i on the first and row ℓ on the second chess board. Similarly for player II let strategy $h = jn + m$ be to choose columns j and m respectively. The game equivalent to the locational assignment problem is the two person zero-sum game whose payoff matrix is given by $b_{gh} = k_{ij} \cdot a_{\ell m}$

where $g = i.n + \chi$ $\qquad h = j.n + m.$

It will have been apparent that both optimal assignment problems are of the nature of permutation problems. The advantage of having an equivalent game formulations is essentially that of a reduction of this difficult type of a problem to a continuous one of manageable properties.

2. Mathematical discussion.

Let $\quad \overset{o}{A} = (\overset{o}{a}_{\chi m})$ $\qquad \overset{o}{a}_{\chi\chi} = 0$

$\qquad a_{\mu m} \geqq 0$

$K = (k_{ij})$, $\qquad k_{ii} = 0$

$\qquad k_{ij} > 0 \quad i \neq j$

be given n x n matrices, P an n x n permutation matrix.[2]

The problem of optimal assignment of locations takes the mathematical form of

Problem 1

Find $\bar{P}$ such that $\text{trace}\,(\overset{o}{A}'\bar{P}'K\,\bar{P}) = \min\limits_{P} \text{trace}\,(\overset{o}{A}'P'K\,P)$.

Let $\quad A = \overset{o}{A} + D$

where D a diagonal matrix. Since the diagonal of P' K P contains only zero elements $\text{trace}\,(\overset{o}{A}\,P'\,K\,P) = \text{trace}\,(A\,P'\,K\,P)$. The subject of this discussion paper is

Lemma 1

For every given matrix $\overset{o}{A}$ there exists a matrix $\overset{o}{A} + D = A = (a_{\chi m})$ (with D a diagonal matrix) such that problem 1 is equivalent to

Problem 2

Find numbers $x_{i\chi}$, y_{jm} $\quad i,j,\chi,m = 1,\dots n$ such that

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2/ Permutations are indicated by letters P, $\bar{P}$, Q in the following ways
if P stands with a matrix it denotes a permutation matrix, as in P'KP
if P is used as a subscript it denotes an index for that permutation as in u_P
if P occurs as a superscript it denotes the operator that transforms the number in question into the number which is assigned to it by the permutation, as in χ^P.

$$\sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{\ell m}\, \bar{x}_{i\ell}\, \bar{y}_{jm} = \min_{x_{i\ell}} \max_{y_{jm}} \sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{\ell m}\, x_{i\ell}\, y_{jm}$$

subject to the conditions

$$(1) \quad x_{i\ell} \geq 0, \qquad y_{jm} \geq 0$$

$$(2) \quad \sum_{\ell} x_{i\ell} = 1, \qquad \sum_{m} y_{jm} = 1$$

$$(3) \quad \sum_{i} x_{i\ell} = 1, \qquad \sum_{j} y_{jm} = 1$$

__Proof:__

Following von Neumann [ℓ.c. p. 4] we consider the two vector spaces R = set of all vectors $z = (z_{ij})$ in n^2 dimensions such that

$$(4) \quad z_{ij} \geq 0$$

$$(5) \quad \sum_{j} z_{ij} = 1, \qquad \sum_{i} z_{ij} = 1$$

S = set[3] of all vectors $z = (z_{ij})$ in n^2 dimensions such that $z_{ij} = \delta_{ij}^Q$ for some permutation Q of the integers 1,...n. Each z in S is thus the direct sum of the permuted columns of the n x n unit matrix.

Lemma 2

[G. Birkhoff, 1946] R = convex hull of S. [For a proof cf. von Neumann: l.c. p. 5 - 8].

Therefore every vector $x = (x_{gh})$ in R can be represented as a linear combination of vectors $z_Q = (z_{gh}) = (\delta_{gh}^Q)$ of S with coefficients u_Q

(6) $$0 \leq u_Q \leq 1$$

$$x = \sum_Q u_Q z_Q.$$ Thus

(7) $$x_{gh} = \sum_Q u_Q \cdot \delta_{gh}^Q$$

and

(8) $$1 = \sum_g x_{gh} = \sum_Q u_Q \sum_g \delta_{gh}^Q$$

$$= \sum_Q u_Q$$

Similarly

$y_{gh} = \sum_Q v_Q \delta_{gh}^Q$ with

(9) $$0 \leq v_Q \leq 1$$

(10) $$\sum_Q v_Q = 1.$$ Hence

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[3] The corresponding notation for S in von Neumann's paper is P.

$$(11) \quad \sum_{i,j,\chi,m=1} k_{ij}\, a_{\chi m}\, x_{i\chi}\, y_{jm} = \sum_{\substack{i,j,\chi,m=1\\P,Q}}^{n} k_{ij}\, a_{\chi m}\, u_P\, \delta_{i\chi}{}^P\, v_Q\, \delta_{jm}{}^Q$$

$$= \sum_{\chi,m=1}^{n} k_{\chi}{}^P{}_m{}^Q\, a_{\chi m}\, u_P\, v_Q$$

with (8) $\sum_P u_P = 1$ (10) $\sum_Q v_Q = 1$

We show next that for fixed $x = (x_{i\chi})$

$$(12) \quad \max_{y_{jm}} \sum_{i,j,\chi,m=1}^{n} k_{ij}\, a_{\chi m}\, x_{i\chi}\, y_{jm} =$$

subject to (1). (2). (3)

$$\max_{v_Q} \sum_{\substack{i,\chi,m=1\\Q}}^{n} k_{im}{}^Q\, a_{\chi m}\, x_{i\chi}\, v_Q$$

subject to (6), (8).

For brevity write for the left side M. Let M be taken on for $y_{jm} = y^o_{jm}$.

According to (7) y^o_{jm} may be represented in the form

$y^o_{jm} = \sum v^o_Q \quad \delta_{jm}{}^Q$ with some v^o_Q satisfying (6) and (8). Hence

$$M = \sum_{\substack{i,j,\chi,m=1\\Q}}^{n} a_{\chi m}\, k_{ij}\, x_{i\chi}\, v^o_Q\, \delta_{jm}{}^Q = \sum_{i,\chi,m=1}^{n} a_{\chi m}\, k_{im}{}^Q\, x_{i\chi}\, v^o_Q$$

The right side is clearly

$$(13) \quad \leq \max_{v_Q} \sum_{\substack{i,\chi,m=1\\Q}}^{n} a_{\chi m}\, k_{im}{}^Q\, x_{i\chi}\, v_Q$$

subject to (6) (8)

Let this maximum be taken on for $v_Q = v^*_Q$. Assume

that the "<" sign holds in (13). Now $\sum_Q v_Q\, \delta_{jm}{}^Q$ defines a vector(y_{jm}) by

(7) which would yield a value of $\Sigma\, a_{\ell m}\, k_{ij}\, x_{i\ell}\, y_{jm} > M$. This is a contradiction, and therefore the equality sign holds in (13) and so in (12).

In the same way it is seen that

$$(14) \qquad \min_{x_{i\ell}} \ \max_{y_{jm}} \ \sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{\ell m}\, x_{i\ell}\, y_{jm}$$

subject to (1),(2),(3)

$$= \min_{u_P} \ \max_{v_Q} \ \sum_{\substack{\ell,m=1 \\ P,Q}}^{n} k_{\ell}{}^{P}{}_{m}{}^{Q}\, a_{\ell m}\, u_P\, v_Q$$

subject to (6), (8).

For the sum $\displaystyle\sum_{\ell,m=1}^{n} k_{\ell}{}^{P}{}_{m}{}^{Q}\, a_{\ell m}$

one may also write trace $(A'\, P'\, K\, Q)$, so that the right side of (14) becomes

$$\min_{u_P} \ \max_{v_Q} \ \sum_{P,Q} \operatorname{trace}\, (A'\, P'\, K\, Q)\, u_P\, v_Q$$

subject to (6), (8).

We proceed to prove that for a suitable D this expression is equal to $\min_P$ trace $(A'\, P'\, K\, P)$. The first step is to show that for fixed u_P

$$(15) \qquad \max_{v_Q} \ \sum_{Q} \operatorname{trace}\, (A'\, P'\, K\, Q)\, v_Q = \operatorname{trace}\, (A'\, P'\, K\, P)$$

~~subject to (9) (10)~~

For this it is sufficient that trace $(A'\, P'\, K\, Q) < 0$ if $P \neq Q$.

Presently we show that this can be achieved by choosing

$$A = \overset{o}{A} + cI$$

where I is the unit matrix and c a sufficiently small constant, say,

$$c < - \frac{\max\limits_{P,Q} \text{ trace } (\overset{o}{A}' \, P' \, K \, Q)}{\min\limits_{i \neq j} \quad k_{ij}}$$

Suppose that $P \neq Q$, hence that $k_{\chi P \chi Q} \neq 0$ for some χ. trace (A'P'KQ) may then be written

$$\sum_{\chi m} k_{\chi \rho m Q} \, \overset{o}{a}_{\chi m} + \sum_{\chi} k_{\chi P \chi Q} \, c$$

$$\leqq \max_{P,Q} \text{ trace } (\overset{o}{A}' \, P' \, K \, Q) + \sum_{\chi} k_{\chi P \chi Q} \, c$$

$$< 0 \leqq \text{ trace } (\overset{o}{A}' \, P' \, K \, P)$$

$$= \text{ trace } (A' \, P' \, K \, P)$$

This establishes that

$$\max_{v_Q} \sum_{Q} \text{ trace } (A' \, P' \, K \, Q) \, v_Q = \text{ trace } (A' \, P' \, K \, P)$$

subject to (9) (10)

It follows now at once that

(16) $$\min_{u_P} \max_{v_Q} \sum_{P,Q} \text{ trace } (A' \, P' \, K \, Q) \, u_P \, v_Q$$

subject to (6), (8)

$$= \min_{P} \text{ trace } (A' \, P' \, K \, P) = \text{ trace } (A' \, \bar{P}' \, K \, \bar{P}).$$

(16) in conjunction with (14) establishes the assertion of lemma 1.

In equation (14) the solution is

(17) $$u_P = \begin{cases} 1 \text{ for } P = \bar{P} \\ 0 \text{ for } P \neq \bar{P} \end{cases}$$

(18) $$v_Q = \begin{cases} 1 \text{ for } Q = \bar{P} \\ 0 \text{ for } Q \neq \bar{P} \end{cases}$$

Returning to equations (7) and (9) of p. 5 we note that the solution of problem 2 becomes

$$(19) \qquad \bar{x}_{i\ell} = \sum_{Q} \bar{u}_{Q} \, \delta_{i\ell Q} = \delta_{i\ell \bar{P}}$$

$$(20) \qquad \bar{y}_{jm} = \sum_{Q} \bar{v}_{Q} \, \delta_{jmQ} = \delta_{jm\bar{P}}.$$

Alternative Continuizations. Efficiency Rents.

3. To find equivalent games is only a special case of the more general task of finding analogous continous problems for the permutation problems considered. For instance in the same way as before it can be shown that

3.1 the personnel assignment problem (p. a. p.) is equivalent to

Problem 4

$$\underset{x_{ij}}{\text{Maximize}} \quad \sum_{i,j=1}^{n} c_{ij} \, x_{ij}$$

subject to

$$(3.1) \qquad \sum_{j} x_{ij} \leqq 1$$

$$(3.2) \qquad \sum_{i} x_{ij} \leqq 1$$

$$(3.3) \qquad x_{ij} \geqq 0$$

3.2 the locational assignment problem (l. a. p.) is equivalent to

Problem 5

$$\underset{x_{ij}}{\text{Minimize}} \quad \sum_{i,j,\ell,m=1}^{n} k_{ij} \, a_{\ell m} \, x_{i\ell} \, x_{jm}$$

subject to

$$(3.1) \qquad \sum_{j} x_{ij} \geqq 1$$

$$(3.2) \qquad \sum_{i} x_{ij} \geqq 1$$

$$(3.3) \qquad x_{ij} \geqq 0$$

The usefulness of problem 4 lies in the fact that it permits the formulation of necessary and sufficient conditions for the solution of the p.a.p. in terma of efficiency prices.

Let the location associated with a plant by the solution of the p.a.p. or the l.a.p. respectively be called the socially optimal location for that plant (with respect to that problem).

Definition A set of non-negative numbers r_i associated with the locations i is called a set of efficiency rents if for each plant the net profit after rents is smallest when the assignment of plants to locations is the socially optimal one. Let P be the permutation which assigns to each plant its socially optimal location. The definition of the efficiency rents r_i for the p.a.p. is

(3.4) $$c_{jQ_j} - r_{jQ} \leq c_{jP_j} - r_{jP} \quad \text{for all } j,\ Q.$$

And for the l.a.p.:

(3.5) $$\sum_m k_{\ell^Q m^Q} a_{\ell m} + r_{\ell^Q} \geq \sum_m k_{\ell^P m^P} a_{\ell m} + r_{\ell^P}$$

for all ℓ, Q.

Suppose that a permutation P satisfies (3.4) Then clearly

$$\sum_j [c_{jQ_j} + r_{jQ}] \leq \sum_j [c_{jP_j} + r_{jP}] \quad \text{or}$$

$$\sum_j c_{jQ_j} \leq \sum_j c_{jP_j}$$

Any solution P of (3.4) is therefore a solution of the p.a.p. Similarly it follows from

$$\sum_{\ell,m} (k_{\ell^Q m^Q} \cdot a_{\ell m} + r_{\ell^Q}) \geq \sum_{\ell,m} (k_{\ell^P m^P} \cdot a_{\ell m} + r_{\ell^P})$$

at once that any solution P of (3.5) solves the l.a.p.

Hence the existence of efficiency prices such that (3.4) or (3.5) hold is a sufficient condition for P to be a solution of the p.a.p. or the l.a.p., respectively. That it is a necessary condition for the p.a.p. can be seen from the equivalent problem 4 as follows.

The minimand of problem 4 is linear and hence convex in x_{ij}. The constraints (3.1), (3.2), (3.3) are linear and hence concave in x_{ij}. By the theorem of Kuhn and Tucker [Second Berkeley Symposium p. 486] on the existence of Lagrangean parameters or by a similar argument in terms of linear activity analysis one has

$$(3.6) \qquad \max \sum_{i,j=1}^{n} c_{ij}\, x_{ij} =$$

subject to (3.1), (3.4), (3.3)

$$= \min_{\lambda_i, \mu_j} \max_{x_{ij}} \left\{ \Sigma\, c_{ij}\, x_{ij} - \sum_i \lambda_i \left(\sum_j x_{ij} - 1\right) - \sum_j \mu_j \left(\sum_i x_{ij} - 1\right) \right\}$$

subject to

$$(3.7) \qquad x_{ij} \geqq 0$$

$$(3.8) \qquad \lambda_i \geqq 0$$

$$(3.9) \qquad \mu_j \geqq 0$$

Necessary for a saddle point $\bar{x}_{ij}$, $\bar{\lambda}_i, \bar{\mu}_j$ of the linear function on the right side of (3.6) is that the following conditions hold

$$(3.10) \qquad c_{ij} - \bar{\lambda}_i - \bar{\mu}_j \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \text{ if } \bar{x}_{ij} \begin{Bmatrix} > \\ = \end{Bmatrix} 0.$$

Making use of the fact that $\bar{x}_{ij}$ is of the form $\bar{x}_{ij} = \delta_{ij}{}^P$ one obtains

$$(3.11) \qquad c_{ij} - \bar{\lambda}_i - \bar{\mu}_j \begin{Bmatrix} = \\ \leqq \end{Bmatrix} 0 \quad \text{if} \quad i \begin{Bmatrix} = \\ \neq \end{Bmatrix} j^P$$

or

$$c_j Q_j - \bar{\lambda}_j Q \geqq c_j P_j - \bar{\lambda}_j P \qquad \text{for all } j,\ Q$$

(3.11) permits the following economic interpretation. The profit of each firm at its socially optimal location can be split into two parts, one attributable to the location i, $\bar{\lambda}_i$, and one associated with the activity j, $\bar{\mu}_j$.

Supposing that a rent $\bar{\lambda}_i$ is **charged** location i, then the profit after rent for plant j is equal to $\bar{\mu}_j$ if the plant is at location j^P and less if it is at any other location. Thus if all plants are at their socially optimal locations no plant can benefit from any locational change.

The same argument as before cannot be used to derive a set of conditions corresponding to (3.11) for the l.a.p. The stumbling block here is the circumstance that the quadratic form

$$\sum_{i,j,\ell,m=1}^{n} k_{ij}\, a_{m\ell}\, x_{i\ell}\, x_{jm}$$

is not necessarily positive semi-definite, and hence that it is not necessarily a convex function of the $x_{i\ell}$, and that therefore the saddle point theorem of Kuhn-Tucker [l.c.] does not apply.

From the following example it is seen that efficiency rents, as defined on p.10, do not exist for the l.a.p. in general.

Let

$$A' = \begin{pmatrix} 0 & 2.1 & 1.5 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

In table 1 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ denotes the permutation that carries 1 into 2, 2 into 1, 3 into 3. Similarly for all other permutations. The table shows the costs of transportation $\sum_m a_{\ell m} k_{\ell^\rho m^\rho}$ for the flows **originating at plant** ℓ **as** under the assignment $i = \ell^\rho$ of locations to plants.

χ \ Q	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$
1	5.7	5.1	3.6
2	6	4	8
3	4	7	4
1 + 2 + 3	15.7	16.1	15.6

Table 1

From the last row it is seen that the two permutations

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$$

are the optimal ones. Under the assignments represented by permutation $\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$ plants 1 and 2 are at locations 1 and 3 respectively; under $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$ they are at 3 and 1 respectively. Suppose now that efficiency rents r_i exist satisfying the defining relation (3.5). With respect to the plants 1,2 and the two assignments just mentioned, (3.5) requires that

$$3.6 + r_3 \leqq 5.1 + r_1$$

$$8 + r_1 \leqq 4 + r_3 .$$

This is a contradiction, and hence efficiency rent in the above sense cannot exist.

This contradiction is not removed if instead of transportation cost absorption by the supplying plant, one assumes some other imputation of transportation costs to plants; for instance, that the receiving plant bears the cost of transportation or that both receiver and supplier pay 1/2 of the transportation cost. In the latter case the entries of the right upper corner in table 1 are changed to

13.1	10.6
7.1	11.1

and the conflicting relations read

$$10.6 + r_3 \leqq 13.1 + r_1$$

$$11.1 + r_1 \leqq 7.1 + r_3 .$$

Clearly these considerations do not settle the question whether efficiency prices of some other kind will exist capable of guiding the computations of the solution.

Of course it is always possible to characterize the solution in terms of some prices. For instance, let $k(\chi, Q)$ be the cost of transportation (for some specified imputation) for plant χ under the assignment represented by permutation Q. If P is the optimal assignment,

$$\sum_{\chi} k(\chi, Q) - \sum_{\chi} k(\chi, P) \geq 0 \quad \text{for all } Q.$$

Thus if $-k(\chi, P)$ is taken to be an efficiency rent (or rather subsidy) for location $i = \chi^P$, then the sum of transportation cost and rent will be positive for some plant unless all plants are at their socially optimal locations. These rents, however, do not induce the individual plants to retain their position at the socially optimal locations. For if all plants are at their socially optimal locations the individual plant or some coalition of plants may nevertheless benefit from an exchange of locations, at the expense, that is, of some other plant not included in the coalition. Only within a coalition of all plants is the socially optimal assignment safeguarded.

A second difference of importance lies in the fact that the efficiency prices λ_i, μ_j can be put to use in the computations of the solution by means of the gradient method, (say), whereas the latter rents are known only to the extent that the solution is known.