

COWLES COMMISSION DISCUSSION PAPER: Economics No. 2041

See CCDP
No. 2034

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Note on Marschak's Cowles Commission Discussion Paper Economics 2034

Organized Decision-Making

Martin Beckmann

April 7, 1952

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This note to be inserted into C.C.D.P. 2034 after Section 15.

15a. Suppose now that the rule f does not prescribe behavior with certainty but is in terms of randomized strategies. Denote by

- $q_i(x_i)$ the probability that partner i commits the firm ($i=1, 2$)
 $p_i(x_i)$ that he phones and
 $o_i(x_i)$ that he does not do anything.

The integral (15.2) reads now explicitly as follows. Expected profits from commitments by partner i are

$$(15a.1) \quad \int_0^1 \int_0^1 [(x_1 + x_2 - 1) q_i(x_i) [1 - p_j(x_j)]] dx_1 dx_2 \quad (i \neq j)$$

and expected profits after communication are

$$(15a.2) \quad \int_0^1 \int_0^1 [2 \max(x_1 + x_2 - 1, 0) - c] [p_1(x_1) + p_2(x_2) - p_1(x_1) p_2(x_2)] dx_1 dx_2$$

If i does not do anything, no profits arise. Hence U is the sum of two expressions (15a.1) and (15a.2). The main observation to be made is that integration with respect to one variable, say x_j , will cause the integrand to become a linear function of $p_i(x_i)$ and $q_i(x_i)$, where $i \neq j$. We have to maximize with respect to the functions $q_i(\cdot)$, $p_i(\cdot)$, the expression

$$U = \int_0^1 [\lambda_1(x_1) \cdot p_1(x) + \mu_1(x_1) \cdot q_1(x_1) + \dots] dx_1,$$

where ... denotes terms not containing $q_i(x_i)$ or $p_i(x_i)$ and therefore of no interest here. Since $q_i(x_i)$, $p_i(x_i)$ satisfy inequalities

$$0 \leq p_i(x_i) \quad 0 \leq q_i(x_i)$$

$$p_i(x_i) + q_i(x_i) \leq 1,$$

it is clear that U will be maximized by the following choices

Table 1

$\lambda_i(x_i)$	$\lambda_i(x_i)$	$q_i(x_i)$	$p_i(x_i)$	$x_i \in$
≤ 0	≤ 0	0	0	do nothing: $x_i \in X_i^0$
> 0	$\lambda_i \leq 1$	1	0	commit: $x_i \in X_i^q$
$\lambda_i \leq 0$	0	0	1	phone: $x_i \in X_i^p$

This proves that the optimal strategies can be chosen among the pure strategies.

We proceed to calculate $\lambda_i(x_i)$ and $\lambda_i(x_i)$. Write

$$\left. \begin{aligned} Q_i(x_i) &= \int_{1-x_i}^1 q_j(t) dt \\ Q_i^*(x_i) &= \int_{1-x_i}^1 t \cdot q_j(t) dt \\ P_i(x_i) &= \int_{1-x_i}^1 p_j(t) dt \\ P_i^*(x_i) &= \int_{1-x_i}^1 t \cdot p_j(t) dt \end{aligned} \right\} i \neq j$$

Hence $Q_i(0) = Q_i^*(0) = P_i(0) = P_i^*(0) = 0$

$$(15a.3) \quad Q_1^*(1) \leq \frac{1}{2} \quad P_1^*(1) \leq \frac{1}{2}$$

$$Q_1(1) - Q_1^*(1) \leq \frac{1}{2} \quad P_1(1) - P_1^*(1) \leq \frac{1}{2}$$

Then from the integral (15a.1)

$$\lambda_1(x_1) = x_1 [1 - P_1(1)] + P_1(1) - P_1^*(1) - \frac{1}{2}$$

and from both (15a.1) and (15a.2)

$$\begin{aligned} \lambda_1(x_1) = & x_1^2 - x_1 [Q_1(1) + 2 P_1(x_1)] + 2 P_1(x_1) \\ & - 2 P_1^*(x_1) + Q_1(1) - Q_1^*(1) - c [1 - P_1(1)] \end{aligned}$$

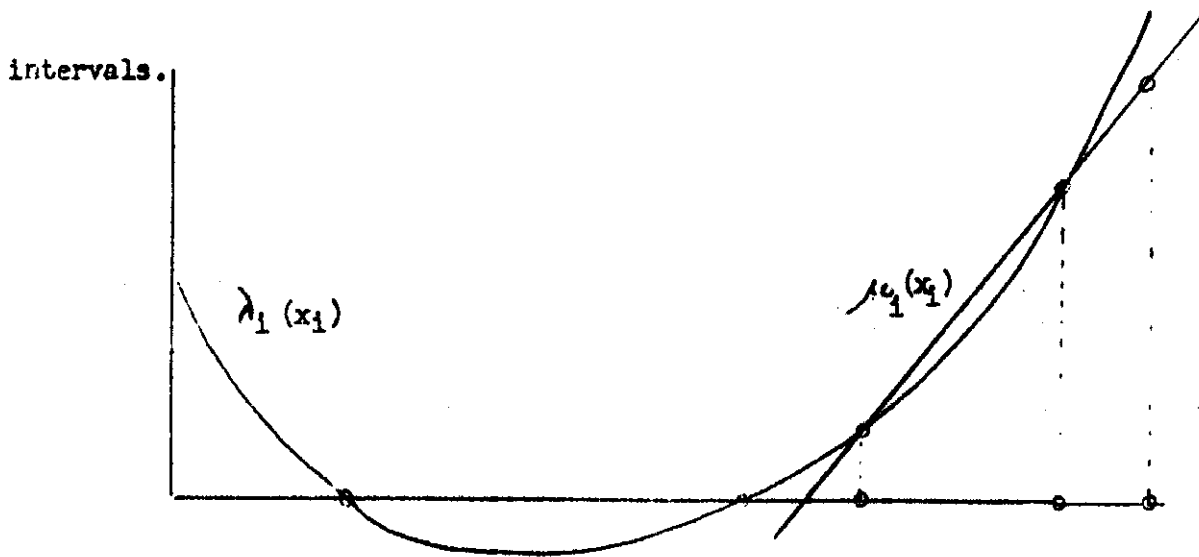
We first note that $\lambda_1(x_1)$ is linear with non-negative slope coefficient.

From the table it is seen now that an interval X_1^q can never be to the left of an interval X_1^o .

Secondly $\lambda_1(x_1)$ is convex since

$$\begin{aligned} \frac{d \lambda_1(x_1)}{dx_1} &= 2 x_1 - 2 P_1(x_1) - Q_1(1) \\ \frac{d^2 \lambda_1(x_1)}{dx_1^2} &= 2 [1 - p_1(1 - x_1)] \geq 0. \end{aligned}$$

Thus the curve $y = \lambda_1(x_1)$ intersects the two lines $y = \lambda_1(x_1)$ and $y = 0$ in at most two points each. These intersections on the other hand contain all the separation points between the sets X_1^k . Hence the sets X_1^k are intervals not exceeding 5 in number. Because of the restriction on the relative location of X_1^o and X_1^q the most general arrangement is p o p q p, and all other cases ^{are} obtained from it by omitting one or several symbols in the sequence. The 5 interval case is actualized by the following configuration, and any other one can be obtained by contraction of some of these



Let the strategy intervals now be labelled as in the following scheme

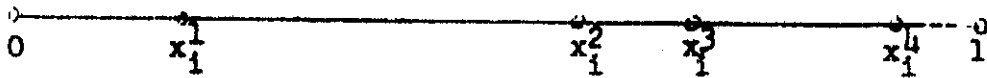


Table 2 lists the relations by which the interval endpoints $x_i^1, x_i^2, x_i^3, x_i^4$ are defined ($i = 1, 2$).

Table 2

(\cdot)	$\lambda_i(\cdot)$	$\frac{d\lambda_i}{dx_i}(\cdot)$	$\lambda_i(\cdot) - \mu_i(\cdot)$	$\frac{d}{dx_i}[\lambda_i(\cdot) - \mu_i(\cdot)]$
x_i^1	$= 0$	≤ 0		
x_i^2	$= 0$	≥ 0		
x_i^3			$= 0$	≤ 0
x_i^4			$= 0$	≥ 0

for $i = 1, 2$.

Any set of numbers $x_j^1 \dots x_1^4, x_j^1 \dots x_j^4$ that solves the relations of Table 2 maximizes U and is thus a solution of our problem. For, the conditions given in Table 1 are sufficient and the intervals as defined in Table 2 are so derived as to satisfy the conditions of Table 1.

We conclude by giving a particular solution of the problem. Let S, t be parameters satisfying

$$0 \leq S; \quad S + 4c \leq t - 4c; \quad t \leq 1.$$

A family of solutions (and presumably the only solution) is then given by

$$\begin{aligned} x_i^1 &= S & x_j^1 &= 1 - t \\ x_i^2 &= S + 4c & x_j^2 &= 1 - t + 4c \\ x_i^3 &= t - 4c & x_j^3 &= 1 - S - 4c \\ x_i^4 &= t & x_j^4 &= 1 - S \end{aligned}$$

(Note the anti-symmetric relationship $x_i^e = 1 - x_j^e$). One has

Table 3

$f(x_i)$ \ x_i	S	S + 4c	t - 4c	t	1
$P_i(\cdot)$	S	S	t - 8c	t - 8c	1 - 8c
$P_i(\cdot) - P_i^*(\cdot)$	$\frac{1}{2} S^2$	$\frac{1}{2} S^2$	$\frac{t^2}{2} - 4cS - 4ct$	$\frac{t^2}{2} - 4cS - 4ct$	$\frac{1}{2} - 4cS - 4ct$
$Q_i(\cdot)$					4c
$Q_i(\cdot) - Q_i^*(\cdot)$					$4cS + 8c^2$

$$\lambda_1(x_1) = x_1^2 - x_1 [Q_1(1) + 2P_1(x_1)] + 2[P_1(x_1) - P_1^*(x_1)] \\ + Q_1(1) - Q_1^*(1) - c[1 - P_1(1)]$$

$$\lambda_1(s) = s^2 - s[4c + 2s] + s^2 \\ + 4cs + 8c^2 - c8c = 0$$

$$\lambda_1(s+4c) = (s+4c)^2 - (s+4c)[2s + 4c] \\ + s^2 + 4cs + 8c^2 - 8c^2 = 0$$

$$\lambda_1(x_1) - \mu_1(x_1) = x_1^2 - x_1 [Q_1(1) + 1 - P_1(1) \\ + 2P_1(x_1)] + 2[P_1(x_1) - P_1^*(x_1)] \\ + [Q_1(1) - Q_1^*(1)] - [P_1(1) - P_1^*(1)] + \frac{1}{2} \\ - c[1 - P_1(1)]$$

$$\lambda_1(t-4c) - \mu_1(t-4c) = (t-4c)^2 - (t-4c)[4c+1-1+8c+2t-16c] + t^2 - 8cs - 8ct \\ + 4cs + 8s^2 - (\frac{1}{2} - 4cs - 4ct) + \frac{1}{2} - 8c^2 = 0$$

$$\lambda_1(t) - \mu_1(t) = t^2 - t[4c+1-1+8c+2t-16c] \\ + t^2 - 8cs - 8ct + 4cs + 8c^2 - (\frac{1}{2} - 4cs - 4ct) + \frac{1}{2} \\ - 8c^2 = 0.$$

With

$$s' = 1 - t$$

$$t' = 1 - s$$

the conditions for the x_j follow. We may skip the verification of the inequalities for the derivatives of $\lambda_1(x_1)$ and $\lambda_1(x_1) - \mu_1(x_1)$ since our

assumptions on S and t , which are preserved for S' , t' , ensure that the x_1^1, \dots, x_1^h are put in the right order. This completes the proof.

While we have not shown that this is the only solution, we have found a reasonable answer to the problem. For it is sufficient that the firm is in possession of one set of strategies ensuring the maximal profit.

The results of this paper are in accordance with those of K. Faxén in Cowles Commission Discussion Paper Economics 2037. In addition we have established that his relative optimum (for the case labelled II) is the absolute optimum.