

Appendix to

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Some Considerations on the Relevance of Entrepreneurial
Anticipations to Current and Future Activity of the Firm

Solution of Certain Problems of Production Planning Over Time

Illustrating the Effect of the Inventory Constraint

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1. Introduction. We shall consider the problem of an organization which wishes to schedule the production of a given commodity x over T equal, successive periods of time in such a way as (1) to meet initially known demands n_1, n_2, \dots, n_T in these periods while (2) incurring the lowest possible cost.

Denote by x_j the production of the j^{th} period. Any production plan which meets condition (1) will be called feasible. The production plan which satisfies both conditions will be called the optimal production plan and will be denoted by $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_T$.

Let h_0 denote beginning inventories and let h_k denote inventories at the end of period k . The relation between h_k , production and sales can be expressed conveniently by the equation

$$(1.1) \quad h_k = h_0 + \sum_{t=1}^k (x_t - n_t), \quad k = 1, 2, \dots, T.$$

The requirement that production should be adequate to meet demand, i.e., that inventories should at all times be nonnegative, is what we call the inventory constraint. It is expressed by the conditions

$$(1.2) \quad h_k = h_0 + \sum_{t=1}^k (x_t - n_t) \geq 0, \quad k = 1, 2, \dots, T.$$

We shall assume that production is planned so that the terminal inventory h_T is zero:

$$(1.3) \quad h_T = h_0 + \sum_{t=1}^T (x_t - n_t) = 0$$

that is, so that the last of (1.2) is an equality.

An additional constraint to be noted is that production cannot be negative:

$$(1.4) \quad x_k \geq 0 \quad k = 1, 2, \dots, T.$$

Finally, we assume throughout that

$$(1.5) \quad h_0 < \sum_{t=1}^T n_t$$

since otherwise there is no problem of planning production.

Let C denote the total cost of carrying out any given production plan. We will regard C as consisting of the sum of the cost C_p of producing the given program, the cost C_a of carrying the resulting inventories, and a constant c independent of the production schedule adopted.

We shall assume that C_p is symmetric in the x 's in such a way that marginal cost in each period is the same, nonnegative, strictly increasing, differentiable function ϕ of production in that period only, that is, that

$$(1.6) \quad C_p = \sum_{t=1}^T \bar{\Phi}(x_t) \quad \text{where} \quad \frac{\partial C_p}{\partial x_t} = \frac{d \bar{\Phi}(x_t)}{d x_t} = \phi(x_t), \quad t = 1, 2, \dots, T.$$

As for storage costs, we shall assume that the cost of storing a unit is proportional to the length of time during which it is stored, and we will denote by α the cost of storing one unit for one period. If we make the further assumption, which is reasonable as a convenient approximation, that production and sales take place at an even rate within each period, then the change in inventories $h_k - h_{k-1}$ within each period k will itself occur at a constant rate and the average amount held for the period will be

$\frac{h_k + h_{k-1}}{2}$. It follows that the total cost of storage, taking (1.3) into account, will be given by $C_s = \alpha \left[\frac{h_0}{2} + \sum_{t=1}^{T-1} h_t \right]$. The total cost C can

therefore be expressed as

$$(1.7) \quad C = \sum_{t=1}^T \Phi(x_t) + \alpha \left[\frac{h_0}{2} + \sum_{t=1}^{T-1} h_t \right] + c.$$

To simplify the notation, we shall put $N_k = \sum_{t=1}^k n_t$ in many of the formulas to follow.

2. The Unconstrained Solution. We shall begin by considering the solution of the problem when the constraints (1.2) and (1.4) involving inequalities are neglected. This solution, which we shall call the "unconstrained solution," can be obtained by equating to zero the derivatives of the Lagrangean function:

$$(2.1) \quad C^* = \sum_{t=1}^T \Phi(x_t) + \alpha \left[\frac{h_0}{2} + \sum_{t=1}^k (x_t - n_t) \right] + \frac{\alpha h_0}{2} \\ + \lambda \left[h_0 + \sum_{t=1}^T (x_t - n_t) \right] + c.$$

This yields the system of equations:

$$(2.2) \quad \frac{\partial C^*}{\partial x_t} = \phi(x_t) + \lambda + (T-t)\alpha = 0 \quad t = 1, 2, \dots, T$$

which imply

$$(2.3) \quad \phi(x_{t+1}) = \phi(x_t) + \alpha, \quad t = 1, 2, \dots, T-1$$

or more generally,

$$(2.4) \quad \phi(x_{t+j}) = \phi(x_t) + j\alpha, \quad 2 \leq t+j \leq T.$$

Condition (2.4) has the following simple economic interpretation.

$\phi(x_{t+j})$ is the cost of making an additional unit available in period $t+j$ by producing it in this period. On the other hand, $\phi(x_t) + j\alpha$

may be thought of as the cost of making an additional unit available in period $t + j$ by producing it in period t and storing it for j periods, thus incurring a storage cost $j\alpha$. If the right member of (2.4) were less than the left member, costs would be reduced by stepping up x_t and reducing x_{t+j} and vice versa. When the constraints are taken into account, the equations (2.4) are replaced by the conditions

$$(2.4') \quad \phi(x_{t+j}) \leq \phi(x_t) + j\alpha.$$

The reason for this is that it is always possible to increase x_t and decrease x_{t+j} correspondingly, the reverse, however, not being true. If the right member exceeds the left member, it may still not be possible to decrease x_t , for such a decrease might make it impossible to meet demand in the first $t+j-1$ periods. Thus (2.4') says that one condition for minimum costs is that the cost of producing one additional unit in any period should be not larger than the cost of producing an extra unit in any earlier period and carrying it over to the given period by storing it.

By means of equation (2.4) we can exhibit more explicitly the "unconstrained solution" of the problem. From (2.4) we have in fact,

$$(2.5) \quad x_t = \phi^{-1}[\phi(x_1) + (t-1)\alpha] \quad t = 2, 3, \dots, T$$

so that substituting into (1.3) and rearranging terms we obtain:

$$(2.6) \quad \sum_{t=1}^T \phi^{-1}[\phi(x_1) + (t-1)\alpha] = N_T - h_0.$$

Since the function ϕ is monotonic, the left member of (2.6) is itself a monotonic function of x_1 so that this equation will in general admit of a single solution, say $x_1 = \xi_1^T$. (Here the superscript T refers to the fact that we are dealing with a T -period problem.) The unconstrained solutions ξ_t^T for the other x 's are then obtained by substituting ξ_1^T

for x_1 in the equations (2.5). Clearly these formulas imply a gradually rising pattern of production which will be inconsistent with the constraints unless sales themselves exhibit a suitably rising pattern.

The rising pattern of the ξ_t^T 's can be exhibited explicitly if we consider the simplest case of increasing marginal cost, namely when $\bar{\Phi}(x)$ is quadratic and therefore $\phi(x)$ is linear. Let us write in this case

$$\phi(x) = \phi' \cdot x + \phi_0, \quad \phi' > 0,$$

so that

$$\phi^{-1}(x) = \frac{x - \phi_0}{\phi'}.$$

Then equation (2.5) becomes

$$(2.5') \quad x_t = x_1 + (t-1) \frac{\alpha}{\phi'}.$$

The solution of (2.6) yields

$$\xi_1^T = \frac{N_T - h_0}{T} - \frac{T-1}{2} \cdot \frac{\alpha}{\phi'}$$

whence, substituting ξ_1^T for x_1 (2.5'), we have:

$$\xi_t^T = \left[\frac{N_T - h_0}{T} - \frac{T+1}{2} \frac{\alpha}{\phi'} \right] + t \cdot \frac{\alpha}{\phi'}$$

in which the rising pattern of the ξ_t^T 's is clear.

The unconstrained solution obtained from (2.5), (2.6) will represent the optimal program if and only if it happens to satisfy the constraints (1.2) and (1.4). Since this solution implies $\xi_t^T \geq \xi_1^T$, $t = 2, 3, \dots, T$, conditions (1.4) will be satisfied if $\xi_1^T \geq 0$, which in turn can be stated as:

$$(2.7) \quad \sum_{t=2}^T \phi^{-1} [\phi(0) + (t-1)\alpha] \leq N_T - h_0.$$

For the conditions (1.2) to be satisfied by the unconstrained solution,

we must further have

$$(2.8) \quad \sum_{t=1}^k \xi_t^T = \sum_{t=1}^k \rho^{-1} [\rho(\xi_t^T) + (t-1)\alpha] \geq N_k - h_0,$$

$$k = 1, 2, \dots, T-1.$$

The nature of the optimal solution when not all of (2.7), (2.8) are satisfied will be indicated in the next section.

3. The Constrained Solution. We shall now show that when the constraints (2.7) and (2.8) are not all satisfied by the unconstrained solution

$\xi_1^T, \xi_2^T, \dots, \xi_T^T$, the optimal solution may be constructed as follows: First of all, by ignoring successively n_T, n_{T-1}, \dots , as far as is necessary, we seek to determine an integer $k_1 \geq 1$, such that for the problem consisting of the first k_1 periods, the unconstrained solution, which we shall designate by $\xi_1^{1,k_1}, \dots, \xi_{k_1}^{1,k_1}$, satisfies the constraints for the k_1 -period problem, whereas for each p such that $T \geq p > k_1$, the unconstrained solution ξ_1^p, \dots, ξ_p^p for the first p periods does not satisfy the constraints for the p -period problem. The integer k_1 is unique if it exists. If no such integer exists, then even $\xi_1^1 = n_1 - h_0$ does not satisfy the constraints for the problem defined by first period demand and initial inventory h_0 . That is, in this event $\xi_1^1 < 0$, $h_0 > n_1$, and we shall put $x_1 = 0$.

If we are thus led to put $x_1 = 0$, then we determine in the same fashion in how many additional periods the production shall be zero. We use in the case of x_2 the demands of the last $T-1$ periods and an initial inventory of $h_0 - n_1$, etc. Continuing in this manner, suppose we find that

$$x_1 = x_2 = \dots = x_{k_0} = 0,$$

but that x_{k_0+1} should not be planned as 0. It is clear that if

$$N_p \leq h_0 < N_{p+1},$$

then

$$k_0 \leq p.$$

In the process of determining k_0 , we determine incidentally the integer k_1 for the problem of the last $T - k_0$ periods with initial inventory $h_0 - N_{k_0}$.

After having determined k_0 and k_1 , we treat the problem defined by the demands of the remaining $T - k_0 - k_1$ periods, with initial inventory zero, in exactly the same manner, determining next the unique integer k_2 , $T - k_0 - k_1 \geq k_2 \geq 1$ such that the unconstrained solution $\xi_1^{2,k_2}, \dots, \xi_{k_2}^{2,k_2}$ of the problem of the first k_2 of the remaining periods satisfies the constraints for these periods, whereas the unconstrained solution ξ_1^q, \dots, ξ_q^q for the q -period problem starting with period $k_0 + k_1 + 1$ does not satisfy the constraints if $q > k_2$.

Continuing thus, we obtain a finite sequence of integers $k_0, k_1, k_2, \dots, k_p$ such that

$$\sum_{j=0}^p k_j = T$$

and a corresponding, feasible production plan, consisting of $p + 1$ blocks of periods:

$$(3.1) \quad x_1 = x_2 = \dots = x_{k_0} = 0, \quad x_{k_0+1} = \xi_1^{1,k_1}, \dots, x_{k_0+k_1} = \xi_{k_1}^{1,k_1}, \\ x_{k_0+k_1+1} = \xi_1^{2,k_2}, \dots, x_{k_0+k_1+k_2} = \xi_{k_2}^{2,k_2}, \dots, x_T = \xi_{k_p}^{p,k_p}$$

which, as we shall show, is the optimal plan. Here the first superscript j on ξ_1^{j,k_j} designates the block of which this production figure is a

number and the second superscript k_j denotes the number of periods in the block. The subscript i has the usual meaning in each block.

First of all, no reallocation purely within the block of zeros is feasible, and any feasible reallocation of production within the blocks of k_1, k_2, \dots, k_p periods would increase costs since the block plans are internally optimal by construction. A reallocation calling for a net decrease in production in any earlier block or blocks, to be offset by increased production later, is not feasible since total production in each block is either zero (in the opening block) or is designed to meet the total demand of that block exactly. It therefore remains only to show that costs are increased by increasing production in earlier blocks in order to reduce production in later blocks.

To show this, we note first that if we have a k -period problem contained, with respect to time, in a p -period problem, and if ξ_s^p and ξ_1^k refer to the same period, then from

$$\phi(\xi_{t+s}^p) = \phi(\xi_s^p) + t\alpha$$

and

$$\phi(\xi_{t+1}^k) = \phi(\xi_1^k) + t\alpha$$

we have

$$\phi(\xi_{t+s}^p) - \phi(\xi_{t+1}^k) = \phi(\xi_s^p) - \phi(\xi_1^k), \quad t = 0, 1, \dots, k-1.$$

Hence, in view of the monotonic character of ϕ , we must have one of the following situations:

$$(3.2a) \quad \xi_{t+s}^p > \xi_{t+1}^k, \quad t = 0, 1, \dots, k-1$$

or

$$(3.2b) \quad \xi_{t+s}^p = \xi_{t+1}^k, \quad t = 0, 1, \dots, k-1$$

or finally

$$(3.2c) \quad \sum_{t+s}^p < \sum_{t+1}^k, \quad t = 0, 1, \dots, k-1.$$

Secondly, from relationships of the type

$$\phi\left(\sum_{t}^k\right) = \phi\left(\sum_{t}^k\right) + (t-1)\alpha$$

we note that if it increases costs to carry inventory from the first period of one block (other than the zero block) to the first period of the next, then it also increases costs to carry inventory from any period of the first mentioned block to any period of the next.

We shall begin by demonstrating that

$$(3.3) \quad \phi\left(\sum_{1}^{1, k_1}\right) + k_1\alpha > \phi\left(\sum_{1}^{2, k_2}\right).$$

By the definition of k_1 , the unconstrained solution $\sum_{1}^{k_1+k_2}, \dots, \sum_{k_1+k_2}^{k_1+k_2}$

does not satisfy the constraints of the problem for the $k_1 + k_2$ periods starting with period $k_0 + 1$. Hence, we must have for some integer, p ,

$1 \leq p \leq k_1 + k_2 - 1$, an inequality $\sum_{1}^{k_1+k_2} + \dots + \sum_p^{k_1+k_2} < N_{p+k_0} - h_0$.

Suppose $p \leq k_1$. Then, $\sum_{1}^{1, k_1} + \dots + \sum_p^{1, k_1} \geq N_{p+k_0} - h_0$, and hence we

have $\sum_{t=1}^p \left(\sum_t^{1, k_1} - \sum_t^{k_1+k_2} \right) > 0$ so that from (3.2) we conclude that in

fact

$$(3.4a) \quad \sum_t^{1, k_1} > \sum_t^{k_1+k_2} \quad t = 1, 2, \dots, k_1.$$

Then, since

$$\sum_{t=1}^{k_1} \sum_t^{1, k_1} + \sum_{t=1}^{k_2} \sum_t^{2, k_2} = \sum_{t=1}^{k_1+k_2} \sum_t^{k_1+k_2} = N_{k_0+k_1+k_2} - h_0$$

we conclude that

$$\sum_{t=1}^{k_2} \left(\sum_t^{2, k_2} - \sum_{t+k_1}^{k_1+k_2} \right) < 0.$$

so that we have, again with the aid of (3.2),

$$(3.4b) \quad \sum_t^{2, k_2} < \sum_{t+k_1}^{k_1+k_2}, \quad t = 1, 2, \dots, k_2$$

If $p > k_1$, a similar computation yields the same conclusions, so that (3.4a,b) hold in either case. Then

$$\phi\left(\sum_1^{1, k_1}\right) > \phi\left(\sum_1^{k_1+k_2}\right)$$

so that

$$\phi\left(\sum_1^{1, k_1}\right) + k_1\alpha > \phi\left(\sum_1^{k_1+k_2}\right) + k_1\alpha = \phi\left(\sum_{k_1+1}^{k_1+k_2}\right) > \phi\left(\sum_1^{2, k_2}\right)$$

as was required.

Similar reasoning shows that

$$\phi\left(\sum_1^{j, k_j}\right) + k_j\alpha > \phi\left(\sum_1^{j+1, k_{j+1}}\right), \quad j = 1, 2, \dots, p-1,$$

which enables us to conclude quite generally that

$$\phi\left(\sum_1^{j, k_j}\right) + (k_j + k_{j+1} + \dots + k_{j+r})\alpha > \phi\left(\sum_1^{j+r+1, k_{j+r+1}}\right)$$

so that in fact it increases costs to produce in any period of any block, except possibly in the zero block, for the purpose of meeting demand in any period of any later block.

Finally, in view of what precedes, to show that a reallocation calling for positive production anywhere in the block of zero productions will increase costs, it will suffice to show that

$$\phi(0) + \alpha > \phi\left(\sum_1^{1, k_1}\right).$$

In the (k_1+1) -period problem defined by the initial inventory $h_0 = N_{k_0-1}$ and the demands $n_{k_0}, n_{k_0+1}, \dots, n_{k_0+k_1}$, the unconstrained solution does not satisfy the constraints, by hypothesis. Hence we have for the unconstrained solution of this problem

$$\xi_1^{k_1+1} + \xi_2^{k_1+1} + \dots + \xi_{k_1+1}^{k_1+1} = N_{k_0+k_1} - h_0$$

but for some $p < k_1$,

$$\xi_1^{k_1+1} + \xi_2^{k_1+1} + \dots + \xi_{p+1}^{k_1+1} < N_{k_0+p} - h_0.$$

On the other hand in the k_1 -period problem with initial inventory $h_0 = N_{k_0}$ and demands $n_{k_0+1}, \dots, n_{k_0+k_1}$ the unconstrained solution does satisfy the constraints, by hypothesis. Hence we have for this problem

$$\xi_1^{1,k_1} + \dots + \xi_{k_1}^{1,k_1} = N_{k_0+k_1} - h_0$$

and, for the same p as above,

$$\xi_1^{1,k_1} + \dots + \xi_p^{1,k_1} \geq N_{k_0+p} - h_0.$$

From these relations we can conclude, in a manner similar to that used earlier, that

$$\xi_{q+1}^{k_1+1} > \xi_q^{1,k_1} \quad q = 1, 2, \dots, k_1$$

and that $0 > \xi_1^{k_1+1}$.

From these last two results we have then

$$\phi(0) + \alpha > \phi\left(\xi_1^{k_1+1}\right) + \alpha = \phi\left(\xi_2^{k_1+1}\right) > \phi\left(\xi_1^{1,k_1}\right)$$

and the fact that the stated plan is optimal is now completely proved.

4. Application to the Two and Three Period Problems. We discuss the two and three-period problems by way of illustrating Section 3.

In the two-period case, the solutions may be detailed as follows:

$$(4.1) \quad \tilde{x}_1 = \xi_1^2, \quad \tilde{x}_2 = \xi_2^2 = N_2 - h_0 - \xi_1^2, \quad \text{if } \xi_1^2 \geq \max(n_1 - h_0, 0)$$

$$(4.2) \quad \check{x}_1 = \max(n_1 - h_0, 0), \quad \check{x}_2 = N_2 - h_0 - \check{x}_1, \quad \text{if } \xi_1^2 < \max(n_1 - h_0, 0).$$

In this case, equation (2.6) reduces to the form

$$(4.3) \quad \phi(x_1) + \alpha = \phi(N_2 - h_0 - x_1),$$

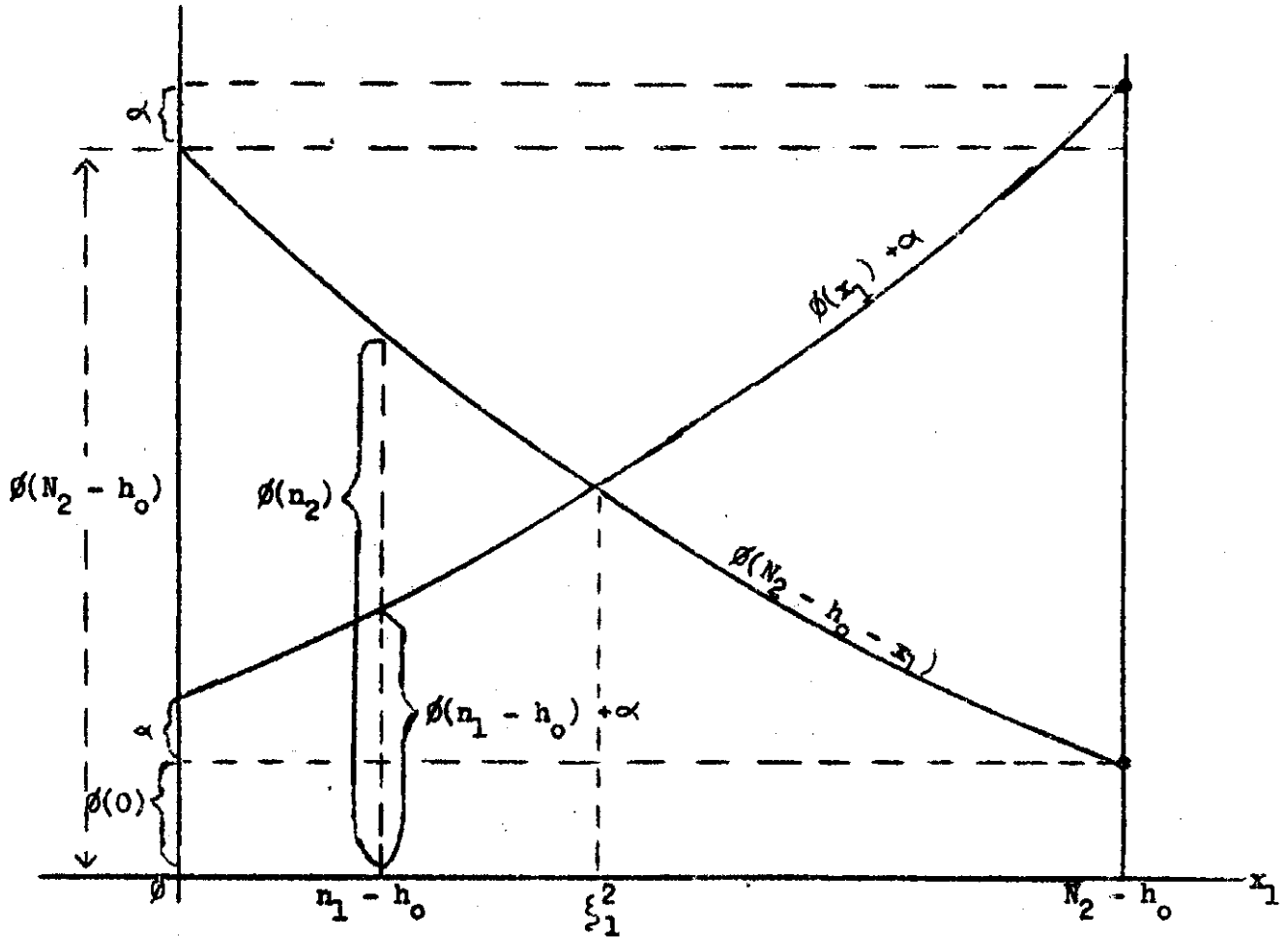
the condition that ξ_1^2 be nonnegative becomes

$$(4.4) \quad \phi(N_2 - h_0) \geq \phi(0) + \alpha,$$

while the condition that ξ_1^2 be not less than $n_1 - h_0$ reduces to

$$(4.5) \quad \phi(n_2) \geq \phi(n_1 - h_0) + \alpha.$$

It is interesting to exhibit these conditions graphically:



$$\phi(n_2) \geq \phi(n_1 - h_0) + \alpha \quad \text{if and only if } \xi_1^2 \geq n_1 - h_0$$

$$\phi(N_2 - h_0) \geq \phi(0) + \alpha \quad \text{if and only if } \xi_1^2 \geq 0$$

In the three-period case, the various solutions may be classified as follows:

$$(4.6) \quad \left\{ \begin{array}{l} \tilde{x}_i = \xi_i^3 \quad i = 1, 2, 3 \\ \text{if} \\ \xi_1^3 \geq \max(n_1 - h_0, 0) \\ \xi_1^3 + \xi_2^3 \geq n_1 + n_2 - h_0 \end{array} \right.$$

If these constraints are not both satisfied, we look at the problem of the first two periods. The solution is, if $\xi_1^3 \geq \max(n_1 - h_0, 0)$,

$$(4.7) \quad \left\{ \begin{array}{l} \tilde{x}_1 = \xi_1^2 = \xi_1^{1,2} \\ \tilde{x}_2 = \xi_2^2 = \xi_2^{1,2} \\ \tilde{x}_3 = n_3 = \xi_1^{2,1} \end{array} \right.$$

and finally, if $\xi_1^2 < \max(n_1 - h_0, 0)$, we have

$$(4.8) \quad \left\{ \begin{array}{l} \tilde{x}_1 = \max(n_1 - h_0, 0) \\ \tilde{x}_2 = J_1 \\ \tilde{x}_3 = J_2 \end{array} \right.$$

where J_1 and J_2 give the optimal solution to the two-period problem remaining after \tilde{x}_1 is fixed. The solution (4.8) may be further detailed, if desired, due consideration being given to the fact that we have

$$0 \leq h_0 < n_1, \quad n_1 \leq h_0 < N_2, \quad \text{or} \quad N_2 \leq h_0 < N_3.$$

5. The Case of No Storage Costs. In this case, equations (2.5) and (2.6) become respectively

$$x_t = x_1 \quad t = 2, 3, \dots, T$$

$$Tx_1 = N_t - h_0$$

so that the unconstrained solution of the problem is

$$(5.1) \quad x_t = \frac{N_t - h_0}{T} \quad t = 1, 2, \dots, T$$

Thus the unconstrained solution calls for production at an even rate throughout the horizon, the fluctuations in sales being absorbed by the accumulation or reduction of inventories. This production plan satisfies the constraint (1.3) and because of (1.5) it also satisfies the constraints (1.4). Hence, if it satisfies the conditions (1.2) it will actually represent the optimal plan.

A simple circumstance in which conditions (1.2) are satisfied is that in which

$$(5.2) \quad \frac{N_k}{k} \leq \frac{N_T}{T}, \quad k = 1, 2, \dots, T$$

that is, in which the terminal sales average is maximal. We have then, in fact

$$\frac{N_k - h_0}{k} \leq \frac{N_T - h_0}{T}, \quad k = 1, 2, \dots, T$$

so that using (5.1) we have

$$h_k = h_0 + \sum_1^k x_t - N_k = h_0 + k \frac{(N_k - h_0)}{k} - N_k \geq h_0 + k \frac{(N_k - h_0)}{k} - N_k = 0, \quad k = 1, 2, \dots, T-1$$

and the constraints are indeed satisfied. A special case of this is the case of nondecreasing sales:

$$n_1 \leq n_2 \leq \dots \leq n_T.$$

In case the constraints (1.2) are not all satisfied by (5.1), then the process of Section 3 leads to the determination of a unique integer k_1 , $1 \leq k_1 \leq T - 1$, such that

$$(5.3) \quad \frac{N_{k_1} - h_0}{k_1} \begin{cases} \geq \frac{N_k - h_0}{k} & \text{if } 1 \leq k < k_1, \\ > \frac{N_k - h_0}{k} & \text{if } T \geq k > k_1, \end{cases}$$

a second integer k_2 such that

$$\frac{N_{k_1+k_2} - N_{k_1}}{k_2} \begin{cases} \geq \frac{N_{k_1+k} - N_{k_1}}{k} & \text{if } 1 \leq k < k_2 \\ > \frac{N_{k_1+k} - N_{k_1}}{k} & \text{if } T - k_1 \geq k > k_2, \end{cases}$$

and so on. Suppose as in Section 3 there are p such integers altogether,

with $\sum_{j=1}^p k_j = T$. The optimal solution is then

$$(5.4) \quad \begin{cases} \tilde{x}_1 = \tilde{x}_2 = \dots = \tilde{x}_{k_1} = \frac{N_{k_1} - h_0}{k_1} \\ \tilde{x}_{k_1+1} = \dots = \tilde{x}_{k_1+k_2} = \frac{N_{k_1+k_2} - N_{k_1}}{k_2} \\ \dots \\ \tilde{x}_{k_1+\dots+k_{p-1}+1} = \dots = \tilde{x}_T = \frac{N_T - N_{k_1+\dots+k_{p-1}}}{k_p} \end{cases}$$

An instructive special case of this is the case in which sales are nonincreasing:

$$n_1 \geq n_2 \geq \dots \geq n_T$$

Here $k_2 = k_3 = \dots = k_p = 1$ and the optimal production plan is

$$(5.5) \quad \tilde{x}_1 = \tilde{x}_2 = \dots = \tilde{x}_{k_1} = \frac{N_{k_1} - h_0}{k_1}, \quad \tilde{x}_t = n_t, \quad t = k_1+1, \dots, T.$$

When $h_0 = 0$, this reduces to

$$(5.6) \quad \tilde{x}_t = n_t, \quad t = 1, 2, \dots, T.$$

It should be noted that in (5.3) we have actually determined k_1 so that

$$n_k \geq \frac{N_{k_1-1} - h_0}{k_1 - 1} \quad \text{whereas} \quad n_{k_1+1} < \frac{N_{k_1} - h_0}{k_1},$$

that is, we have used h_0 to smooth production so far as possible. Since there are no costs to storage, we will never have an initial block of periods in which the production is zero.

It is also interesting to note that the solution in the case of no storage costs is independent of the exact form of the cost function $\bar{\Phi}$.

6. An Operational Procedure. The preceding solution is a backward-looking one, so to speak, starting as it does at the horizon and working towards the present until at least a part of the solution has been determined. We may transform it into the following, forward-looking procedure which has significant implications for the planning process.

In the first period, production must be at least $\max(n_1 - h_0, 0)$, which we designate by \underline{I}_1^1 . Production will need to be scheduled at a higher level only if it pays to store for a later period or periods. For example, it will not pay to increase first period production over the minimum, for the purpose of meeting second period demands, unless \tilde{x}_2 exceeds a level \underline{I}_2^1 defined by

$$(6.1) \quad \phi(\underline{I}_2^1) = \phi(\underline{I}_1^1) + \alpha.$$

In general, if \underline{I}_1^1 is not optimal, but is to be increased to take care of demands in period k , then \tilde{x}_k must exceed the level \underline{I}_k^1 defined by

$$(6.2) \quad \phi(\underline{I}_k^1) = \phi(\underline{I}_1^1) + (k-1)\alpha$$

and furthermore, for every intervening period, we must have

$$(6.3) \quad \tilde{x}_t \geq \underline{I}_t^1, \quad t = 2, 3, \dots, t-1$$

since if $x_t < \mathcal{B}_t^1$, it pays to increase x_t to the level \mathcal{B}_t^1 before increasing x_1 above the level \mathcal{B}_1^1 .

Suppose now that

$$(6.4) \quad h_0 + \sum_{t=1}^k \mathcal{B}_t^1 - N_k \geq 0 \quad k = 1, 2, \dots, T.$$

Then \mathcal{B}_1^1 as above defined is in fact \tilde{x}_1 , for if $\tilde{x}_1 > \mathcal{B}_1^1$, then by Section 3, $\tilde{x}_1 = \xi_1^k$, where $1 < k \leq T$. Next, from the monotonic character of ϕ we have

$$\xi_t^k > \mathcal{B}_t^1, \quad t = 1, 2, \dots, k$$

so that

$$(6.5) \quad \sum_{t=1}^k \xi_t^k > \sum_{t=1}^k \mathcal{B}_t^1.$$

From (6.4) and (6.5) now follows

$$h_0 + \sum_{t=1}^k \xi_t^k - N_k > 0$$

whereas the values $\xi_1^k, \xi_2^k, \dots, \xi_k^k$ satisfy the constraint

$$h_0 + \sum_{t=1}^k \xi_t^k - N_k = 0.$$

The contradiction shows that in fact $\tilde{x}_1 = \mathcal{B}_1^1$.

If $\tilde{x}_1 = \mathcal{B}_1^1$, then we seek next to determine x_2 in the same manner, using an initial inventory equal to $\max(h_0 - n_1, 0)$. The sequence of zero productions of the solution described in Section 3 will be found first of all by this process.

On the other hand, if condition (6.4) fails to hold for the first time at $t = t_1$, i.e., if

$$(6.6) \quad \left\{ \begin{array}{l} h_0 + \sum_{t=1}^k \mathcal{I}_t^1 - N_t \geq 0 \quad k < t_1 \\ \text{but} \\ h_0 + \sum_{t=1}^{t_1} \mathcal{I}_t^1 - N_{t_1} < 0 \end{array} \right.$$

we then compute $\xi_1^{t_1}, \xi_2^{t_1}, \dots, \xi_{t_1}^{t_1}$, so that

$$(6.7) \quad h_0 + \sum_{t=1}^{t_1} \xi_t^{t_1} - N_{t_1} = 0.$$

From (6.6) and (6.7) and from the monotonicity of ϕ we have

$$\xi_t^{t_1} > \mathcal{I}_t^1, \quad t = 1, 2, \dots, t_1.$$

Hence we have also

$$(6.8) \quad h_0 + \sum_{t=1}^k \xi_t^{t_1} - N_k \geq 0 \quad k = 1, 2, \dots, t_1-1$$

so that, since (6.7) and (6.8) are satisfied, for the problem of the first t_1 periods, $\xi_1^{t_1}, \dots, \xi_{t_1}^{t_1}$ is the optimal plan.

Reasoning now as above, we see that this solution is optimal also for the first t_1 periods in the T-period problem unless conditions indicate positive inventories should be held at point t_1 . This will not be the case unless \tilde{x}_k exceeds the level $\mathcal{I}_k^{t_1}$ defined by

$$(6.9) \quad \phi(\mathcal{I}_k^{t_1}) = \phi(\mathcal{I}_1^{t_1}) + (k-1)\alpha$$

for some $k > t_1$ and also, for all t such that $t_1 < t < k$,

$$(6.10) \quad \tilde{x}_t \geq \mathcal{I}_t^{t_1}.$$

Hence if

$$(6.11) \quad \sum_{t=t_1+1}^k \mathcal{I}_t^{t_1} \geq N_k - N_{t_1}, \quad k = t_1 + 1, \dots, T$$

then $\xi_1^{t_1}, \xi_2^{t_1}, \dots, \xi_{t_1}^{t_1}$ is optimal for the first t_1 periods in the T -period problem, the opposite assumption leading to a contradiction just as before.

Proceeding in this manner, we eventually determine an integer $k_1 \leq T$ such that $\xi_1^{k_1}, \xi_2^{k_1}, \dots, \xi_{k_1}^{k_1}$ is optimal for the first k_1 periods in the T -period problem. This k_1 is precisely the k_1 of our original solution, as is not difficult to show.

After finding k_1 , we can handle by the same method the remaining $T-k_1$ period problem, with initial inventory zero and demands $n_{k_1+1}, n_{k_1+2}, \dots, n_T$. Continuing thus, we can determine the entire sequence of k 's.

7. Economic Implications. Of first importance in the solutions to the problem given in preceding sections is the fact that the optimal plan for the entire problem may break up into a sequence of plans optimal for certain blocks of periods. Furthermore, the plan for any one block of periods finally depends only on the expected demands in that block of periods. In particular, the optimal production plan for the first period will depend only on demand for the first t_1 periods and, unless demand has a sufficiently rising pattern, t_1 will tend to be small. Thus the optimal production plan can be arrived at without solving the problem for periods later than t_1 . The problem for later periods need not be solved until the first t_1 periods have elapsed, and it is only at this point that precise knowledge of demands in periods later than t_1 is required. It is true that in order to determine t_1 we must have some knowledge of demand for periods later than t_1 ; but we do not need any more information than is necessary to establish that condition (6.11) holds.

An important illustration of the preceding remarks is the case in

which demand exhibits a systematic and marked wave-like pattern as a result of seasonal fluctuations. Let us suppose that the seasonal cycles are so defined that the periods of heavy demand come at the end of each cycle and are immediately followed by the periods of low demand of the next cycle. Consider now the situation at the beginning of such a cycle, and let θ denote the last period of heavy demand which closes the first cycle. Because of the forthcoming seasonal rise in sales, condition (6.4) is not likely to be satisfied. In fact, provided marginal costs are sufficiently high in relation to storage costs, t_1 is not likely to be less than θ . In other words, the optimum plan will tend to cover at least the entire first seasonal cycle. At the same time, t_1 is not likely to exceed θ . In order for t_1 to exceed θ it is necessary that condition (6.11) with $\int_t^{t_1}$ replaced by \int_t^θ should fail to hold. Now period θ will be followed by periods of seasonally low sales for which the conditions just stated will necessarily hold. Clearly therefore this condition could only fail once we reach the heavy shipment part of the next cycle, say in the neighborhood of 2θ . However, even here the condition can fail to hold only if sales in the entire second cycle are expected to exceed by a substantial margin sales in the entire first cycle. Just how much higher these later demands should be, depends of course on storage costs and the nature of the marginal cost function. In this connection it should be remembered that "storage" costs include such factors as deterioration, shrinkage, obsolescence, and so on. A similar reasoning suggests that it is even less likely that the plan will have to be extended to include periods later than the end of the second season.

We therefore reach the conclusion that the optimum plan is likely to extend over the entire first seasonal cycle but at the same time is not

likely to extend beyond this cycle except in the presence of a rapidly rising overall trend. Furthermore, if it extends beyond the current cycle this extension is likely to proceed by whole cycles. We may finally note that to the extent that new information will make it necessary to re-plan in the course of a seasonal cycle, the revised plan itself will tend to extend to the balance of the given cycle, (plus, only rarely, complete later cycles).