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Note on Marschak's Model of a Arbitrage Firm.

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I. General Comments on Decentralized Decision-Making.

1. Consider, as in Cowles Commission Discussion Paper, Economics 2034, an organization, consisting of n members, $i=1, \ldots, n$. Each member observes the value of a variable x_i and controls a parameter of action a_i. The organization has a common goal, to maximize the function

$$V(a_1, \ldots, a_n | x_1, \ldots, x_n),$$

given the values $(x_1, \ldots, x_n)$, by selecting suitable values of $(a_1, \ldots, a_n)$.

2. Evidently, if these values were fixed centrally, they would be found by maximizing

$$V^{\prime}(x_1, \ldots, x_n) = \max_{a_1, \ldots, a_n} V(a_1, \ldots, a_n | x_1, \ldots, x_n).$$

The value V' is reached when $(a_1, \ldots, a_n)$ is determined according to some function

$$(a_1, \ldots, a_n) = f(x_1, \ldots, x_n) \qquad (\text{Rule } \gamma^c).$$

3. In a decentralized case, each member has to fix the value of his a_i, knowing his x_i, according to some rule $a_i = f_i(x_i)$, that is given by the management of the organization. Knowing the joint probability distribution of $(x_1, \ldots, x_n)$, the management could select the rule

$$[a_1 = f_1(x_1);\ a_2 = f_2(x_2);\ \ldots\ ;\ a_n = f_n(x_n)] \qquad (\text{Rule } \gamma^d)$$

so as to maximize the expectation $E(V)$.

Let this happen when $V = V[f_1(x_1) \ldots f_n(x_n) | x_1, \ldots, x_n] = V''$. Since the set of all rules of type γ^d is a subset of the set of all rules of type γ^c, $V'' \leqq V'$, and $E(V'') \leqq E(V')$.

4. As a device in decentralizing, the management could give each member a bonus function $V_i(a_i | x_i)$ such that the i-th member will maximize V_i, given x_i. The optimal choice of V_i (from the point of view of the management) is to choose it such that $a_i = f_i(x_i)$ maximizes V_i.

Denote the profit, that is reached by the organization under this system (rule γ^b) by V'''. Obviously,

$$V''' \leqq V'' \quad \text{and } E(V''') \leqq E(V''),$$

where the equality sign holds for some γ^b.

5. The team rule could also provide for a kind of sequential decision-making in the following way: If $x_i \in X_i^{pj}$, $f_i(x_i)$ means that member i should initiate communication with member j, after which the $(a_i \; a_j)$ are determined according to a new function

$$(a_i \; a_j) = f_{ij}(x_j \; x_j).$$

This could also be represented by bonus functions $V_{ij}(a_i | x_i \; x_j)$ and $V_{ji}(a_j | x_i \; x_j)$, to be used after communication has taken place. For some bonus functions, $V''' = V''$, and for all the others, $V''' < V''$.

In general, there is a system of bonus functions $(V_1 \; V_2 \ldots V_n, V_{1\cdot 2} \; V_{1\cdot 3} \ldots V_{1\cdot 23})$ that give the same profit as a system of rules $(f_1 \; f_2 \ldots f_n, f_{12} \; f_{13} \ldots f_{123} \ldots)$.

6. An example, on which the reasoning above could be applied, is given in CCDP 2034, Sections II and III. In Section II, the firm of two partners has a set of rules $(f_1 \; f_2 \; f_{12})$ and in Section III, a set of bonus functions. A joint distribution of $(x_1 \; x_2)$ is introduced, and the expected profit is shown to be lower if the bonus functions are used

then if the rules are used.

The reasoning above suggests that there should be some other bonus functions, that give the same profit as the rules in Section II.

7. The same general reasoning could also be applied to the problem of decentralized decisions on inventory movements in an organization, such as a chain store system. The management could

a) charge the retail stores a "cost-of-ordering," and make this cost such a function of the orders as to maximize profits for the system as a whole when the retail stores maximize their profits.

b) construct a rule, according to which the retail stores have to send in their orders, even if some other ordering policy should appear more profitable to them.

c) collect information at the center on the demand and inventory situation in the various retail stores and let the central management allocate the inventories.

The problem of how a retail store (with given demand, etc.) should adjust to a given "cost-of-ordering function" has been studied, and solved for certain types of such functions. Thus, we can find the distribution function for the demand at the wholesale level. This distribution will have as parameters the parameters in the "cost-of-ordering function," and it will determine costs at the wholesale level.

The bonus function for each retail store is determined by its "cost-of-ordering function" and existing accounting rules. Thus, we are equipped to find the "cost-of-ordering functions" that maximize profits for the organization as a whole. Except in special cases, this is not the most general class of bonus functions, and we should thus expect that there are rules of ordering (case b), that give higher profit than the best "cost-of-ordering-function" with optimal adjustment to it by the stores (case a).

II. Comments on Marschak's Example.

Note: The notations are taken from CCDP 2034.

1. Consider a partitioning of the intervals $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$ into subsets x_1^o, x_1^p, x_1^q, x_2^o, x_2^p, x_2^q. Denote the set of $(x_1\ x_2)$ such that $x_1 \in x_1^o$ and $x_2 \in x_2^o$ by $(x_1^o\ x_2^o)$, and introduce similar notations $(x_1^o\ x_2^p)$, $(x_1^o\ x_2^q)$, (x_1^p, x_2^o), ..., etc. Denote the subset of $(x_1^o\ x_2^o)$, where $x_1+x_2 > 1$, by $(x_1^o\ x_2^o)^+$ and the subset where $x_1+x_2 < 1$ by $(x_1^o\ x_2^o)^-$, and similarly $(x_1^o\ x_2^p)^+$, $(x_1^o\ x_2^p)^-$, etc.

The expected profit of the firm is

$$\iint_{A_1} (x_1+x_2-1)dx_1\ dx_2 + \iint_{A_2} 2(x_1+x_2-1)dx_1\ dx_2 + \iint_{A_3} \max[0;2(x_1+x_2-1)]-c\ dx_1\ dx_2$$

where $A_1 = (x_1^o\ x_2^q) \cup (x_1^q\ x_2^o)$

$A_2 = (x_1^q\ x_2^q)$

$A_3 = (x_1^p\ x_2^o) \cup (x_1^p\ x_2^p) \cup (x_1^p\ x_2^q) \cup (x_1^o\ x_2^p) \cup (x_1^q\ x_2^p)$.

This can be rearranged into:

$$\iint_{B_1} 2(x_1+x_2-1)dx_1\ dx_2 - \iint_{B_2} c\ dx_1\ dx_2 + \iint_{B_3} (x_1+x_2-1)dx_1\ dx_2$$

$$- \iint_{B_4} (x_1+x_2-1)dx_1\ dx_2 + 2\iint_{B_5} (x_1+x_2-1)dx_1\ dx_2 - 2\iint_{B_6} (x_1+x_2-1)dx_1\ dx_2$$

where B_1 = the set for which $0 < x_1 < 1$, $0 < x_2 < 1$ and $x_1+x_2 > 1$.

$B_2 = A_3$

$B_3 = (x_1^o\ x_2^q)^- \cup (x_1^q\ x_2^o)^-$

$B_4 = (x_1^o\ x_2^q)^+ \cup (x_1^q\ x_2^o)^+$

$B_5 = (x_1^q\ x_2^q)^-$

$B_6 = (x_1^o\ x_2^o)^+$

The first integral is independent of the partitioning, and will be disregarded in the following. If the length of x_1^p is (1-a) and of x_2^p is (1-b), then the second integral is c(1-ab).

2. It can further be shown that if ${}^p x_2$ is an arbitrary partitioning of $0 < x_2 < 1$, and ${}^p x_1^* = (x_1^{o*};\ x_1^{p*};\ x_1^{q*})$ is a partitioning of $0 < x_1 < 1$ that maximises profits for ${}^p x_2$, and if $x_1' \in x_1^{o*}$ and $x_1'' \in x_1^{q*}$, then $x_1'' > x_1'$.

3. Denote: $\max x_1(x_1 \in x_1^o) = x_1'$; $\min x_1(x_1 \in x_1^q) = x_1''$

$\max x_2(x_2 \in x_2^o) = x_2'$; $\min x_2(x_2 \in x_2^q) = x_2''$.

The two points $(x_1'\ x_2')$ and $(x_1''\ x_2'')$ can be located in relation to the line $x_1 + x_2 = 1$ in three ways; illustrated by cases I, II and II on Chart 1.

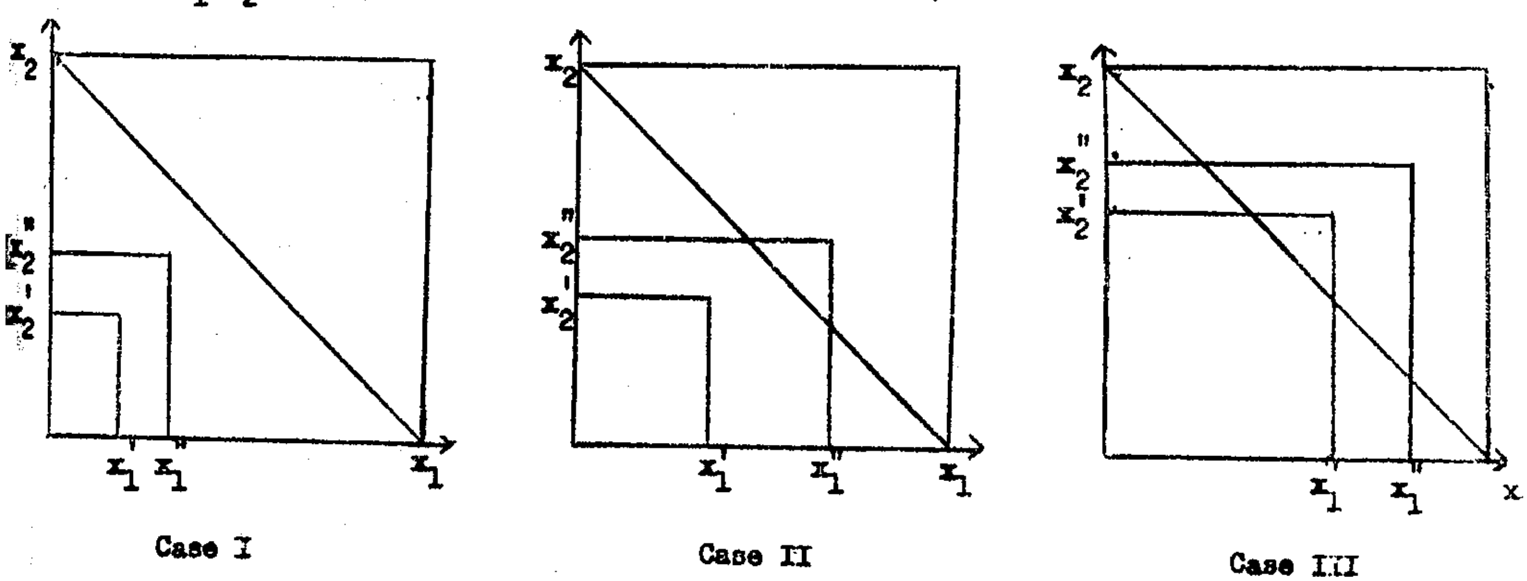

Chart 1.

In Case I, B_6 is empty and in Case III, B_5 is empty. In Case II, both B_5 and B_6 are empty.

Cases I and III have not yet been investigated completely, but preliminary considerations suggest that Cases I or III could not represent optimal partitionings. The following discussion will be confined to Case II.

4. The problem is now to maximize

$$\iint_{B_3} (x_2+x_1-1)dx_1\, dx_2 - \iint_{B_4} (x_1+x_2-1)dx_1\, dx_2 + c\cdot ab = -L.$$

Assume that x_1^o x_1^q x_2^o and x_2^q are intervals (but not x_1^p and x_2^p). The case when x_i^o and x_i^q are not intervals has not yet been investigated completely, but preliminary considerations suggest that they are intervals in an optimal situation.

An example of a case when x_i^o and x_i^q are intervals is given in Chart 2. B_3 is shadowed /// and B_4 is shadowed \\\\\\ .

5. Lemma. If $\Delta\alpha$ is the set of $(x_1\ x_2)$ such that

$x_1 > x_1^*$, $x_2 > 1 - x_1^* - \alpha$ and $x_1 + x_2 - 1 < 0$, then

$$\iint_{\Delta\alpha} (x_1+x_2-1)dx_1\, dx_2 = -\frac{1}{6}\alpha^3. \quad \text{(Chart 3)}.$$

Note that the integral does not depend upon x_1^*.

Also, if $\Delta\alpha'$ is the set of $(x_1\ x_2)$ such that

$x_1 < x_1^*$, $x_2 < 1 - x_1^* + \alpha$ and $x_1 + x_2 > 1$,

$$\iint_{\Delta\alpha'} (x_1+x_2-1)dx_1\, dx_2 = \frac{1}{6}\alpha^3.$$

6. Denote the lengths of x_1^o by a_1, of x_1^q by a_2, of x_2^o by b_2, and of x_2^q by b_1. It is seen that $a_1 + a_2 = a$ and $b_1 + b_2 = b$.

L can now be written

$$L = \iint_{(x_1^o\ x_2^q)^-} -(x_1+x_2-1)dx_1\, dx_2 + \iint_{(x_1^o\ x_2^q)^+} (x_1+x_2-1)dx_1\, dx_2 + \iint_{(x_1^q\ x_2^o)^-} -(x_1+x_2-1)dx_1\, dx_2$$
$$+ \iint_{(x_1^q\ x_2^o)^+} (x_1+x_2-1)dx_1\, dx_2 - c(a_1+a_2)(b_1+b_2).$$

Assume first that the diagonal $x_1+x_2 = 1$ cuts two sides, which form right angles, in one of the shaded rectangles (Chart 4). The sum of the integrals over the shaded areas in Chart 4 is (d_1 as defined in Chart 4)

$$\frac{1}{6}[(a_1+b_1-d_1)^3 + d_1^3 - (b_1-d_1)^3 - (a_1-d_1)^3] \qquad \text{where } 0 \leq d_1 \leq a_1.$$

This expression is minimum for $d_1 = a_1$.

7. From Chart 2 it is obvious that d_1 does not affect the integral over $(x_1^q\ x_2^o)$ and vice versa. It is further obvious that a situation, in which the diagonal does not cut both $(x_1^o\ x_2^q)$ and $(x_1^q\ x_2^o)$ cannot be optimal. Thus, we are left with the case when the diagonal cuts parallell sides of both $(x_1^o\ x_2^q)$ and $(x_1^q\ x_2^o)$ (Chart 5).

Assume that $b_1 \geq a_1$. The sum of the integrals over $(x_1^o\ x_2^q)$ is now (d_1' as defined in Chart 5),

$$\frac{1}{6}\left[(b_1-d_1')^3 - (b_1-a_1-d_1')^3 + (a_1+d')^3 - d_1'^3\right] \quad 0 \leq d_1' \leq b_1 - a_1$$

which is minimum for $d_1 = \dfrac{b_1-a_1}{2}$. The minimum value is

$$\frac{1}{24}\left[(a_1+b_1)^3 - (b_1-a_1)^3\right].$$

A similar analysis can be carried through for the rectangle $(x_1^q\ x_2^o)$.

8. If, for instance, $b_1 \geq a_1$ and $a_2 \geq b_2$, we can write

$$L = \frac{1}{24}\left[(a_1+b_1)^3 + (a_2+b_2)^3 - (b_1-a_1)^3 - (a_2 - b_2)^3\right] - c(a_1+a_2)\cdot(b_1+b_2).$$

It is then assumed that the rectangle $(x_1^o\ x_2^q)$ with sides a_1 and b_1 and the rectangle $(x_1^q\ x_2^o)$ with sides a_2 and b_2 are located optimally in relation to the diagonal $x_1 + x_2 - 1 = 0$.

If L is differentiated with respect to a_1 a_2 b_1 and b_2, and the first derivatives put equal to zero, the following equations are obtained:

$$\left.\begin{array}{l}(a_1+b_1)^2 + (b_1-a_1)^2 = 8c(b_1+b_2)\\ (a_2+b_2)^2 - (a_2-b_2)^2 = 8c(b_1+b_2)\\ (a_1+b_1)^2 - (b_1-a_1)^2 = 8c(a_1+a_2)\\ (a_2+b_2)^2 + (a_2-b_2)^2 = 8c(a_1+a_2)\end{array}\right\}$$

The only root of that system, for which a_1 a_2 b_1 b_2 are all positive is

$$a_1 = b_1 = a_2 = b_2 = 4c .$$

An investigation of the second-order derivatives, which has not yet been carried out, would be needed in order to establish that this root gives a minimum.

Since $0 \leqq a_1 \leqq \frac{1}{2}$, $c \leqq \frac{1}{8}$. If $c > \frac{1}{8}$, it does not pay to phone.

9. The solution in CCDP 2029, where it was assumed from the beginning that $a_1 = a_2 = b_1 = b_2$, is a special case of the solution obtained here. It is obtained when the sets $(x_1^o\ x_2^q)$ and $(x_1^q\ x_2^o)$ are moved to the corners (0;1) and (1;0) respectively. But in general, they could be located anywhere along the diagonal (Chart 6). $x_1^{'}$ and $x_1^{''}$ on Chart 6 could be given any values such that $0 \leqq x_1^{'} \leqq 1 - 8c$ and $x_1^{'} + 4c \leqq x_1^{''} \leqq 1 - 4c$.

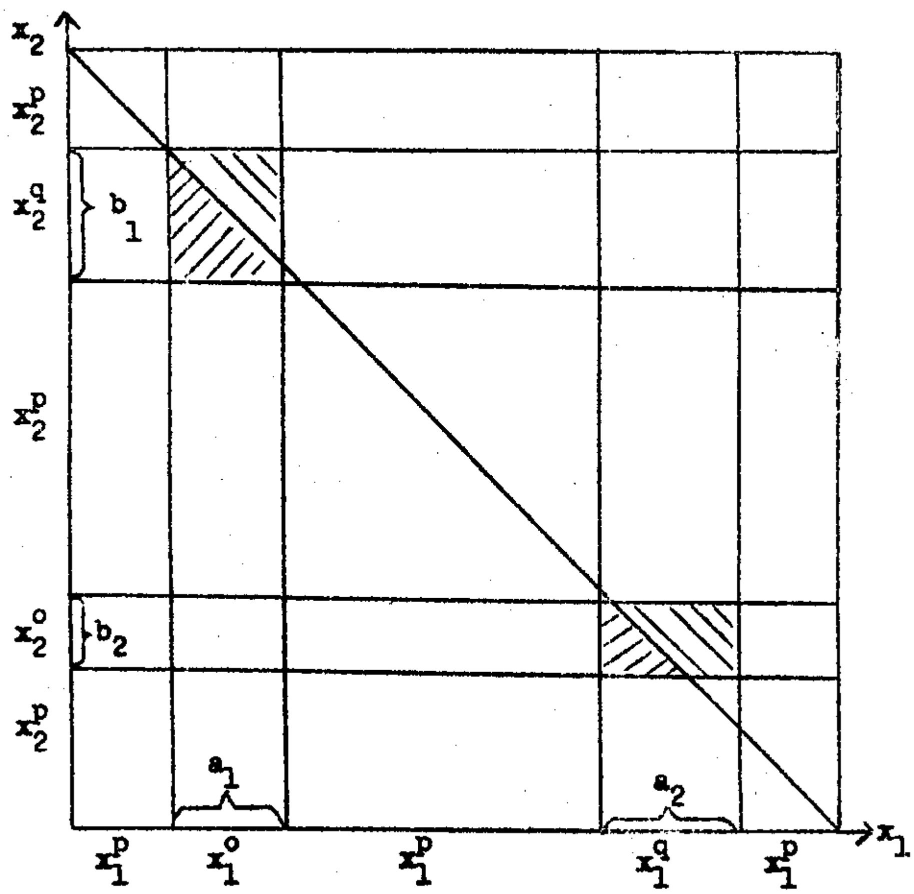

Chart 2

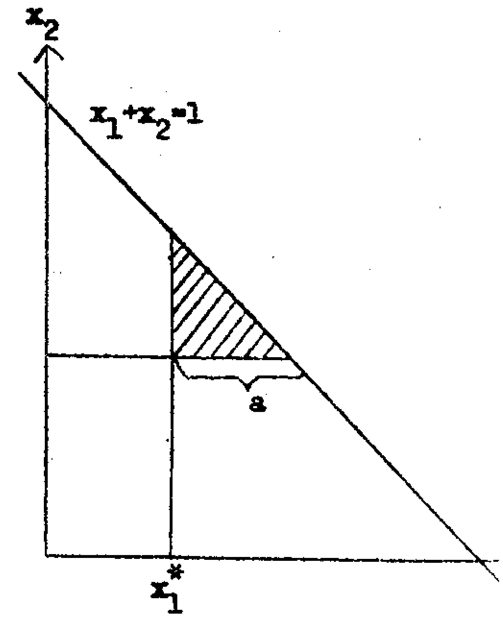

Chart 3

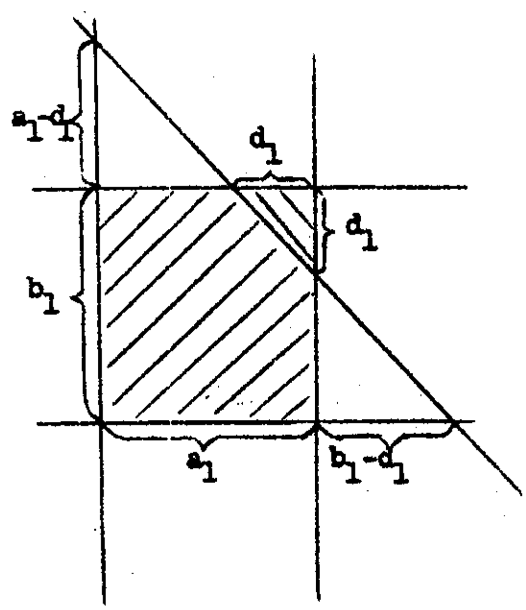

Chart 4

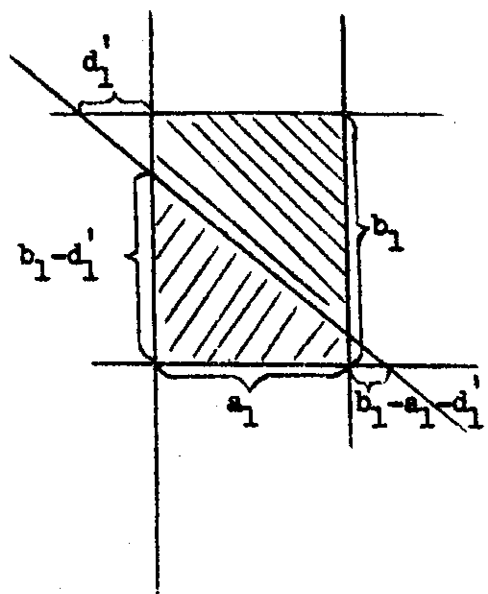

Chart 5

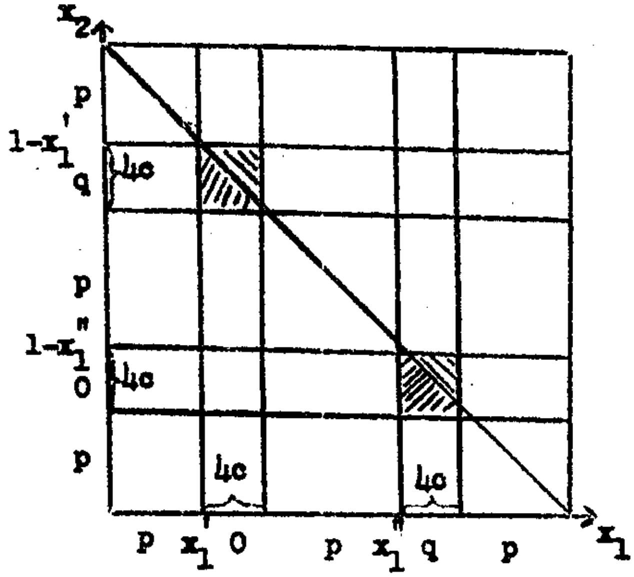

Chart 6