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A Continuous Model of Transportation[1]

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Summary: This paper is a first attempt toward the construction of general models of optimum transportation and location. In this end it is inspired and closely related to the model advanced by T. C. Koopmans [1, Chapter XIV]. A major difference lies, however, in its mathematical set up. This treatise is couched in terms of (sectionally) smooth [3, I, p. 438] spatial flow distributions, so that methods of the calculus of variations become applicable.[2] Under those assumptions the present paper studies only the optimum pattern of interlocal flows within the framework of a given production program. It is shown that the optimum conditions on a spatial flow system are realized under ideal competition, a systematic

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1. The author is greatly indebted to Professor N. Georgescu-Roegen and Professor G. Tintner for the careful reading of a first draft of this paper and most valuable comments and criticism. He has benefitted also from discussions with Cowles Commission staff members; in particular with Professors Marschak and Koopmans. Further criticisms are invited.

2. The notion of continuous geographical distributions of demand or supply is well known in location theory. Usually, continuity is introduced with the assumption of market and supply areas or of a continuous family of isotimes (isoprice-curves). It is not mentioned in most cases, that then a continuous transportation or flow system is implied. This does not in all respects imply a sacrifice of economic actuality. Whenever the traffic pattern is local, or dense for other reasons, or whenever the networks of shipping are flexible enough from the view point of long run analysis, the continuous approach is meaningful. In any case, it may be defended as a bird's eye view of the space economy, based on the same principles that underlie the notions of market and supply areas in traditional location theory. [For these notions cf. 2, pp. 71-73 and passim.]

treatment of which is, however, deferred to a separate paper.

1. Suppose[3] a centrally directed economy to be confronted with the following problem. The production of a small number of goods for general consumption is scattered all over the region so that it may be considered to have a continuous distribution with, of course, varying density. In the short run, the output of each of these commodities shall be fixed, and the amount to be received by each consumer or storage house, both being distributed continuously, shall also be fixed. If the imput for transportation can be appraised in terms of a cost function, and the cost varies over the region in a known way, depending on the position and the quantities of shipments there, what is the optimum arrangement of shipments, that is, the system of interlocal flows which has the least sum of transportation costs?

A problem of this kind is in the nature of the calculus of variations and leads into the intricate field of boundary value problems. Only in the simplest cases can an explicit solution be expected. Nevertheless, a treatment may be of elemental usefulness as a study of an economic optimum under extreme conditions. This paper derives the system of differential equations that describes the optimum situation (Section 2), and shows the uniqueness of their solution (Section 4). A theorem is proved which leads to a comparison with conditions on a market (Section 3). In conclusion a hypothetical example is outlined (Section 5).

2. Basic for the treatment in this paper is the assumption that the central authority can with each transportation performance associate an expense, called transportation cost. Its task becomes simply to minimize

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3. The imbedding of the transportation problem into this institutional frame is unessential and made only for easier presentation.

transportation costs subject to the given program of production and consumption over space. For this problem to be determined, data are needed on the spatial distribution of production and consumption of the commodity considered, on exports and imports of this commodity, and on the transportation costs. The region R itself is assumed to be bounded and to possess a boundary Γ with a continuously turning tangent. To introduce the locational data, let q(x, y, u) be the surface density function that equals the excess of production over consumption at point (x, y) of the region. u is a parameter to be specified later. For an excess of consumption, consequently, q is negative. Of primary importance is the form in which we assume transportation costs to occur. Here the transportation costs shall be attributed to each (infinitesimal) way element such that K(x, y, a)ds are the cost of transportation for a commodity flow of density a to be carried over a distance ds at a width dt. For the total of transportation costs we obtain

$$\iint_R K(x, y, a)ds\,dt = \iint_R K(x, y, a) \frac{\partial(s, t)}{\partial(x, y)} dx\,dy =$$
$$\iint_R K(x, y, a)dx\,dy$$

since the mapping $(s, t) \longrightarrow (x, y)$ is area preserving, that is $\frac{\partial(s, t)}{\partial(x, y)} \equiv 1$. The density a is related to the system of shipments as follows. Let the shipments, or flows in the terminology of interregional analysis, be associated with a field of vectors $\varphi = \varphi(x, y)$ giving the direction of flows at every point and having length equal to the density of flow (density = amount /unit area). Then $|\varphi|$ is the density a, introduced above, and $K = K(x, y, |\varphi|)$.

It should be noted that in this set up all costs of loading or unloading and the general overhead of the transportation systems are neglected. It is not difficult to see, however, that given the program q, these cost components become constants and do not bear on the minimum problem.

In terms of the field φ of shipments, henceforth also referred to as the flow field φ, the present problem becomes that of finding a flow field which minimizes a cost integral $\iint_R K(|\varphi|)dx\,dy$ and is subject to the conditions of the program q(x, y). These latter conditions will presently be formulated. The excess supply functions q(x, y) act, in fact, as sources and sinks in the flow field φ. This relationship is well treated in general vector analysis [3, II; p. 371], giving rise to equations div φ = q in the interior of R. Similarly, on the boundary the normal component of outflow is found to equal the export density, taken per line element of the contour. Thus $\varphi_n = g$, where the subscript n denotes the normal component of φ directed to the exterior of R on Γ, and g denotes the export (or import, if negative) density on the contour Γ. We thus obtain an analytical formulation of the original problem as follows: To find a field of (sectionally smooth) [3, I; p. 439, 5, I; p. 171] vectors φ such that $\iint_R K(|\varphi|)dx\,dy$ is minimal and satisfies conditions

(2.1) $\quad$ $\operatorname{div} \varphi - q = 0$ in R

(2.2) $\quad$ $\varphi_n - g = 0$ on Γ.

From equation (2.1) it is seen that, if the flow field φ is known, so is the geographical distribution of excess production. The converse is not generally true. This indicates that flow analysis properly includes the spatial distribution of production, the objective of location theory, and is thus slightly more general.

Everthing so far holds true if instead of one commodity several commodities are considered. Let a subscript i refer to the particular commodity. Then the problem becomes to find vector fields φ_i(i = 1, ..., n) such that $\iint_R \Sigma_i K_i\,(x, y\,|\varphi_i|)dx\,dy$ attains a minimum subject to the conditions

(2.3) $$\operatorname{div} \varphi_i - q_i = 0 \text{ in } R$$

(2.4) $$(\varphi_i)_n - q_i = 0 \text{ on } \Gamma .$$

In the following it is convenient to suppress the indices and to lead the argumentation in terms of only one commodity. Subsequent generalizations involve no difficulties.

In order to solve this problem in the calculus of variations we introduce the side condition (2.1) under the integral, making use of the Lagrange Multiplier Rule.[4/] With h as a Lagrangean multiplier we have

$$\int\int_R [K(|\varphi|) + h(\operatorname{div} \varphi - q)]dx\, dy .$$

The Euler equations [4, p. 12, equa. 61.] are readily found to be

(2.5) $$\frac{dK}{d|\varphi|} \cdot \frac{\varphi}{|\varphi|} - \operatorname{grad} h = 0$$

for which we may write

$$k \frac{\varphi}{|\varphi|} - \operatorname{grad} h = 0$$

putting

$$k = \frac{dK}{d|\varphi|} .$$

Equations(2.1), (2.5) are literally preserved in a 3-dimensional economy, if we should prefer it as the more general frame. Since k has the dimension price/length and the expression $\frac{\varphi}{|\varphi|}$ is a dimensionless unit vector, h has the dimension of a price. The nature of h is further investigated below.[5/]

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4. This has been proved admissible by Max Coral, The Euler Lagrange Multiplier Rule for Double Integrals, Contributions to the Calculus of Variations, Vol. II, p. 63. The author is indebted to Professor N. Georgescu-Roegen for this reference.

5. A similar problem as the present one, in physics, that came to the author's attention after the present question had been treated, is the derivation of the hydrodynamic pressure from the equations of movement for an incompressible frictionless fluid. The Lagrangean multiplier turns out to be the pressure. Cf. A. Sommerfeld, Vorlesungen ueber theoretische Physik, 1946, Bd. II, pp. 87-88, in particular equation 9.

Note that in equation (2.5) attention is shifted to a vector field $\chi = k \frac{\varphi}{|\varphi|}$ which arises from the original φ field by multiplication with a nonnegative scalar function $\frac{k}{|\varphi|}$. The vectors give the marginal cost of transportation along the routes of φ. They constitute a gradient field so that the identity

(2.6) $\quad$ curl $\chi = 0$

holds [3, II, p. 93]. From this it may be seen that (2.5) puts indeed a restriction on φ and does not merely introduce a new unknown h.

Equation (2.6) in particular excludes closed field lines of χ. Since the factor of φ in (2.5) is nonnegative, it holds true for φ. The curl itself has no economic meaning. It was, of course, to be expected that circular shipments, which is the meaning of closed circuits of φ, are inefficient.

3. More interesting is the fact that the field lines of commodity flows are perpendicular to the curves h = const., a well known property of gradient fields [3, II; p. 90]. The curves h = const. will be called potential curves. They are related to the locational potential introduced by Koopmans [1, p. 236], as is seen from the following theorem:
For any two points P_1, P_2, of R, the (marginal) costs of transportation from P_1, to P_2, are not smaller than is the difference of the potentials $h(P_2) - h(P_1)$. (Marginal) costs of transportation and potential differences are equal if and only if the points P_1 and P_2 are on one field line of φ and the route of transportation is the field line. [Compare with 1, p. 244, equa. 2.12, 2.13].

The proof of this is suggested by well known facts in the calculus of variations and will be sketched only in brief. Choose an arbitrary potential curve Π and a point S not on it. Consider the extremal E which minimizes a (one-dimensional) transportation cost integral

$$dt \cdot \int_E k\, ds = dt \cdot \int_E k \sqrt{1 + (y')^2}\, dx$$

between the curve Π and the point S. dt denotes the width of the flow thread and stands here only for dimensional reasons. Since it does not affect the minimum problem it will be suppressed subsequently. For clarity it should be mentioned that the present problem in the calculus of variation is entirely different from the original one of Section 2.

E intersects the curve Π transversally (by the transversality theorem [4, p. 25]), that is in this case orthogonally. Let S now vary over this particular extremal and denote the variable value of $\int_{E(S)} k\, ds$ by $b = b(x, y)$ where (x, y) are the coordinates of S. By a well known theorem [4, corollary 8, p. 20] the value of b is equal to that of the Hilbert integral

$$\int_E (f - y' f_{y'})dx + f_{y'}\, dy \qquad \text{with} \qquad f = k(x, y)\sqrt{1 + y'^2}\,.$$

Since the Hilbert integral is independent of the path it may be differentiated like an ordinary Riemann integral. This yields at once

$$(3.1) \qquad (\frac{\partial b}{\partial x})^2 + (\frac{\partial b}{\partial y})^2 - k^2 = 0\,.$$

From (2.5) it follows easily that also

$$(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2 - k^2 = 0.$$

We want to show that, indeed, b = h + const. in a neighborhood of the arc Π. This would mean that the potential curves, h = const., have constant minimal transportation cost distances, equal to the differences of the potential h (first part of the theorem). And since field lines and extremals are perpendicular to the curve systems h = const. and b = const. respectively -- the latter by the transversality theorem, the first because of the gradient equation (2.5) -- field lines of and extremals E would coincide (second part of the theorem).

To complete the proof we have to show therefore that the partial differential equations (3.1) in b and h have only one solution. The following facts are straight-forward.

3.1. On π b = 0, h = constant.

3.2. Let t, n denote the tangential and normal direction on π, respectively. Then

$$\frac{\partial b}{\partial t} = 0 \qquad \frac{\partial h}{\partial t} = 0$$

$$\frac{\partial b}{\partial n} = k \qquad \frac{\partial h}{\partial n} = k \qquad \text{on } \pi,$$

therefore

$$\frac{\partial b}{\partial x} = \frac{\partial h}{\partial x} \qquad \frac{\partial b}{\partial n} = \frac{\partial h}{\partial n} \qquad \text{on } \pi.$$

3.3. Let the differential equation (3.1) be denoted by $F = F(u, u_x, u_y, x, y)$. Then $F_{u_x} \cdot y_t - F_{u_y} \cdot x_t = 2(u_x y_t - u_y x_t) = 2\begin{vmatrix} u_x & u_y \\ x_t & y_t \end{vmatrix} \neq 0$. For the direction of grad u for u = b, h does not coincide with the tangential vector on π (on the contrary it is perpendicular to it for both h and b). 3.1, 3.2, 3.3 together with the differential equation (3.1) are sufficient for the identity h = b + const. in a neighborhood of π.

This is a well known theorem on the uniqueness of solution of first order partial differential equations [5, II; p. 68, 69]. It might be surprising that this proof does not employ equation (2.1) explicitly. This equation plays, however, its part in determining the arcs π, to start with.

Of particular interest is the case k = const., the normal assumption in general location theory. It is useful to study the proof for this case. The extremals of the variation problems $\int k \, ds = k \int ds$ are known to be straight lines.

$$ds = m\,dx + n\,dy \qquad , \qquad m, n = \text{constant}$$
$$m^2 + n^2 = 1 .$$

Therefore

$$\frac{\partial b}{\partial x} = k\,m \qquad \frac{\partial b}{\partial y} = k\,n$$

$\left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 = k^2$ without recourse to the Hilbert integral. The further proof remains the same. However, also a direct proof can be given for the fact that the flow field consists of straight lines. A sufficient condition for straightness of field lines is that the directional derivative of the unit field vector taken in the direction of the field vanishes everywhere. In geometrical interpretation: that a vector transported along a field line does not change its direction. Note that the derivative $D_{(\psi)}$ of a function g in the direction of a unit vector ψ is equal to ψ grad g. If g is a vector $\begin{pmatrix} g^1 \\ g^2 \end{pmatrix}$ then

$$D_{(\psi)}\, g = \begin{pmatrix} \psi \text{ grad } g^1 \\ \psi \text{ grad } g^2 \end{pmatrix} . \quad \text{In particular}$$

$$D_{(\psi)}\psi = \begin{pmatrix} \text{grad } \psi^1 \\ \text{grad } \psi^2 \end{pmatrix} \cdot \psi \quad .$$

Suppose now ψ = grad h. This is equation (2.5) for constant k, if we put without loss of generality k = 1, and if $\frac{\varphi}{|\varphi|} = \psi$, which is a unit vector as required.

The proof consists now in showing that

$$D_{(\psi)}\psi = D_{(\text{grad } h)} \text{ grad } h = 0 .$$

$$D_{(\text{grad } h)}\text{grad } h = \begin{pmatrix} \text{grad } h_x \\ \text{grad } h_y \end{pmatrix} \text{grad } h = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} \begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

$$= \frac{1}{2} \text{grad } (h_x^2 + h_y^2) = \frac{1}{2} \text{grad}[(\text{grad } h)^2] .$$

But $(\text{grad } h)^2 = k^2 = \text{const.}$ by (3.1).

$$D_{(\text{grad } h)} \text{ grad } h = \frac{1}{2} \text{grad } [(\text{grad } h)^2] = 0 .$$

Thus every field line of grad h is straight, but the field lines of grad h coincide with those of φ by (2.5). This completes the proof.

Even in this case of straight flow lines the total pattern may be complicated. For there occur, in general, curves of discontinuity where the directions of flow change.[6] It is exactly in the determination of these subregions (or of the lack of any) that the first equation (2.1) enters. This is made plausible by the fact that a subregion S, bounded everywhere by other subregions, satisfies an equation $\iint_S \text{div } \varphi \, dx \, dy = 0$. This follows from the Gauss theorem [3, II; 360] and the observation that $\varphi = 0$ on the contour of S. For if $\varphi \neq 0$ at a point of discontinuity, shorter extremals could be constructed between points on two sides of the discontinuity,

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6. An exact proof for this comes only with an existence proof for sectionally smooth solutions φ. Counterexamples to the existence of continously differentiable solutions in all cases are easily given.

Consider the square region bounded by arcs

II	$0 \leq x \leq 1$,	$y = 0$
IV	"	$y = 1$
I	$0 \leq y \leq 1$	$x = 0$
III	"	$x = 1$

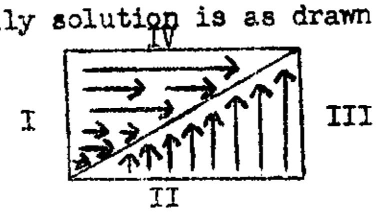
Let $q = c = \text{const.} < 0$.

$\varphi_n = 0$ on arcs III, IV

$\varphi_n = -cx$ on arc II

$\varphi_n = -cy$ on arc I.

Then the only solution is as drawn in Figure 1.



with the diagonal (0, 0) to (1,1) an arc of discontinuity.

but this would contradict the theorem. Economically we have the result that, in general, self-sufficient subregions, defined for a specific commodity, emerge in an optimal situation. These are nothing but the well known market and supply areas of location theory. In the present model the centers of such economic regions turn up with the problem solution while in the traditional theories they are given data. Thus the range of investigation is broader than in these other models.

The fact that our field lines are routes of smallest transportation costs between point sets of given potential and that the potential measures the transportation costs, points to a close analogy of potential curves and the isoprice-curves in a market economy. For under conditions of free interlocal trade the price difference for a commodity between two places equals exactly the least transportation costs, if there is positive trade in this commodity between the two points, and it does not exceed them otherwise. Thus the geographical price distribution $p = p(x, y)$ has to satisfy condition (2.5) on λ. However, in the market case the excess productions q are also dependent on p. Now nothing can prevent us from identifying the parameter u [p. 3] in q with p. $q = q(x, y, p)$. It should be emphasized that this does not take recourse to the original minimum problem. We rather specify the equations (2.1), (2.5) as they stand. The economic reinterpretation is as follows. Suppose that a competitive market economy has the same excess production and the same transportation cost function as the centrally directed economy. Or rather, let a centrally directed economy be confronted with the production program that resulted in the equilibrium of a competitive economy under the same spatial conditions. Suppose furthermore that a potential curve in the one case coincides with an isoprice curve in the other.[7] Then the entire dis-

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7. For this it is sufficient that the same boundary conditions hold in either case.

tribution of prices, or potentials, and flows is equal in both economies. In particular the price distribution must satisfy an equation

(3.2) $$\text{grad } p - k \frac{\Psi}{|\Psi|} = 0$$

where Ψ stands for the interlocal shipments and is subject to

(3.3) $$\text{div } \Psi - q(x, y, p) = 0$$

and where p is the price. The question, whether these (necessary) conditions are in fact sufficient to determine the price distribution under free interlocal trade, is, of course, not answered by the preceding analysis. Note, for instance, that h was determined only up to an addition constant.[8,9]

4. The sufficiency of equations (2.1), (2.5) for the determination of the present problem follows from the fact, that their solution, if it exists,[10]

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8. The author possesses a proof for the sufficiency of equations (3.2),(3.3) but this lies beyond the scope of the present investigation.

9. A notable exception from the suggested equivalence of optimum transportation and the pattern under competition occurs when, instead of $k_i = k_i(x, y, |\varphi_i|)$ in the integral above (2.3) the following transportation cost function is introduced: $k_i = k_i(x, y; |\varphi_1|, |\varphi_2|, \ldots, |\varphi_n|)$ and the transportation cost minimum is considered for the n commodities simultaneously. That is, (2.5) becomes then

(2.5a) $$\left(\sum_j \frac{dk_j}{d|\varphi_i|}\right) \frac{\varphi_i}{|\varphi_i|} - \text{grad } h_i = 0$$

which says that the marginal costs of transportation with respect to the i^{th} commodity take the form of an increase in the cost of transportation for each one commodity. Under conditions of a market economy, only the part $\frac{dk_i}{d|\varphi_i|}$ is imputed to the interpreneur. Therefore the market situation is described by

(2.5b) $$\frac{dk_i}{d|\varphi_i|} \cdot \frac{\varphi_i}{|\varphi_i|} - \text{grad } h_i = 0$$

in contradistinction to (2.5a). A popular example for such a case would be provided by the cost of congestion in urban traffic. Each additional (or marginal) car causes a congestion effect to all other vehicles, the cost of which it does not bear.

It is not astounding that the principal differences of a centrally directed and a market economy should be tied up with the notion of transportation costs, social and individual. Here seems to lie a fruitful vantage point for further studies in properties of optimal location.

10. The existence of a solution, always an intricate matter in boundary value problems, cannot be settled here. Some clues are contained in [5, II, Chapter 7].

is unique, as we shall show in the present solution.

Let φ, ψ be two different solutions satisfying equations

(2.5) $$k \frac{\varphi}{|\varphi|} - \text{grad } h = 0 \qquad k \frac{\psi}{|\psi|} - \text{grad } b = 0$$

and being sectionally continuously differentiable. [Cf. for this notion 3, III; p. 438]. Moreover, it shall be prescribed that on curves of jump discontinuities Γ^*, [3, II; p. 51] of φ (or ψ) the normal flows φ_n (ψ_n) remain continuous. This, according to equation (2.2) means nothing else but that exports from one subregion shall equal imports to the neighbor subregion, or in other words, that no flows shall vanish or originate on curves of changing flow direction.[11] These conditions are normal and imply no restrictions of economic consequence.[12]

Consider $A = \iint_R \varphi \text{ grad } b \, dx \, dy$. Then the Gauss theorem is applicable [3, II; pp. 360-364].

(4.1) $$A = \int_{\Gamma, \Gamma^*} b \, \varphi_n \, ds - \iint_R b \text{ div } \varphi \, dx \, dy .$$

Now the boundary conditions provide that $\varphi_n = \psi_n$ on Γ. In the interior the contour integrals over Γ^* (along the curves of jump discontinuities of φ) vanish because of the above arrangements. Finally equation (2.1) stipulates that div φ = q = div ψ.

$$A = \int_{\Gamma, \Gamma^*} b \, \psi \, ds - \iint_R b \text{ div } \psi \, dx \, dy = \iint_R \psi \text{ grad } b \, dx \, dy = \iint_R k \, |\psi| \, dx \, dy$$

the latter equality following by (2.5). Thus

$$\iint_R \varphi \text{ grad } b \, dx \, dy = \iint_R k \, |\psi| \, dx \, dy = \iint_R k \, |\varphi| \, dx \, dy$$

since the minimum is by hypothesis the same for φ and ψ. Now by Schwarz's inequality [3, I; p. 12].

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11. Except of course for the infinitesimal amounts of flow from local excess production on the curves.

12. Actually more was shown already on p. 10, namely that $\varphi = 0$ at all discontinuities of the field φ. But it may be desirable to have this uniqueness proof independent of the theorem of page 6.

$$(4.2) \qquad \iint \varphi \text{ grad } b \, dx\, dy \leq \iint |\varphi| |\text{grad } b| \, dx\, dy \leq \iint |\varphi| \, k \, dx\, dy$$

and the equality sign in the first inequality holds if and only if $\varphi \parallel$ grad b. It follows that $\varphi = \frac{|\varphi|}{k}$ grad b. Now with (2.5) $0 = \frac{|\varphi|}{k}$ (grad b - grad h). Thus grad (b-h) = 0 wherever $|\varphi| \neq 0$. By the same argument $\frac{\Psi}{k}$ grad (b-h) = 0.

$$(4.3) \qquad \text{grad } (b-h) = 0 \text{ if } |\Psi| \neq 0 \text{ or } |\varphi| \neq 0 .$$

This proves that the direction of flows is uniquely determined. We proceed to show, that also $|\varphi| = |\Psi|$ everywhere. Since $\varphi \parallel \Psi$ wherever $\varphi \neq 0$ or $\Psi \neq 0$ their difference has a direction parallel to grad h. Or $\varphi - \Psi = a$ grad h, grad h, with scalar a. This remains true trivially if both $\varphi = \Psi = 0$; hence it holds everywhere. Consider the sets where a changes its sign. Because of the sectional smoothness of φ, Ψ, these sets form contours consisting of curves where either a = 0 or a jumps. Since $(\varphi - \Psi)_n$ is continuous on each curve (cf. above) it follows from $(\varphi - \Psi)_n = a(\text{grad } h)_n$ that $(\text{grad } h)_n$ vanishes or is itself discontinuous on these curves. Suppose $\varphi_n \neq 0$, then because of (2.5) $(\text{grad } h)_n \neq 0$ and is therefore discontinuous. From $k \frac{\varphi_n}{|\varphi|} = (\text{grad } h)_n$ it follows that φ_n would be discontinuous in contradiction to previous assumptions. Thus $\varphi_n = 0$ on all curves where a jumps. Suppose there are points on these curves where $\Psi_n \neq 0$. Then grad b = grad h by (4.3) and therefore $(\varphi - \Psi) = a$ grad b. By applying the same arguments as before one obtains that also $\Psi_n = 0$ wherever a jumps.

The contours Γ'' where a changes its sign divide the interior of R into subregions, R_j. Without restriction assume $a \geq 0$ on one such R_j. Then, since $\varphi_n = \Psi_n = 0$ on Γ_j''

$$0 = \int_{\Gamma_j''} h\, (\varphi - \Psi)_n \, ds = \iint_{R_j} \text{div } [h\, (\varphi - \Psi)] \, dx\, dy = \iint_{R_j} h \text{ div } (\varphi - \Psi) dx\, dy$$

$$+ \iint_{R_j} \text{grad } h \, (\varphi - \Psi) \, dx = \iint_{R_j} \text{grad } h \cdot a \text{ grad } h \geq 0.$$

The equality can only hold for $a \equiv 0$. By the same argument for regions with $a \leq 0$ it follows eventually that $a \equiv 0$ everywhere on R. Hence $|\varphi| = |\Psi|$. In particular $\varphi = 0$ if and only if $\Psi = 0$. This completes the proof that $\varphi \equiv \Psi$. Eventually it follows that $h \equiv b + \text{const.}$

5. The following hypothetical example may serve as an illustration. We assume a region, bounded by the following arcs

(1) $\tilde{x} = -c \qquad -b \leq \tilde{y} \leq b$

(2) $\tilde{y} = \begin{cases} b \\ -b \end{cases} \qquad -b \leq \tilde{x} \leq c - \frac{b^2}{4c}$

(3) $\tilde{y} = \sqrt{4c(c - \tilde{x})}$.

It shall have an export surplus in commodity X. Let the local excess production be given an equal to $\frac{1}{\sqrt{x^2 + y^2}}$. This expression becomes infinite at the origin, but it is easily seen that the excess production of any circle surrounding the singularity remains finite. For, if C is such a circle

$$\iint_C q \, dx \, dy = \int_0^{r_o} \int_0^{2\pi} \frac{1}{r} \, r \, dy \, dr = r_o \, 2\pi ,$$

by use of polar coordinates.

On the boundary we prescribe the following values of the outflow:

$$\varphi_n = p^2 + 2pc \qquad \text{on (1)}$$

$$\varphi_n = \frac{4p^2}{b(p + 1)} \qquad \text{or (2), (4).} \quad \varphi_n = 0 \qquad \text{on (3).}$$

Here p is a parameter which has the following values

$$p = \tilde{x} + \sqrt{b^2 + \tilde{x}^2} \qquad \text{on (2), (4)}$$

$$p = c + \sqrt{c^2 + \tilde{y}^2} \qquad \text{on (1).}$$

Furthermore, let $k = \frac{|\varphi|^2}{2}$. Then the equation system (2.1), (3.2) becomes φ - grad h = 0.

(5.1) $\quad \text{div}\,\varphi = \text{div grad } h = \Delta h = q = \frac{1}{\sqrt{x^2 + y^2}}$.

It can be verified that the differential equation and boundary conditions are satisfied by the family of confocal parabolas $y^2 = 2h\,(x - \frac{h}{2})$ giving rise to potential functions

(5.2) $\quad h = x + \sqrt{x^2 + y^2}$.

The proof that h from (5.2) satisfies the equation (5.1) is straight-forward. To check the boundary conditions note first that the streamlines of the φ - vector field are given by the family of orthogonal trajectories

(5.3) $\quad y^2 = 2p\,(\frac{p}{2} - x)$ where p is a current parameter.

Now the volume of flow streaming out of a strip bounded by two streamlines belonging to parameters p and p + dp respectively, is given by $\frac{1}{2}\,\frac{dI}{dp}\,dp$ where I is the area of the parabola of parameter p, measured from the vertex to the boundary of R at its left. The normal component of φ on a line element ds of the boundary is the amount of flow streaming out of a strip which occupies the length ds on this boundary. Thus $\varphi_n = \frac{1}{2}\,\frac{dI}{dp} \cdot \frac{dp}{ds}$.

For the various sides of R we obtain

$\varphi_n = p^2 + 2pc$ (1)

$\varphi_n = \frac{4p^2}{b\,(p+1)}$ (2), (4)

$\varphi_n = 0$ (3) as required.

From the uniqueness proof of Section 4 it now follows that $h = x + \sqrt{x^2 + y^2}$ is also the only solution h. The geometric picture is roughly as follows

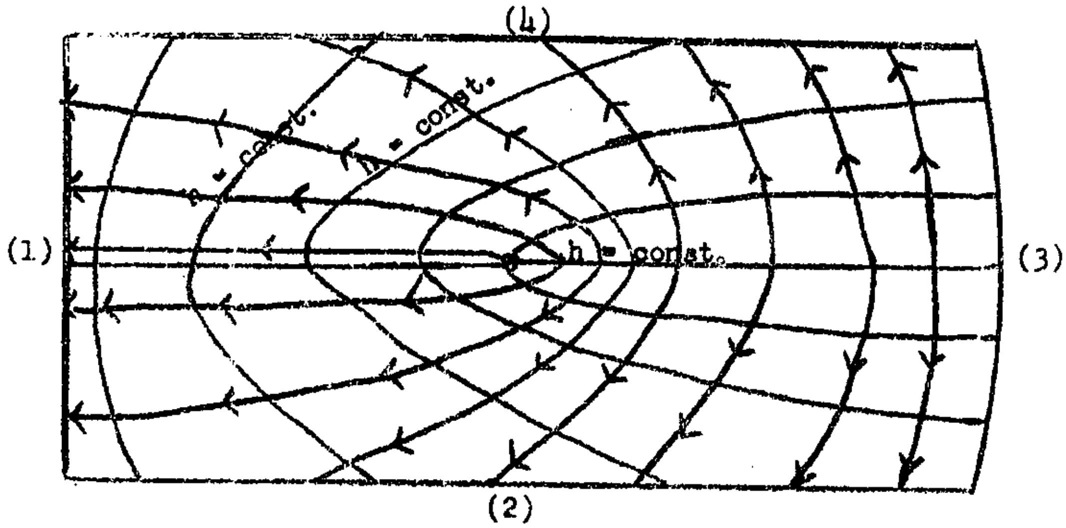

Figure 2.

It shows two families of normally intersecting parabolas with their common focus at the origin. The potential curve of maximal h is the straight line from the origin to the right. The boundary (3) coincides with a potential curve $\tilde{y} = \sqrt{4c(c - \tilde{x})}$. Clearly the geodesics of minimal transportation costs are curvilinear here because the marginal transportation costs are dependent on the density of flows.

Conclusion:

In a continuously distributed space economy with a given production program the optimum system of interlocal commodity flows is characterized by two simple sets of differential equations. There exists a family of potential curves with properties similar to those of isoprice curves (isotimes) in a market economy. The optimum solution is unique. It turns out to minimize transportation costs in the large (i.e., for the whole economy) as well as in the small (i.e., between the points connected). A difference between this model of optimum location and a model designed for a competitive market economy lies only in the fact that in the present one, production (and consumption) programs are given data. It suggests itself to identify the local potential with the local price of the market economy and to introduce a production program that depends upon it. This model which would only slightly differ from the present one (in that q then depends on h) gives a representation of the spatial equilibrium in a continuously distributed economy under conditions of perfect market competition. It occurs that models of this kind might find a place in a mathematical framework of general location theory. The exploration of these questions constitutes a problem in its own right, and is not to be treated here.

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