Social Welfare Functions Based on Rankings

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A social welfare function is a rule which gives "community preferences" as a function of "individual preferences". Our election system, for example, gives a "community choice" of candidates as a function of voters' choices. Kenneth Arrow [1] has shown that if (1) individual preferences are expressed as rankings of various alternatives, and if (2) we require certain apparently innocuous properties (described later) of any "acceptable" social welfare function, then "acceptable" welfare functions do not exist. No welfare function has the properties which, it would seem at first sight, we would require of, e.g., a reasonable voting system.

Clifford Hildreth [2] has shown one way of avoiding the Arrow paradox. In the Hildreth system the individual preferences are to be expressed, not as rankings, but as numerical utilities. Hildreth suggests that in particular these numerical utilities might be the von Neumann utilities used in the explanation of action in the face of risk. The present writers feel that the Hildreth results do not completely remove the sting from the Arrow paradox. In many important practical situations we only know individuals' rankings. In an election, for example, we usually know only which candidate is most preferred by each voter. We might reasonably ask each voter to rank all candidates.

1. The authors are indebted to Clifford Hildreth and James Templeton for valuable discussions.

But it seems unreasonable to ask what are the ratios or differences among the voters' preferences; or to require each voter to state what probability combination of, e.g., his first and third preferences would be just as desirable to him as his second favorite candidate.

In this paper we consider social welfare functions which, as in Arrow's system, give "social rankings" as a function of individual rankings. In the first part of this paper we argue that the Arrow postulates are not as plausible as they, at first, appear. The Arrow postulates can be modified somewhat to meet our objections, but then many social welfare functions satisfy the modified postulates.

In the second part of this paper we consider which of the many functions, not rejected by the modified postulates, seems most reasonable.

The Arrow conditions, to be satisfied by any "acceptable" Social Welfare Function may be paraphrazed as follows (cf. [1]):

Condition 1. The social welfare function is defined for a sufficiently wide range of individual orderings (rankings)

Condition 2. If alternative (a) rises or remains still in the ordering of every individual and no other change takes place in those orderings then alternative (a) rises, or at least does not fall, in the social ordering.

A social welfare function gives a social choice (or set of choices) for every set of available alternatives. We may consider the "choice function" associated with a given social welfare function, and we may consider how this choice-function changes as voter's preferences change or as changes take place in the considered (tho not necessarily available) candidates. Arrow requires:

Condition 3. (Independence of irrelevant alternatives) If each voter ranks each <u>available</u> candidate exactly the same in one situation as he does in another, then--no matter what be true about the rankings of the other candidates or the number of non-available candidates which have been considered--the choice among the available candidates is the same in both situations.

Condition 4. The Social welfare function must not be "imposed"; i.e., it must not be given independently of individual preferences.

Condition 5. The social welfare function must not be dictatorial; i. e., it must not be identical with the preferences of one individual-- irrespective of all other individuals' preferences.

Arrow has shown that no social welfare function satisfies the above, apparently reasonable, conditions.

We will consider an example of a welfare function which, like all welfare functions, contradicts the Arrow conditions. This example will be useful later when we consider the plausibility of the Arrow conditions.

Consider the welfare function which prescribes: for each "candidate" sum the ranks given it by the various "voters"; one candidate is preferred to another if his sum of ranks is less than that of the other. Thus if we have three candidates a, b and c, and they are ranked by two voters A and B as in table 1, then b is preferred to a and a is preferred to c.

voter \ candidate	a	b	c
A	1	2	3
B	3	1	2
$\sum$	4	3	5

Table 1

	a	b
A	1	2
B	2	1
$\sum$	3	3

Table 2

But if c had not been there, the rankings would have been as in table 2; a and b would be socially indifferent. This contradicts condition 3, the independence of irrelevant alternatives.

We will first present our objections to the Arrow conditions informally and intuitively. We will then state our position more formally.

Suppose you intended to serve refreshments to two friends. You could serve them either coffee or tea but not both. Mr. A preferred coffee, Mr. B preferred tea; it seems clear that a symmetric ("democratic") welfare function would rank coffee and tea equally. Suppose you had other information concerning the preferences of A and B. While A prefers coffee to tea, he prefers tea to cocoa and cocoa to milk. B on the other hand not only prefers tea to coffee, he prefers cocoa to coffee, milk to coffee, tomato juice to coffee; he would rather drink water than coffee; while tea was preferred to cocoa, milk, tomato juice and water. Given this added information it seems plausible to serve tea rather than coffee; for it doesn't make "much difference" to A and makes quite a bit of difference to B.

In terms of the example of tables one and two, if we had the information in table one then this information should have been used even when only a and b are available. If we had only initially the information in table two (i.e., the preferences of A and B for a and b) and then had been given the information of table one (i.e., the preferences of A and B for a, b and c) we might wish to use the fact that it seems to make "more difference" to B than to A. Thus the "irrelevant alternative" is not necessarily irrelevant.[1]

Still looking at the problem intuitively, it may be objected that although B prefers c more than a but less than b, while A prefers c more than b but less than a, still, B may feel less "difference" between a and b than does A. This argument might be put forth no matter how many objects were

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1. Also, L. J. Savage [3] points out in a different context "when the new act is admitted the group may well change its choice to arrive at a compromise with some members who prefer the new possibility, without actually adopting the new possibility itself".

found preferred by B to b but not to a. We can avoid some of the consequences of this argument if we assume that each individual has only a finite number of indifference levels or "levels of discretion". That is, for some N, once we find N states or candidates none of which are indifferent in the individual's preferences, then every other state or candidate is indifferent to one of these. This assumption is not unreasonable; we can not expect individuals to have more than 10^3 or 10^6 or 10^9 levels of discretion.[1]

Let us state our position more formally: Suppose that there are M voters, $1,\ldots,i,\ldots,M$. Each has a finite number of "levels of discretion", $1, 2, \ldots, L_i$. Level one is best; L_i, worst. The number L_i of levels may differ from person to person.

Suppose we are considering two candidates (1 and 2) and suppose further - for a moment - that we know the level at which each voter i ranks 1 and 2; i.e., we know l_{i1}, l_{i2}. Given the matrix l_{ij}, a social welfare function will rank the candidates 1 and 2. If $l_{11} < l_{12}$ and if candidate 2 fell in the opinion of voter 1 (i.e., l_{12} increases) everything else remaining the same, we would require that candidate 2 should not rise in the social ordering. This does not contradict the Arrow requirements. We would also admit the following

Resolution 1, a social welfare function shall not be rejected as unreasonable on the sole grounds that candidate j^o falls in the social ordering, when, for some $i = i^o$, $l_{i^o j^o}$ increases - the other l_{ij} remaining the same.

Typically we will not know the exact levels l_{ij}, or even the number of levels L_i. All we will know is the rankings a_{ij} of n candidates by m voters.

1. We realise that continuity assumptions are often made, but these, we feel, are made for reasons of mathematical convenience rather than out of the conviction that the individual has a non-countable -- or even denumerable -- number of "discretion levels".

This information may be expressed by a matrix $A = (a_{ij})$. (We assume that each voter has ranked all of the n candidates). Our previous discussion justifies

Resolution 2, a social welfare function shall not be rejected on the sole grounds that it changes the ordering of j_1 and j_2 as the state of information changes.

The condition that the welfare function should be independent of irrelevant alternatives may be preserved somewhat; it seems reasonable to require

Resolution 3, for a given state of information the welfare function should order the "candidates" independently of their availability.

Welfare Functions

If we modify the Arrow conditions to satisfy our three "resolutions", then we have a set of conditions satisfied by many welfare functions. We will now consider which functions are "most plausible". We will follow Arrow in the use of the axiomatic approach; that is, we will lay down conditions to be required of any welfare function, and then will seek those welfare functions which satisfy these conditions. Our conditions will be somewhat different from those of Arrow.

Our first three conditions, we feel, are self-explanatory. They are:

Condition 1 For any given state of information (given by a matrix $A = (a_{ij})$ of orderings) the social welfare function gives a simple ordering of alternatives, independently of their availability. I.e., the social welfare function orders the vectors $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Condition 2 (Pareto optimality) If nobody prefers j_2 to j_1 and somebody prefers j_1 to j_2, then j_1 is socially preferred to j_2. I.e., if $a_{ij_1} \le a_{ij_2}$ for all i and $a_{ij_1} < a_{ij_2}$ for some i, then $\{a_{ij_1}\} < \{a_{ij_2}\}$

Condition 3 (Universal Applicability) Our social welfare function is defined for all states of information. I.e., the ordering is defined for all matrices (a_{ij}) of positive integers.

The next condition needs some explanation and justification. It says, roughly, that if a constant c is added to every entry in the i^{th} row of the matrix A this does not change the social ranking of the candidates (columns). Intuitively, if voter i has, say, 80 levels of discretion it makes no difference whether we give him an alternative from his 2nd discretion level instead of his 1st, or his 80th discretion level instead of his 79th. It makes the same "difference" to him.[1] In other words

Condition 4 Suppose voter i has exhibited k_i levels of discretion. The social ordering among candidates 1 and 2 remains unchanged if we replace a_{i1} and a_{i2} by $a_{i1} + c$ and $a_{i2} + c$ respectively. c must be an integer such that $1 \le a_{ij} + c \le k_i$ for all j.

In some cases the following condition is desirable.

Condition 5 (Symmetry) The social ordering is unchanged if the rows of A are interchanged.

If we insist on conditions 1 to 5 then one and only one social welfare function is acceptable. This prescribes that $\{a_{ij_1}\} > \{a_{ij_2}\}$ if and only if $\sum_i a_{ij_1} > \sum_i a_{ij_2}$.

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1. The assumption that we can multiply by a constant would give us an analogue of the Hildreth social welfare function. This would say that it makes no difference whether we give him his 2nd discretion level rather than his 1st, or his 80th discretion level instead of his 40th. In our context, our assumption seems more plausible. If instead of ranks a_{ij} we took $e^{a_{ij}}$, then we would have the Hildreth analogue. Using logarithms the Hildreth case reduces to the additivity condition. Hence, our results have a dual character.

If we require only conditions 1-3 any monotonic ordering function defines a social welfare function and these are the only acceptable ones. The above results will be demonstrated in the next section of this paper.

The authors also conjecture that if only conditions 1-4 are required, then a simple class of welfare functions is acceptable. A welfare function is a member of this acceptable class if and only if there exists a set of weights w_i such that $\{a_{ij_1}\} > \{a_{ij_2}\}$ if and only if $\sum_i w_i a_{ij_1} > \sum_i w_i a_{ij_2}$.

<u>Proofs:</u>

Let there be m voters. A acceptable social welfare function is then a weak ordering of the points $\{a_i\}$ in a subset of m dimensional Euclidian space such that

1) Pareto optimality : If $a_i' \geq a_i''$ for all values of i, except i_0; then $\{a_i'\} > \{a_i''\}$ (a_i'' preferred to a_i') when $a_{i_0}' > a_{i_0}''$.

2) Symmetry: $\{a_i\} = \{b_i\}$ when there exists a one to one mapping of the indices $i \rightleftarrows j$ such that $a_i = b_j$.

3a) If $\{a_i\} > \{b_i\}$ if and only if $\{a_i\} + \{c_i\} > \{b_i\} + \{c_i\}$.

Theorem 1: A. The ordering relation defined by $\{a_i\} = \{b_i\}$ if $\sum a_i = \sum b_i$, $\{a_i\} > \{b_i\}$ if $\sum a_i > \sum b_i$ is a acceptable social welfare function.

B. It is the only acceptable social welfare function.

<u>Proof:</u> A is trivial since conditions 1, 2, 3a, are easily verified. B is more difficult.

Let us first consider the case (a) where $\sum a_i = \sum b_i$. We must show that any social welfare function must necessarily prescribe $\{a_i\} = \{b_i\}$. Let us add $\left\{-a_i + \sum_{s=1}^{i-1}(b_s - a_s)\right\} = \{c_i\}$ to the two vectors. We then have $a_i + c_i = \sum_s^{i-1}(b_s - a_s) = b_{i-1} - a_{i-1} + \sum^{i-2}(b_s - a_s) = b_{i-1} + c_{i-1}$, and $a_0 + c_0 = b_m + c_m = 0$.

Hence, by (3a) we need only consider the case where $a_i = b_{i-1}$ and $a_0 = b_F = 0$ when $\sum a_i = \sum b_i$. Hence by (2), $\{a_i\} = \{b_i\}$.

Let us now consider the case when $\sum a_i > \sum b_i$. We must show that any social welfare function must necessarily prescribe $\{a_i\} > \{b_i\}$. Let us consider a vector $\{c_i\}$ such that $\sum c_i = \sum (a_i - b_i)$, $c_i = 0$ for $a_i < b_i$ and, $c_i + b_i \leqq a_i$, otherwise. Then $\{b_i + c_i\} = \{a_i\}$, by the preceding result, and $\{b_i + c_i\} > \{b_i\}$ by (1). Hence, $\{a_i\} > \{b_i\}$.

In the proof of Theorem 1, we added, using (3a), constants to the vectors $\{a_i\}$ and $\{b_i\}$ in proving the first case (a). It is interesting to note that we need not postulate the existence of any numbers a_i, b_i greater than those observed.

We prove this fact by means of mathematical induction on m, the number of voters. For $m = 2$, it is clear since the constants added are such that the resulting vectors contain only the numers 0, and $b_1 - a_1$, which may be taken as non-negative. Let us assume the result is true for vectors in $m - 1$ space, and let us choose two vectors satisfying case (a) in m space. Rearrange these vectors so that $a_m - b_m > 0$. Now consider the vectors

$$\{a_1, a_2, \ldots, a_{m-2}, a_{m-1}\} \text{ and } \{b_1, b_2, \ldots, b_{m-2}, b_{m-1} - (a_m - b_m)\}$$

which satisfy case (a) in m-1 space. By the induction hypothesis, we assume that constants may be added to these vectors such that none of the resulting numbers are greater than those appearing in the two vectors and, a fortiori in the vectors $\{a_i\}$, $\{b_i\}$. In the original vectors b_{m-1}, and a_m are transformed into $a_m - b_m$, and b_m is transfored into 0. Hence the result is proven.

The preceding argument suffices to show also that the constants added need never be such so as to obtain negative numbers.

If the original subset in the m space is restricted, furthermore, by the condition that a_i should be non-negative integers (rankings) the results still hold since, in that case, all constants added in the proof of Theorem 1 were also integers. Suppose condition (3a) is changed to read

3b) The a_i are determined except for arbitrary scale factors; e.g. $\left\{a_i\right\} > \left\{b_i\right\}$, if and only if $\left\{c_i\ a_i\right\} > \left\{c_i\ b_i\right\}$.

Then by replacing a_i by $a_i = e^{x_i}$ and using Theorem 1 in the x space we obtain

Theorem 2: A When (3a) is replaced by (3b) then the ordering relation defined by $\left\{a_i\right\} = \left\{b_i\right\}$ if $\prod a_i = \prod b_i$, $\left\{a_i\right\} > \left\{b_i\right\}$ if $\prod a_i > \prod b_i$ is a acceptable social welfare function. B. It is the only acceptable social welfare function [when (3a) is replaced by (3b)].

From Theorems 1 and 2 we have

Theorem 3: When (3a) is replaced by

3c) The a_i are determined except for arbitrary linear transformations; e.g., if $\left\{a_i\right\} > \left\{b_i\right\}$, if and only if $\left\{c_i + d_i a_i\right\} > \left\{c_i + d_i b_i\right\}$, then no social welfare function is possible.

Hence, these results give us some insight into the "Possibility Theorem" of Arrow.

Suppose now that a social welfare function is not required to satisfy conditions (2) and (3a). Then we have

Theorem 4: A If (2) and (3a) are not required, the ordering relation defined by $\left\{a_i\right\} = \left\{b_i\right\}$ if $f\left[\left\{a_i\right\}\right] = f\left[\left\{b_i\right\}\right]$, $\left\{a_i\right\} > \left\{b_i\right\}$ if $f\left[\left\{a_i\right\}\right] > f\left[\left\{b_i\right\}\right]$, is a acceptable social welfare function, when f is monotonic; i.e. for $c_{i_1} > 0$ and $c_i \geqq 0$, $f\left[\left\{b_i + c_i\right\}\right] > f\left[\left\{b_i\right\}\right]$.

B. These are the only acceptable social welfare functions (if (2) and (3a) are not required).

<u>Proof</u>: A is trivial. Now for B. Suppose we are given a social welfare function (i.e. an ordering of the points, in a subset of m space). By ordering we may define a real value function $g\left[\{a_i\}\right]$ which has the property that $g\left[\{a_i\}\right] = g\left[\{b_i\}\right]$ if and only if $\{a_i\} = \{b_i\}$ and $g\left[\{a_i\}\right] > g\left[\{b_i\}\right]$ if and only if $\{a_i\} > \{b_i\}$. Since the social welfare function satisfies (1) g is monotonic.

References

[1] Arrow, K. J., "Social Choice and Individual Values", John Wiley and Sons, New York (1951), Chap. 3.

[2] Hildreth, C. Cowles Commission Discussion Paper, Economics No. 2002, p. 1.

[3] Savage, L. J. "The Theory of Statistical Decisions", Journal of the American Statistical Association, Volume 46, No. 253, p. 64.