COWLES COMMISSION DISCUSSION PAPER: ECONOMICS NO. 278

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Towards A Theory of Financial Behavior[1]

by Harry Markowitz

May 8, 1950

Part I

Part one of this paper discusses the plans and goal of a project which the writer intends to undertake. Part two of this paper takes the first small steps toward the attainment of this goal.

The writer is attempting to formulate a set of equations whose purpose is to describe the behavior of financial institutions (particularly Commercial Banks and Insurance Companies).[2] These equations are to be fitted to and tested by the relevant data. Present banking theory and experience suggest some properties of such a set of equations. They suggest variables which should enter these equations; the signs of some derivatives; some homogeniety properties. They do not suggest, however, the form of these equations, or the distribution of the "unexplained residual."

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1. My thanks to Professor Marschak, Carl Christ, and various other people (in and out of the Cowles Commission). Discussions with them about this paper and its contents have been most helpful. Needless to say, my mistakes are my own.

2. Previous studies with similar purposes include:
A. J. Brown, "The Liquidity Preference Schedules of The London Clearing Banks," Oxford Economic Papers October 1938.
J. Tinbergen, Business Cycles in the U.S.A. 1919-1937 Chapter III B.

It seemed to the writer that there were two alternative methods by means of which a set of equations, to be fitted and tested by data, could be obtained: a) Such a set could be "arbitrarily" selected, perhaps on the basis of simplicity, from the infinite possible sets of equations which meet the suggested properties; or b) the behavior equations could be deduced from more fundamental postulates. The second alternative was chosen. Specifically, the behavior equations to be fitted will be derived from a set of assumptions which may be characterized as describing a rational individual who likes profits and dislikes risks, and who acts on the basis of subjective probability beliefs.[1]

Our assumptions must be narrow enough to have interesting verifiable consequences. It is too much to hope that these consequences will be consistent with observations in all respects. But we must start somewhere; and the writer feels that the fitting and testing of equations derived from such assumptions will be more useful and informative than the fitting of "arbitrarily" selected equations.

Our basic set of assumptions will include assumptions concerning 1) the opportunities, 2) the beliefs, and 3) the behavior of the individual given his opportunities and beliefs.

1) We will assume that the individual acts at discrete time points $t=\ldots,-1,0,1,\ldots$. At time t he must choose values of variables $X_{1\cdot t},\ldots,X_{n\cdot t}$ subject to constraints $C_{1\cdot t},\ldots,C_{h_t\cdot t}$. C_{jt} may take the form of equalities, inequalities or other restrictions on the vector $(X_{1\cdot t},\ldots,X_{n\cdot t})$. Given the values of $(X_{1\cdot t},\ldots,X_{n\cdot t})=(X)_t$ for $t<1$; and given the values of certain "other things" $(Y_{1\cdot t},\ldots,Y_{R\cdot t})=(Y)_t$ for $t\leq 1$, then the opportunities available at time $t=2$

1. The general implications of such assumptions, and their relation to other assumptions which could be made, has been discussed by Professor Marschak; for example, see "Role of Liquidity under Complete and Incomplete Information" Cowles Commission Papers, New Series No. 37.

(i.e., $C_{1.2},\ldots,C_{h_2.2}$) depend upon the choices $(X)_1$ and upon events $(Y)_2$. The opportunities available at t=3, given the conditions at t=1, depend upon $(X)_1$, $(Y)_2$, $(X)_2$, $(Y)_3$. And so on.

The opportunities available to (say) a Commercial Bank differ from those of an Insurance Company which, in turn, differ from those of (say) an Investment Company. The "institutional assumptions", from which we derive the behavior equations of such different institutions, must differ accordingly.

2) We will assume that the individual's beliefs concerning the variables Y_{it} consists of a subjective probability density distribution

$$P\ (,\ldots,(Y)_{-1},\ (Y)_o,\ (Y)_1,\ldots;\ (W)_{-1},\ (W)_o,\ (W)_1\ldots)$$

where $(W)_t$ is a vector of variables, observable at time t, which "influence" the Y's. To give our model empirical content, we will have to make narrow assumptions concerning $P(Y_{it};W_{st})$.

3) It will be assumed that there are desired variables $D_1,\ldots,D_q$. If the Y's were known with certainty, the X's would be chosen so as to maximize $U=U(D_1,\ldots,D_q)$. D_i is, in general, a function of X's and Y's. (The C's, we have said, are also functions of X's and Y's). Thus D_i may be, for example, dividends paid out at time t (an X), or "value" of the enterprise at time t (a C).

When the Y's are not thought of as certain, then $U=U(\mathcal{f}(D_1,\ldots,D_q))$; where $\mathcal{f}$ is the subjective probability distribution of $(D_1,\ldots,D_q)$. A strategy S prescribes, for every t, a function $X_{it}=G\ (Y_{1t},\ Y_{2t},\ldots,Y_{Rt},\ Y_{1t-1},\ldots,Y_{R\ t-1},\ldots;\ W_{1t},\ldots,W_{st-\Delta},\ldots)$ S and P (Y,W) determine $\mathcal{f}(D)$. We assume that the individual chooses a strategy S_o which maximizes U.

When q=1 we will assume that

$$U\ (\mathcal{f}_1\ (D)\ ) > U\ (\mathcal{f}_2\ (D)\ )$$

if $E_1(D) \geq E_2(D)$

$V_1(D) \leq V_2(D)$

where E is expected value; V is variance; and where the inequality holds in at least one instance. In other words, expected D is considered desirable; variance of D (risk) is considered undesirable. This can be generalized to $S > 1$.[1]

It seems expedient to begin by thinking through a system in which assumptions of type 2) and 3) are combined with simple assumptions of type 1). The writer has started with a model describing the behavior of an investor in securities. Market imperfections and operating costs have been assumed away. Such a simplified model has two uses: a) It raises, in a simple context, problems whose solutions will be useful when we deal with more complicated models which take into account the opportunities facing a Commercial Bank or Insurance Company. b) The conclusions of this simple model could be applied (perhaps after taking costs of transactions into account) to individual investors or Investment Companies.

In part II we set up the premises and take the first steps toward deriving the verifiable implications of this model of investment behavior.

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1. Another assumption which the writer would be willing to make concerning U is that $U(f_1(D)) > U(f_2(D))$ if $\int_{-\infty}^{D} f_1(D)\,dD \geq \int_{-\infty}^{D} f_2(D)\,dD$ where $\geq$ applies for all values of D, and $>$ applies for at least some values of D. This assumption has not been useful in our analysis so far.

Part II

1) For notational simplicity let us assume that every security can be distinguished by two criteria: the issuer and the length of time before maturity. Let S_{ijt} stand for the phrase "securities, issued by i, with j time-periods to go before maturity, available to the investor at time t." For stocks, $j=\infty$. At time t+1, S_{ijt} becomes $S_{i,j-1,t+1}$. Let Q_{ijt} be the "quantity" of S_{ijt} held by the investor. This is defined to be

$$\begin{cases} \text{a) The number of shares held,} & \text{if } j=\infty \\ \text{b) The maturity value,} & \text{if } j<\infty . \end{cases}$$

Let P_{ijt} be the "price" of S_{ijt}. This refers to

$$\begin{cases} \text{price per share,} & \text{if } j=\infty \\ \text{price per dollar of maturity value} & \\ = \dfrac{\text{dollar value at time t}}{\text{dollar value at maturity}}, & \text{if } j<\infty . \end{cases}$$

We will assume, in effect, that there are no costs involved in buying or selling, that the price at which a particular security could be bought by the investor is equal to the price at which it could be sold by him, and is given independently of his decisions. Let C_{ijt} be

$$\begin{cases} \text{dividend per share,} & \text{if } j=\infty \\ \dfrac{\text{amount of coupon payment,}}{\text{maturity value}} & \text{if } j<\infty . \end{cases}$$

C_{ijt} belongs to the party that holds S_{ij} from t-1 to t. The "profit" obtained from holding S_{ij} from t to t+1 is

$$\Pi_{ijt} = Q_{ijt}\,(P_{i,j-1,t+1} + C_{i,j-1,t+1} - P_{ijt})$$

$$\Pi_{ijt} = (P\cdot Q)_{ijt}\left(\frac{(P+C)_{i\cdot j-1\cdot t+1}}{P_{ijt}} - 1\right)$$

Let $X_{ijt} = (P_{ijt})(Q_{ijt})$. X_{ijt} is the amount of funds which the individual holds at time t in the form of securities S_{ij}. Let $Y_{ijt} = \left(\frac{(P+C)_{i\cdot j-1\cdot t+1}}{P_{ijt}} - 1\right)$ be the yield of S_{ijt}.

It will be convenient from now on to replace $(i\cdot j)$ by (h). Total profit from holding securities from t to t+1 is $\Pi_t = \sum_h X_{ht} Y_{ht}$.

The amount of assets tied up in securities from t to t+1 is $A_t = \sum_h X_{ht}$.

2) We assume that at time t=1, the beliefs of the investor concerning Y_{ht} can be expressed as a joint probability density distribution

$$P(\ldots, Y_{1\cdot 1}, \ldots, Y_{n\cdot 1}, Y_{1\cdot 2}, \ldots, Y_{ht}; \ldots, W_{1\cdot 1}, \ldots, W_{st}, \ldots).$$

a) We will assume that $(EY_{ht} \mid Y_{1\cdot t-1}, \ldots, Y_{i\cdot t-\Delta}, \ldots; W_{1\cdot t-1}, \ldots, W_{j\cdot t-\Delta}, \ldots) = Y^o_{ht} = \phi_h(Y_{1\cdot t-1}, \ldots; W_{1\cdot t-1}, \ldots) = \phi_h(Z_{1t}, \ldots, Z_{st})$ where ϕ, we will assume, is of a known form and is linear in any unknown parameters $\alpha_1, \ldots, \alpha_r$. For example, perhaps $Y^o_{ht} = \alpha_0 + \alpha_1 Y_{h,t-1} + \alpha_2 Y_{h,t-2} + \alpha_3$ (nation income).

b) We will assume that the distribution

$$P_t(Y_{1t} - Y^o_{1t}, Y_{2t} - Y^o_{2t}, \ldots, Y_{nt} - Y^o_{nt} \mid Z_{1t}, \ldots, Z_{st})$$ is the same for all t. In particular we will use below the assumption that $\sigma_{h\ell} = \sigma_{Y_{ht} Y_{\ell t}}$ is the same for all t.

Roughly speaking we assume that while the subjective expected yield of various securities may change through time, their pattern of "riskiness" will remain the same[1].

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1. Of course, in applying such a model to actual time series we would have to take into account events (such as changes in Federal Reserve support policy) which had significant effects on the σ's and α's. We could fit our derived equations only to data from periods of time during which the σ's and α's seemed about constant. Or we could generalize our theory to let the σ's and α's be variables.

3) One assumption we could make as to the investor's behavior is that, given his beliefs, his opportunities and the amount of funds to be invested in securities[1], he allocates his funds so as to make $E(\Pi_t)$ large, but keep $\sigma(\Pi_t)$ small. In other words he acts to maximize $U_t = U_t(E_t, V_t)$ where $E_t = E(\Pi_t)$, $V = \sigma_{\Pi_t}^2$, $\frac{\partial U}{\partial E} > 0$, $\frac{\partial U}{\partial V} < 0$.

We will argue later that this assumption is perhaps not realistic, but its implications will be useful in deriving those of other behavior assumptions.

Let F be an allocation of funds among securities. We define F_0 to be a step-wise efficient allocation at time t, if there does not exist an allocation F_1 such that

$$(E_t \mid F_1, P_t, W) \geq (E_t \mid F_0, P_t, W)$$

$$(V_t \mid F_1, P_t, W) \leq (V_t \mid F_0, P_t, W);$$

where the inequality holds in at least one case. The word "step-wise" is used because Π_t only refers to the profit made by holding securities from t to t+1. The word "efficient" is used in the usual sense where E and -V are desired "commodities". If $U_t = U_t(E_t, V_t)$ then the allocation of funds will always be step-wise efficient. The assumption of step-wise efficiency, plus the assumptions as to opportunities and beliefs stated in Part II 1) and 2) imply relationships between decisions $(X)_t$ and observable variables $(V)_t$ which can be expressed as an incomplete set of simultaneous equations. These equations are linear in their parameters, and if we make the usual (but dubious) assumptions as to how random disturbances enter our equations, we can estimate these parameters by the Limited Information Method[2]. Note, the assumption of step-wise efficiency not only makes no assumption concerning $U_t = U_t(E_t, V_t)$.

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1. We can assume that the quantity of funds allocated to securities depends upon what can be done with such funds if they are so allocated. In that case the equations which show what can be obtained -- given the amount of funds devoted to securities -- are an incomplete subset of a set of equations which explain the amount of funds devoted to securities as well as its allocation among securities.

2. This assumes that other economic relations permit the identification of these equations.

except as to its derivatives; it also does not assume that this function necessarily remains constant through time.

We will now derive some implications of step-wise efficiency.

We have the following definitions:

Assets devoted to securities:

1) $A = \sum_{h=1}^{n} X_h$

Expected profits:

2) $E = \sum_h X_h Y_h^o$

Variance of profits:

3) $V = \sum_h X_h^2 \sigma_{hh} + 2\sum_{h>\ell} X_h X_\ell \sigma_{h\ell}$

We will add the restrictions:

4) $X_h \geq 0 \quad h=1,\ldots,n.$

A necessary condition for an allocation of funds to be efficient is that no other allocation with the same A and E has a smaller V. A necessary condition for V to have a relative minimum (A and E held constant, ignoring restrictions 4) for a moment) is that

$$\frac{\partial}{\partial X_i}\left\{ V + 2\lambda_1 \left(E - \sum_h X_h Y_h^o\right) + 2\lambda_2 \left(A - \sum_h X_h\right)\right\} = 0 \qquad \text{for } i=1,\ldots,n$$

where $2\lambda_1$, $2\lambda_2$ are Lagrangian multipliers

Explicitly:

$$\frac{\partial}{\partial X_i}\left\{ \sum_h X_h^2 \sigma_{hh} + 2\sum_{h>\ell} X_h X_\ell \sigma_{h\ell} + 2\lambda_1 \left(E - \sum XY^o\right) + 2\lambda_2 (A - \sum X)\right\} = 0$$

$$2X_i \sigma_{ii} + 2\sum_{j \neq i} X_j \sigma_{ij} + 2\lambda_1 Y_i + 2\lambda_2 = 0 \quad i=1,\ldots,n$$

We can express the above equations and the two additional (constraint) equations 1) and 2) as follows:

$$5)\quad \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} & Y_1^o & 1 \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} & Y_2^o & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} & Y_n^o & 1 \\ Y_1^o & Y_2^o & \cdots & Y_n^o & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E \\ A \end{pmatrix}$$

We restrict X_h to be non-negative. If the minimum V (given A and E) is obtained by an allocation $(X_1^*,\ldots,X^*)$ with $X_h^* > 0$ h=1,...,n, then V must have at this point a relative minimum over the space of $X_h \gtreqless 0$ h=1,...,n. Then $(X_1^*,\ldots,X_n^*)$ satisfies equations 5). If (say) $X_1^* = 0$; $X_2^* > 0,\ldots,X_n^* > 0$, then V may not have a relative minimum over the space of $X_h^* \gtreqless 0$ but must have a relative minimum over the space: $X_1 = 0$; $X_h \gtreqless 0$ h = 2,...,n. This means that $(X_1^*,\ldots,X_n^*)$ may not satisfy 5), but will satisfy a set of equations which ignore X_1:

$$\begin{pmatrix} \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} & Y_2 & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \sigma_{2n} & \cdot & \cdots & \sigma_{n} & Y_n^o & 1 \\ Y_2^o & \cdot & \cdots & Y_n^o & 0 & 0 \\ 1 & \cdot & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_2 \\ \vdots \\ X_n \\ \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E \\ A \end{pmatrix}$$

Similar results hold if any $X_i^* = 0$ or more than one $X_i^* = 0$.

This implies that: if there are n+q securities, $S_1, \ldots, S_{n+q}$, in which the individual could invest, and if he invests in only n of them, say $S_1, \ldots, S_n$, then the positive amounts invested are related to their variances, covariances and expected yields as in equation 5)[1]. For, by our assumptions, the individual chooses an efficient allocation; an efficient allocation must have the least V, given A and E; V must have a relative minimum at the point $(X_1^*, \ldots, X_n^*, 0, \ldots, 0)$ over the space

$$X_h \gtreqless 0 \quad h=1, \ldots, n; \qquad X_\ell = 0 \quad \ell = n+1, \ldots, n+q;$$

and equation 5) is a necessary condition for such a minimum.

We may assume that the matrix in equation 5) has an inverse; otherwise the individual could not "solve for" X_h^*. Solving for 5) we get

$$6) \quad \begin{pmatrix} X_1 \\ \vdots \\ X_n \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} & Y_1^o & 1_1 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \sigma_{1n} & \cdots & \sigma_{nn} & Y_n^o & 1_n \\ Y_1^o & \cdots & Y_n^o & 0 & 0 \\ 1_1 & \cdots & 1_n & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E \\ A \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E \\ A \end{pmatrix}$$

From 6) we get

$$7) \quad X_i = \frac{E \cdot C\,(Y_i) + A \cdot C\,(1_i)}{|M|}$$

where C ( ) stands for "co-factor of".

From 7) we get

$$8) \quad X_i = \frac{(\Sigma X_h Y_h^o)\ (\Sigma \sigma_{ih}^{1} Y_h^o) + A \cdot \sum_{h=1}^{n} \sum_{\ell \neq i} \sigma_{ih\ell}^{"} Y_h^o Y_\ell^o}{\sum_{h,\ell} \sigma_{h\ell}^{"'} Y_h^o Y_\ell^o}$$

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1. 5) also applies if we allow $X_h < 0$. This would be a "short sale".

δ'_{ih}, $\delta''_{ih\lambda}$, $\delta'''_{h\ell}$ are sums of products of σ's. We will show this explicitly for $|M|$. The argument is similar for C (Y_i^o) and C (1_i). $|M|$ is a sum of terms, each term being a product of elements of the matrix. Each row and column of the matrix is represented once and only once in each product. Each non-zero product from $|M|$ must have an element Y_h^o from the row containing only Y^o's and zeros, and it must have an element Y_ℓ^o from the column containing only Y^o's and zeros. Combining terms, we get a quadratic in Y_h^o, with sums of products of σ's for coefficients. 8) may also be written

8a) $$X_i = \frac{(\Sigma X_h Y_h^o) \ L_i (Y^o) + A \cdot Q_i (Y^o)}{P (Y^o)}$$

where $L_i (Y^o)$ is linear in Y_h^o; $P (Y^o)$ and $Q_i(Y^o)$ are quadratic in Y_h^o. This is equivalent to

9) $X_i \cdot P \ (Y^o) = (\Sigma X_h Y_h^o) \ L_i \ (Y^o) + A \cdot Q_i \ (Y^o)$; $i=1,\ldots,n$.

If we substitute $Y_{ht}^o = \phi_h (V_{1t},\ldots,V_{St})$ see II 2)a we get a set of equations

10) $\xi_i(V_{1t},\ldots,V_{St}, \ X_{1t},\ldots,X_{nt}) = 0$ $i=1,\ldots,n$ which are linear in their parameters and contain only observable variables. This suggests estimation by the Limited Information Method. Before attempting such estimation we would have to think through two points:

a) Our equations, as they now stand, do not admit random variation in behavior. Such random variation exists; otherwise "estimation" would be unnecessary. What assumptions are reasonable as to how such random elements enter our equations?

b) The parameters of 10) are not unrelated. An efficient method of estimation would take this into account.

Another point to think through is this: Suppose we had estimates of the parameters of 10). Could we identify the σ's and α's of which these parameters are functions?

The σ's and α's would not add to the information in 10); but they might be interesting for some purposes.

The following objection can be raised against the assumption of step-wise efficient allocation: Is the unit of time equal to a quarter or a year? Certainly the investor acts more frequently than that. Is the unit of time equal to a month, week or day? Certainly the investor does not, in general, act this week for the rewards to be had next week.

Another assumption -- more realistic in some cases -- is that a manager of the invested funds takes as given the value of assets as of (say) January 1. He reinvests earnings until, and tries to maximize the value of assets as of (say) June 31. Similarly for the period July 1 to December 31. Or, in general, there exist m and t_o such that the individual acts at time t, where $t_o + (\lambda-1)\, m \leq t < t_o + \lambda m$, so as to maximize $U_\lambda = U_\lambda\,(E_{t_o+\lambda m},\ V_{t_o+\lambda m})$; where λ is an integer, E and V refer to expectation and variance of $A_{f(\lambda)}$ - assets as of $t_o + \lambda m$, before dividends are paid. It can be shown (see footnote) that if Y_{ht_1} was independent of Y_{ht_2} for all $t_1 \neq t_2$(fantastic assumption) then the above assumption implies step-wise efficient allocation[1].

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1. At time $t_o + \lambda m-1 = f-1$, given A_{f-1}, the investor's allocation will be step-wise efficient so as to maximize $U\,(E_f,\ V_f)$. Let

$$M = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} & r_1^o & 1_1 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \sigma_{1n} & \cdots & \sigma_{nn} & r_n^o & 1_n \\ r_1^o & \cdots & r_n^o & 0 & 0 \\ 1_1 & \cdots & 1_n & 0 & 0 \end{pmatrix} \quad \text{where } r_i^o = \left(E\ \frac{P_{i \cdot f} + C_{i \cdot f}}{P_{i \cdot f-1}} \,\middle|\, V_{1 \cdot f-1}, \ldots, V_{S \cdot f-1} \right)$$

I cannot present the implications when Y_{ht_1} is not independent of Y_{ht_2}. I can, I believe, present a method for finding these implications: At time $t_o + m\lambda - 1 = f - 1$, the investor will act in a step-wise efficient manner. At time

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Let

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ \sigma_{1n} & \cdots & \sigma_{nn} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} .$$

$$(V_f \mid A_{f-1}) = (X_1, \ldots, X_n, \lambda_1, \lambda_2)\ \Sigma \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = (0 \ldots 0\ E,\ A_{f-1}) M^{-1'}\ \ \Sigma\ M^{-1} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ E \\ A_{f-1} \end{pmatrix}$$

Given E, $(V_f \mid A_{f-1}) = (\mathcal{E} A_f^2 \mid A_{f-1}) - E^2 = \alpha E^2 + \beta E A_{f-1} + \gamma A_{f-1}^2$

where $\gamma = (C(1_1), \ldots, C(1_n), C(0), C(0))\ \ \Sigma \begin{pmatrix} C(1_1) \\ \cdot \\ \cdot \\ \cdot \\ C(0) \end{pmatrix} \geq 0$

For an efficient point $\frac{\partial V}{\partial A} < 0$: you can't get less V and the same E with the same or less A.

$$\frac{\partial \alpha E^2 + \beta E A_{f-1} + \gamma A_{f-1}^2}{\partial A_{f-1}} < 0 \qquad\qquad \beta E + 2\gamma A_{f-1} < 0.$$

Since E, γ, A_{f-1}, ≥ 0, β is negative. Since $Er_{f-1} = r^o_{f-1}$ is given independently of r_{f-2} and therefore independently of A_{f-1}, we have

$$VA_f^2 = \mathcal{E}(E\ A_f^2 \mid A_{f-1}) - E^2 = \alpha E^2 + \beta E\, \mathcal{E} A_{f-1} + \gamma\, \mathcal{E} A_{f-1}^2$$

Since $\gamma > 0$, $\beta E < 0$, this cannot be minimum unless the distribution of A_{f-1} has minimum variance for the given expected value. Note that the choice of $\mathcal{E} A_{f-1}$ is no longer arbitrary.

A similar argument holds for $f-2$, $f-3$, ..., $f-m$.

τ-2, given any achievable $E(A_\tau)$, given that the individuals action at $t = \tau - 1$ will be step-wise efficient, and given the distribution

$P\{(Y)_{\tau-1}; (Y)_\tau; (w)_{\tau-1} / V_{1.\tau-2}, \ldots, V_{S.\tau-2}\}$ then $V(A_\tau)$ depends on

$(X_{1.\tau-2}, \ldots, X_{n.\tau-2})$. $(X)_{\tau-2}$ is chosen so as to minimize V. A similar argument applies to τ-3, τ-4, ..., τ-m. We will get in general $X_{ht-\Delta} = G(V_{1.t-\Delta}, \ldots, V_{S.t-\Delta})$.

It obviously cannot be claimed that we have exhausted the assumptions which might reasonably be made concerning the investor's utility function. As we extend our results we must remember: It is not enough to have general or "reasonable" premises; we must be able to derive empirically verifiable - contradictable - conclusions. Otherwise - while our symbols may fit together elegantly - they have no meaning as economic theory.

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The symbol V has been given two meanings. This must cause confusion, as the particular meaning used is not always clear from context. With apologies for this oversight, the writer suggests the following changes:

p. 6) for $\phi_h \ (V_{1t},\ldots,V_{St})$

read $\phi_h \ (Z_{1t},\ldots,Z_{St})$

also, add $(Z_{1t},\ldots,Z_{St}) = (Y_{1.t-1},\ldots; W_{1.t-1}\ldots)$

p. 7) lines 9 and 10) read

$(E_t \mid F_1, P, Z_t) \geq (E_t \mid F_o, P, Z_t)$

$(V_t \mid F_1, P, Z_t) \leq (V_t \mid F_o, P, Z_t)$

line 17 for $(V)_t$, read $(Z)_t$

p. 9 line 5) after $X^*_h \gtreqless 0$

insert $h=1,\ldots,n$

p. 11 lines 12 and 13) change V's to Z's

p. 12 footnote) for $V_{1.f-1},\ldots,V_{S.f-1}$

read $Z_{1.f-1},\ldots,Z_{S.f-1}$

p. 14 line 3) for $V_{1.f-2},\ldots,V_{S.f-2}$

read $Z_{1.f-1},\ldots,Z_{S.f-1}$